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An ε-Uniform Numerical Method for a System of Convection-Diffusion Equations with Discontinuous Convection Coefficients and Source Terms

T. Valanarasu[1], R. Mythili Priyadharshini[2], N. Ramanujam[3] and A. Tamilselvan[3]

[1]Department of Mathematics
Bharathidasan University College
Perambalur - 621 107, Tamilnadu, India
valan_tmj@yahoo.co.in

[2]Department of MACS
National Institute of Technology Karnataka
Mangalore - 575 025, Karnataka, India
mythiliroy7777@yahoo.co.in

[3]Department of Mathematics
Bharathidasan University
Tiruchirappalli - 620 024, Tamilnadu, India
matram2k3@yahoo.com; atsga@yahoo.com

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Abstract

In this paper, a parameter-uniform numerical method is suggested to solve a system of singularly perturbed convection-diffusion equations with discontinuous convection coefficients and source terms subject to the Dirichlet boundary condition. The second derivative of each equation is multiplied by a distinctly small parameter, which leads to an overlap and interacting interior layer. A numerical method based on a piecewise uniform Shishkin mesh is constructed. Numerical results are presented to support the theoretical results.

Keywords: Singular perturbation problems, Shishkin mesh, discontinuous convection coefficients

MSC 2010 No.: 65L10

1. Introduction

Singular perturbation problems (SPPs) arise in various fields of applied mathematics such as fluid dynamics, elasticity, quantum mechanics, electrical networks, chemical reactor-theory, biochemical kinetics, gas porous electrodes theory, aerodynamics, plasma dynamics, oceanography, diffraction theory, reaction-diffusion processes and many other areas. Examples of SPPs include the linearized Navier-Stokes equation of fluid at high Reynolds number, heat transportation problem with large Peclet numbers, magneto-hydrodynamics duct problems at Hartman number and drift diffusion equation of semiconductor device modeling. It is a well-known fact that the solutions of the SPPs have a multi-scale character (non-uniform behaviour), that is, there are thin layer(s)(Boundary layer region) where the solution varies rapidly but when distant from the layer(s)(Outer region) the solution behaves regularly and varies slowly. There is a vast literature dealing with SPPs with smooth coefficients and source term for single equation (see Miller et al. (1996) – Farrel et al. (2000a) and references therein) and for system of differential equations (see Mathews (2000) – Tamilselvan and Ramanujam (2010) and references therein). Recently, a few authors have developed uniform numerical methods for SPPs with non-smooth data, that is, discontinuous source term and/or discontinuous convection coefficient and/or discontinuous diffusion coefficient for single equation Farrell et al. (2000b) – Mythili and Ramanujam (2009) and for system of equations Tamilselvan et al. (2007, 2010).

In Matthews et al. (2000a, 2000b), the authors studied a system of two coupled singularly perturbed reaction-diffusion equations for the cases $0<\varepsilon_1=\varepsilon_2<<1$ and $0<\varepsilon_1<<\varepsilon_2=1$. It is shown that a parameter robust numerical method can be constructed which gives first order convergence. Madden and Stynes (2003) examined the same problem for the cases $0<\varepsilon_1\le\varepsilon_2\le 1$. The solution to the system has boundary layers that overlap and interact. The structure of these layers was analysed and this led to the construction of a piecewise uniform mesh that is a variant of the usual Shishkin mesh. They showed that the scheme is almost first order uniform convergence in the perturbation parameters. In Valanarasu and Ramanujam (2004), the authors proposed an asymptotic numerical initial value method to solve a system of two coupled singularly perturbed convection-diffusion equations which involves solving a set of initial value problems and a system of terminal value problem by fitted operator method. Cen (2005) has examined the same for the case $0<\varepsilon_1\le\varepsilon_2<1$, which leads to an overlap and interacting boundary layer.

In this paper, a system of singularly perturbed convection-diffusion equations with discontinuous convection coefficients and source terms are considered on the unit interval $\Omega=(0,1)$. A single discontinuity in the convection coefficients and source terms are assumed to occur at a point $d\in\Omega$. It is convenient to introduce the notation $\Omega^-=(0,d),\Omega^+=(d,1)$ and to denote the jump at d in any function with $[w](d)=w(d+)-w(d-)$. In fact, we consider the following class of problems: find

$$y_1,\ y_2\in C^0(\overline{\Omega})\cap C^1(\Omega)\cap C^2(\Omega^-\cup\Omega^+)$$

such that

$$\begin{aligned} L_1\overline{y} &\equiv -\varepsilon_1 y_1''+a_1(x)\,y_1' + b_{11}(x)\,y_1+b_{12}(x)y_2 = f_1(x),\\ L_2\overline{y} &\equiv -\varepsilon_2 y_2''+a_2(x)y_2'+b_{21}(x)y_1+b_{22}(x)y_2 = f_2(x), \quad x\in\Omega^-\cup\Omega^+, \end{aligned} \tag{1}$$

with the boundary conditions

$$\begin{cases} y_1(0) = p, & y_1(1) = r, \\ y_2(0) = q, & y_2(1) = s. \end{cases} \tag{2}$$

Assume that

$$|[a_1](d)| \le C, \quad |[a_2](d)| \le C, \quad |[f_1](d)| \le C, \quad |[f_2](d)| \le C,$$

$$\alpha_1^* > a_1(x) > \alpha_1 > 0, \quad x < d, \qquad -\alpha_2^* < a_1(x) < -\alpha_2 < 0, \quad d < x,$$

$$\alpha_1^* > a_2(x) > \alpha_1 > 0, \quad x < d, \qquad -\alpha_2^* < a_2(x) < -\alpha_2 < 0, \quad d < x,$$

$$b_{12}(x) \le 0, \qquad b_{21}(x) \le 0,$$

$$b_{11}(x) + b_{12}(x) \ge 0, \qquad b_{21}(x) + b_{22}(x) \ge 0, \quad \forall \quad x \in \overline{\Omega}.$$

The parameters $\varepsilon_1, \varepsilon_2 \in (0,1)$ and without loss of generality we shall assume that $0 < \varepsilon_1 \le \varepsilon_2 < 1$. The functions $a_i(x),\ f_i(x)$ for $i = 1, 2$ are assumed to be sufficiently smooth on $\Omega^- \cup \Omega^+ \cup \{0,1\}$ and the function $b_{ij}(x)$ for $i,\ j = 1, 2$ are to be sufficiently smooth on $\overline{\Omega}$. The function $a_i(x),\ f_i(x)$ for $i = 1, 2$ are assumed to have a single jump discontinuity at the point $d \in \Omega$. In general this discontinuity give raise to interior layers in the solutions of the problems. Because $a_i(x),\ f_i(x)$ for $i = 1, 2$ are discontinuous at d, the solution $\bar{y} = (y_1, y_2)^T$ of (1) - (2) does not necessarily have continuous second derivative at the point d. The above weakly coupled system of singularly perturbed boundary value problem can be written in the vector form as

$$L\bar{y} \equiv \begin{pmatrix} L_1\bar{y} \\ L_2\bar{y} \end{pmatrix} \equiv \begin{pmatrix} -\varepsilon_1 \dfrac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon_2 \dfrac{d^2}{dx^2} \end{pmatrix} \bar{y} + A(x)\,\bar{y}' + B(x)\,\bar{y} = \bar{f}(x), \quad x \in \Omega^- \cup \Omega^+,$$

with the boundary conditions

$$\bar{y}(0) = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \bar{y}(1) = \begin{pmatrix} r \\ s \end{pmatrix},$$

where

$$A(x)=\begin{pmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{pmatrix},\quad B(x)=\begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix} \text{ and } \bar{f}(x)=\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

***Note*:**

Throughout this paper, C denotes a generic constant (sometimes subscripted) which is independent of the singular perturbation parameters $\varepsilon_1, \varepsilon_2$ and the dimension of the discrete problem N. Let $y: D \rightarrow \Re$, $D \subset \Re$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the maximum norm $\| y \|_D = \sup_{x \in D} | y(x) |$. In case of vectors $\bar{y}$, we define $| \bar{y}(x) | = (| y_1(x) |, | y_2(x) |)^T$ and $\| \bar{y} \|_D = \max \{\| y_1 \|_D, \| y_2 \|_D\}$. Throughout the paper, we shall also assume that $\varepsilon_1 \leq C N^{-1}$ and $\varepsilon_2 \leq C N^{-1}$ as is often assumed for convection dominated problems.

Remark 1.1.

For simplicity, we are considering the functions b_{11}, b_{12}, b_{21} and b_{22} are sufficiently smooth on $\overline{\Omega}$. If we allow a simple jump discontinuity at $x = d$ for those functions, the results of this paper remain true. The sign condition imposed on a_1, a_2 is motivated by the argument given in Farrel et al. (2004a) for single equation.

2. Preliminaries

In this section, first we prove the existence of a solution of the BVP (1)-(2). Then we derive a maximum principle and stability result for the same. Further bounds for the solution, smooth and singular components and their derivatives are derived.

Theorem 2.1.

The BVP (1)-(2) has a solution such that y_1, $y_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.

Proof:

The proof is by construction. Let $\bar{u}^- = (u_1^-, u_2^-)^T$ and $\bar{u}^+ = (u_1^+, u_2^+)^T$ be particular solutions of the following systems of equations

$$\begin{pmatrix} -\varepsilon_1 \dfrac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon_2 \dfrac{d^2}{dx^2} \end{pmatrix} \bar{u}^-(x) + A(x)\, \bar{u}^{-'}(x) + B(x)\, \bar{u}^-(x) = \bar{f}(x), \quad x \in \Omega^-,$$

and

$$\begin{pmatrix} -\varepsilon_1 \dfrac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon_2 \dfrac{d^2}{dx^2} \end{pmatrix} \bar{u}^+(x) + A(x)\,\bar{u}^{+'}(x) + B(x)\,\bar{u}^+(x) = \bar{f}(x), \quad x \in \Omega^+.$$

Also let $\bar{\phi}$ and $\bar{\psi}$ be the solutions of the following BVPs

$$\begin{pmatrix} -\varepsilon_1 \dfrac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon_2 \dfrac{d^2}{dx^2} \end{pmatrix} \bar{\phi}(x) + A(x)\,\bar{\phi}'(x) + B(x)\,\bar{\phi}(x) = \bar{0}, \ \ x \in \Omega, \ \bar{\phi}(0) = \bar{1}, \quad \bar{\phi}(1) = \bar{0},$$

and

$$\begin{pmatrix} -\varepsilon_1 \dfrac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon_2 \dfrac{d^2}{dx^2} \end{pmatrix} \bar{\psi}(x) + A(x)\,\bar{\psi}'(x) + B(x)\,\bar{\psi}(x) = \bar{0}, \quad x \in \Omega, \ \bar{\psi}(0) = \bar{0}, \quad \bar{\psi}(1) = \bar{1},$$

respectively. Here $\bar{0} = (0, 0)^T$ and $\bar{1} = (1, 1)^T$. Then, $\bar{u}$ can be written as

$$\bar{y}(x) = \begin{cases} \bar{u}^-(x) + \begin{pmatrix} y_1(0) - u_1^-(0) & 0 \\ 0 & y_2(0) - u_2^-(0) \end{pmatrix} \bar{\phi}(x) + K\,\bar{\psi}(x), & x \in \Omega^-, \\[2ex] \bar{u}^+(x) + \begin{pmatrix} y_1(1) - u_1^+(1) & 0 \\ 0 & y_2(1) - u_2^+(1) \end{pmatrix} \bar{\psi}(x) + K^*\,\bar{\phi}(x), & x \in \Omega^+, \end{cases}$$

where K and K^* are matrices with constant entries to be chosen suitably. Note that on the open interval Ω, $\bar{0} < \bar{\phi}, \ \bar{\psi} < \bar{1}$, and $\bar{\phi}, \bar{\psi}$ cannot have internal maximum or minimum and hence

$$\bar{\phi}' < \bar{0}, \quad \bar{\psi}' > \bar{0}, \quad x \in \Omega.$$

We wish to choose the matrices $K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ and $K^* = \begin{pmatrix} k_1^* & 0 \\ 0 & k_2^* \end{pmatrix}$ so that $y_1, y_2 \in C^1(\Omega)$. That is, we impose the conditions

$$\bar{y}(d-) = \bar{y}(d+) \quad \text{and} \quad \bar{y}'(d-) = \bar{y}'(d+).$$

For the matrices K and K^* to exist is required that

$$\begin{vmatrix} \psi_1(d) & 0 & -\phi_1(d) & 0 \\ 0 & \psi_2(d) & 0 & -\phi_2(d) \\ \psi_{1'}(d) & 0 & -\phi_{1'}(d) & 0 \\ 0 & \psi_{2'}(d) & 0 & -\phi_{2'}(d) \end{vmatrix} \neq 0.$$

This follows from observing the fact that $(\psi_{2'}\,\phi_2 - \psi_2\,\phi_{2'})\,(\phi_1\,\psi_{1'} - \psi_1\,\phi_{1'}) > 0.$

Theorem 2.2. (Maximum Principle)

Suppose that a function $\bar{y}(x) = (y_1(x), y_2(x))^T$, $y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ satisfies $\bar{y}(0) \geq \bar{0}$, $\bar{y}(1) \geq \bar{0}$, $L_1\bar{y}(x) \geq 0$, $L_2\bar{y}(x) \geq 0$, $\forall \quad x \in \Omega^- \cup \Omega^+$ and $[\bar{y}'](d) \leq \bar{0}$. Then, $\bar{y}(x) \geq \bar{0}$, $\forall \quad x \in \bar{\Omega}$.

Proof:

Define $\bar{s}(x) = (s_1(x), s_2(x))^T$ as

$$s_1(x) = s_2(x) = \begin{cases} 1/2 + x/8 - d/8, & x \in \Omega^- \cup \{0, d\} \\ 1/2 - x/4 + d/4, & x \in \Omega^+ \cup \{1\}, \end{cases}$$

where $s_1, s_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$.

Then, $\bar{s}(x) > \bar{0}$, for all $x \in \bar{\Omega}$ and $L\bar{s}(x) > \bar{0}$, $\quad x \in \Omega^- \cup \Omega^+$. We define

$$\zeta = \max\left\{ max_{x\in\bar{\Omega}}(-\frac{y_1}{s_1})(x), \quad max_{x\in\bar{\Omega}}(-\frac{y_2}{s_2})(x) \right\}.$$

Then, $\zeta > 0$ and there exists a point x_0 such that either $(\frac{-y_1}{s_1})(x_0) = \zeta$ or $(\frac{-y_2}{s_2})(x_0) = \zeta$ or both. Further, $x_0 \in \Omega^- \cup \Omega^+$ or $x_0 = d$. Also $(\bar{y} + \zeta\bar{s})(x) \geq \bar{0}$, $\forall \quad x \in \bar{\Omega}$.

Case (i): $(y_1 + \zeta s_1)(x_0) = 0$, for $\Omega^- \cup \Omega^+$. Therefore, $(y_1 + \zeta s_1)$ attains its minimum at x_0. Then,

$$\begin{aligned} 0 < L_1\bar{y}(x) \equiv -\varepsilon_1(y_1 + \zeta s_1)''(x) + a_1(x)(y_1 + \zeta s_1)'(x) + b_{11}(x)(y_1 + \zeta s_1)(x) \\ + b_{12}(x)(y_2 + \zeta s_2)(x) \leq 0, \end{aligned}$$

which is a contradiction.

Case (ii): Similarly, we can consider the case $(y_2 + \zeta s_2)(x_0) = 0$, for $x_0 \in \Omega^- \cup \Omega^+$ and arrive at a contradiction.

Case (iii): $(y_1+\zeta s_1)(x_0)=0,$ for $x_0=d$. Therefore $(y_1+\zeta s_1)$ attains its minimum at x_0. Then,

$$0\le [(y_1+\zeta s_1)'](x_0)=[y_1'](d)+\zeta[s_1'](d)<0,$$

which is a contradiction.

Case (iv): Similarly, we can consider the case $(y_2+\zeta s_2)(x_0)=0,$ for $x_0=d$ and arrive at a contradiction.

Hence, $\bar{y}(x)\ge\bar{0}, \quad \forall \quad x\in\bar{\Omega}.$

Theorem 2.3. (Stability Result)

If

$$y_1, y_2 \in C^0(\bar{\Omega})\cap C^1(\Omega)\cap C^2(\Omega^-\cup\Omega^+),$$

then

$$|y_j(x)|\le C\max\{|y_1(0)|,|y_1(1)|,|y_2(0)|,|y_2(1)|,\|L_1\bar{y}\|_{\Omega^-\cup\Omega^+},\|L_2\bar{y}\|_{\Omega^-\cup\Omega^+}\},$$
$$x\in\bar{\Omega}, \quad j=1,2.$$

Proof:

Set

$$A=C\max\{|y_1(0)|,|y_1(1)|,|y_2(0)|,|y_2(1)|,\|L_1\bar{y}\|_{\Omega^-\cup\Omega^+},\|L_2\bar{y}\|_{\Omega^-\cup\Omega^+}\}.$$

Define two barrier functions $\bar{w}^{\pm}(x)=(w_1^{\pm}(x),w_2^{\pm}(x))^T$ as

$$w_j^{\pm}(x)=\begin{cases} A(1/2+x/8-d/8)\pm y_j(x), & x\le d, \\ A(1/2-x/4+d/4)\pm y_j(x), & x>d, \quad j=1,2. \end{cases}$$

Further, we observe that

$$\bar{w}^{\pm}(0)\ge\bar{0}, \quad \bar{w}^{\pm}(1)\ge\bar{0}, \quad L_1\bar{w}^{\pm}(x)\ge 0, \quad L_2\bar{w}^{\pm}(x)\ge 0$$

and $[\bar{w}^{\pm'}](d)\le\bar{0}$, by a proper choice of A. Then, $\bar{w}^{\pm}(x)\ge\bar{0}, \quad \forall \quad x\in\bar{\Omega}$, by Theorem 2.2, which completes the proof.

To derive parameter uniform error estimates we need sharper bounds on the derivatives of the solution $\bar{y}$. Consider the following decomposition of the solution $\bar{y} = \bar{v} + \bar{w}$, into a non-layer component $\bar{v} = (v_1, v_2)^T$ and an interior layer component $\bar{w} = (w_1, w_2)^T$. Define the discontinuous functions $\bar{v}_0 = (v_{01}, v_{02})^T$ and $\bar{v}_1 = (v_{11}, v_{12})^T$ to be respectively the solutions of the problems

$$A(x)\,\bar{v}_{0'} + B(x)\,\bar{v}_0 = \bar{f}(x), \quad x \in \Omega^- \cup \Omega^+, \ \bar{v}_0(0) = \bar{y}(0), \ \bar{v}_0(1) = \bar{y}(1),$$

and

$$A(x)\,\bar{v}_{1'} + B(x)\,\bar{v}_1 = \begin{pmatrix} \frac{1}{\varepsilon_2}\frac{d^2}{dx^2} & 0 \\ 0 & \frac{1}{\varepsilon_1}\frac{d^2}{dx^2} \end{pmatrix} \bar{v}_0, \quad x \in \Omega^- \cup \Omega^+, \ \bar{v}_1(0) = \bar{0}, \ \bar{v}_1(1) = \bar{0}.$$

We define the discontinuous functions $\bar{v}$ and $\bar{w}$ respectively by

$$L\,\bar{v} = \bar{f}(x), \quad x \in \Omega^- \cup \Omega^+, \tag{3}$$

$$\begin{cases} \bar{v}(0) = \bar{y}(0), \quad \bar{v}(d-) = \bar{v}_0(d-) + \varepsilon_1\varepsilon_2\,\bar{v}_1(d-), \\ \bar{v}(d+) = \bar{v}_0(d+) + \varepsilon_1\varepsilon_2\,\bar{v}_1(d+), \quad \bar{v}(1) = \bar{y}(1), \end{cases} \tag{4}$$

and

$$L\,\bar{w} = \bar{0}, \quad x \in \Omega^- \cup \Omega^+, \tag{5}$$

$$\bar{w}(0) = \bar{0}, \quad [\bar{w}](d) = -[\bar{v}](d), \quad [\bar{w}'](d) = -[\bar{v}'](d), \quad \bar{w}(1) = \bar{0}. \tag{6}$$

Note that $\bar{y} = \bar{v} + \bar{w}$ is in $C^1(\Omega)$. The following lemma provides the bound on the derivatives of the nonlayer and interior components of the solution $\bar{y}$.

Lemma 2.4.

Let $\bar{v}$ and $\bar{w}$ be the solution of the BVP (3) - (4) and (5) - (6) respectively. Then there exists a constant C such that for all $x \in \Omega^- \cup \Omega^+$ we have for $j = 1,2$,

$$\| v_j^{(k)} \| \le C, \quad k = 0, 1, 2, 3,$$

and

$$| w_j(x) | \le \begin{cases} CB^-_{\varepsilon_2}, & x \in \Omega^-, \\ CB^+_{\varepsilon_2}, & x \in \Omega^+, \end{cases}$$

$$|w_1'(x)| \le \begin{cases} C(\varepsilon_1^{-1}B_{\varepsilon_1}^- + \varepsilon_2^{-1}B_{\varepsilon_2}^-), & x \in \Omega^-, \\ C(\varepsilon_1^{-1}B_{\varepsilon_1}^+ + \varepsilon_2^{-1}B_{\varepsilon_2}^+), & x \in \Omega^+, \end{cases} \qquad |w_2'(x)| \le \begin{cases} C\varepsilon_2^{-1}B_{\varepsilon_2}^-, & x \in \Omega^-, \\ C\varepsilon_2^{-1}B_{\varepsilon_2}^+, & x \in \Omega^+, \end{cases}$$

$$|w_1''(x)| \le \begin{cases} C(\varepsilon_1^{-2}B_{\varepsilon_1}^- + \varepsilon_2^{-2}B_{\varepsilon_2}^-), & x \in \Omega^-, \\ C(\varepsilon_1^{-2}B_{\varepsilon_1}^+ + \varepsilon_2^{-2}B_{\varepsilon_2}^+), & x \in \Omega^+, \end{cases} \qquad |w_2''(x)| \le \begin{cases} C\varepsilon_2^{-2}B_{\varepsilon_2}^-, & x \in \Omega^-, \\ C\varepsilon_2^{-1}B_{\varepsilon_2}^+, & x \in \Omega^+, \end{cases}$$

$$|w_1'''(x)| \le \begin{cases} C(\varepsilon_1^{-3}B_{\varepsilon_1}^- + \varepsilon_2^{-3}B_{\varepsilon_2}^-), & x \in \Omega^-, \\ C(\varepsilon_1^{-3}B_{\varepsilon_1}^+ + \varepsilon_2^{-3}B_{\varepsilon_2}^+), & x \in \Omega^+, \end{cases} \qquad |w_2'''(x)| \le \begin{cases} C\varepsilon_2^{-1}(\varepsilon_1^{-2}B_{\varepsilon_1}^- + \varepsilon_2^{-2}B_{\varepsilon_2}^-), x \in \Omega^-, \\ C\varepsilon_2^{-1}(\varepsilon_1^{-2}B_{\varepsilon_1}^+ + \varepsilon_2^{-2}B_{\varepsilon_2}^+), x \in \Omega^+, \end{cases}$$

where

$$B_{\varepsilon_1}^- = \exp(-\alpha(d-x)/\varepsilon_1), \qquad B_{\varepsilon_2}^- = \exp(-\alpha(d-x)/\varepsilon_2),$$

and

$$B_{\varepsilon_1}^+ = \exp(-\alpha(x-d)/\varepsilon_1), \qquad B_{\varepsilon_2}^+ = \exp(-\alpha(x-d)/\varepsilon_2) \quad \text{and} \quad \alpha = \min\{\alpha_1, \alpha_2\}.$$

Proof:

Using appropriate barrier functions, applying Theorem 2.2 and adopting the method of proof used in Cen (2005), the present lemma can be proved.

3. Discrete Problem

The BVP (1) - (2) is discretised using a fitted mesh method composed of a finite difference operator on a piecewise uniform mesh. When $0 < \varepsilon_1 < \varepsilon_2 = 1$, the solution of (1) - (2) has overlapping interior layer at $x = d$. This necessitates the construction of a mesh that is uniform on each of six subintervals. We define

$$\sigma_{\varepsilon_2}^- = \min\left\{\frac{d}{2}, \frac{2\varepsilon_2}{\alpha}\ln N\right\}, \qquad \sigma_{\varepsilon_1}^- = \min\left\{\frac{d}{4}, \frac{\sigma_{\varepsilon_2}^-}{2}, \frac{2\varepsilon_1}{\alpha}\ln N\right\}$$

$$\sigma_{\varepsilon_2}^+ = \min\left\{\frac{1-d}{2}, \frac{2\varepsilon_2}{\alpha}\ln N\right\}, \quad \sigma_{\varepsilon_1}^+ = \min\left\{\frac{1-d}{4}, \frac{\sigma_{\varepsilon_2}^+}{2}, \frac{2\varepsilon_1}{\alpha}\ln N\right\}.$$

A piecewise uniform mesh $\overline{\Omega}_{\varepsilon_1,\varepsilon_2}^N$ is constructed by dividing [0,1] into six subintervals

$[0, d-\sigma_{\varepsilon_2}^-]$, $[d-\sigma_{\varepsilon_2}^-, d-\sigma_{\varepsilon_1}^-]$, $[d-\sigma_{\varepsilon_1}^-, d]$, $[d, d+\sigma_{\varepsilon_1}^+]$, $[d+\sigma_{\varepsilon_1}^+, d+\sigma_{\varepsilon_2}^+]$ and $[d+\sigma_{\varepsilon_2}^+, 1]$.

Then, subdivide $[0, d-\sigma_{\varepsilon_2}^-]$ and $[d+\sigma_{\varepsilon_2}^+, 1]$ into N/4 mesh intervals and subdivide each of the other four intervals into $N/8$ mesh intervals. The interior points of the mesh are denoted by

$$\Omega^N_{\varepsilon_1,\varepsilon_2} = \{x_i : 1 \le i \le \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \le i \le N-1\}.$$

Clearly $x_{N/2} = d$ and $\overline{\Omega}^N_{\varepsilon_1,\varepsilon_2} = \{x_i\}_0^N$. The step sizes of the mesh $\overline{\Omega}^N_{\varepsilon_1,\varepsilon_2}$ satisfy

$$h_i = \begin{cases} H_1 = \dfrac{4(d - \sigma^-_{\varepsilon_2})}{N}, & 1 \le i \le N/4, \\ H_2 = \dfrac{8(\sigma^-_{\varepsilon_2} - \sigma^-_{\varepsilon_1})}{N}, & N/4 \le i \le 3N/8, \\ H_3 = \dfrac{8\sigma^-_{\varepsilon_1}}{N}, & 3N/8 \le i \le N/2, \\ H_4 = \dfrac{8\sigma^+_{\varepsilon_1}}{N}, & N/2 \le i \le 5N/8, \\ H_5 = \dfrac{8(\sigma^+_{\varepsilon_2} - \sigma^+_{\varepsilon_1})}{N}, & 5N/8 \le i \le 6N/8, \\ H_6 = \dfrac{4(1 - d - \sigma^+_{\varepsilon_2})}{N}, & 6N/8 \le i \le N. \end{cases}$$

When $\sigma^-_{\varepsilon_1} = \sigma^-_{\varepsilon_2}/2$ and $\sigma^+_{\varepsilon_1} = \sigma^+_{\varepsilon_2}/2$, then $\varepsilon_2 = O(\varepsilon_1)$ and the result can be easily obtained. Therefore, we only consider the cases $\sigma^-_{\varepsilon_1} < \sigma^-_{\varepsilon_2}/2$ and $\sigma^+_{\varepsilon_1} < \sigma^+_{\varepsilon_2}/2$. Then the fitted mesh finite difference method is to find $\overline{Y}(x_i) = (Y_1(x_i), Y_2(x_i))^T$ for $i = 0,1,\cdots,N$ such that for $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$,

$$\begin{cases} L_1^N \overline{Y}(x_i) \equiv -\varepsilon_1 \delta^2 Y_1(x_i) + a_1(x_i)\, DY_1(x_i) + b_{11}(x_i)\, Y_1(x_i) + b_{12}(x_i)\, Y_2(x_i) = f_1(x_i), \\ L_2^N \overline{Y}(x_i) \equiv -\varepsilon_2 \delta^2 Y_2(x_i) + a_2(x_i)\, DY_2(x_i) + b_{21}(x_i)\, Y_1(x_i) + b_{22}(x_i)\, Y_2(x_i) = f_2(x_i), \end{cases} \tag{7}$$

$$\begin{cases} Y_1(x_0) = y_1(0), & D^- Y_1(x_{N/2}) = D^+ Y_1(x_{N/2}), & Y_1(x_N) = y_1(1), \\ Y_2(x_0) = y_2(0), & D^- Y_2(x_{N/2}) = D^+ Y_2(x_{N/2}), & Y_2(x_N) = y_2(1), \end{cases} \tag{8}$$

where

$$DY_j(x_i) = \begin{cases} D^- Y_j(x_i), & i < N/2 \\ D^+ Y_j(x_i), & i > N/2 \end{cases}, \quad \delta^2 Y_j(x_i) = \frac{(D^+ - D^-) Y_j(x_i)}{(x_{i+1} - x_{i-1})/2},$$

$$D^+ Y_j(x_i) = \frac{Y_j(x_{i+1}) - Y_j(x_i)}{x_{i+1} - x_i}, \text{ and } D^- Y_j(x_i) = \frac{Y_j(x_i) - Y_j(x_{i-1})}{x_i - x_{i-1}}, \text{ for } j = 1,2.$$

The difference operator L^N can be defined as

$$L^N\overline{Y}(x_i) \equiv \begin{pmatrix} L_1^N\overline{Y}(x_i) \\ L_2^N\overline{Y}(x_i) \end{pmatrix} \equiv \begin{pmatrix} -\varepsilon_1\delta^2 & 0 \\ 0 & -\varepsilon_2\delta^2 \end{pmatrix}\overline{Y}(x_i) + A(x_i)\, D\overline{Y}(x_i) + B(x_i)\,\overline{Y}(x_i)$$
$$= \overline{f}(x_i), \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2},$$
$$\overline{Y}(x_0) = \overline{y}(0), D^-\overline{Y}(x_{N/2}) = D^+\overline{Y}(x_{N/2}), \overline{Y}(x_N) = \overline{y}(1).$$

3.1. Numerical Solution Estimates

Analogous to the continuous results stated in Theorem 2.2 and Theorem 2.3 one can prove the following result.

Theorem 3.1.

For any mesh function $\overline{Z}(x_i)$, assume that

$$Z_j(x_0) \geq 0, \quad Z_j(x_N) \geq 0, \quad D^+Z_j(x_{N/2}) - D^-Z_j(x_{N/2}) \leq 0,$$

for $j = 1, 2$, and $L^N\overline{Z}(x_i) \geq \overline{0}$, $\forall\, x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$. Then $\overline{Z}(x_i) \geq \overline{0}$, $\forall \quad x_i \in \overline{\Omega}^N_{\varepsilon_1,\varepsilon_2}$.

Proof:

As in the continuous case, assume that the theorem is not true. Let $\overline{S} = (S_1, S_2)^T$, be

$$S_1(x_i) = S_2(x_i) = \begin{cases} 1/2 + x_i/8 - d/8, & 1 \leq i \leq N/2, \\ 1/2 - x_i/4 + d/4, & N/2 + 1 \leq i \leq N. \end{cases}$$

Define

$$\xi = \max\left\{ \max_{0\leq i\leq N}\left(\frac{-Z_1}{S_1}\right)(x_i), \quad \max_{0\leq i\leq N}\left(\frac{-Z_2}{S_2}\right)(x_i) \right\}.$$

It is obvious that, $\xi > 0$ and $\overline{Z}(x_i) + \xi\,\overline{S}(x_i) \geq \overline{0}$ for $i = 0(1)N$. Further, there exists one $i^* \in \{1, 2, ..., N\}$ such that either $Z_1(x_{i^*}) + \xi\, S_1(x_{i^*}) = 0$ or $Z_2(x_{i^*}) + \xi\, S_2(x_{i^*}) = 0$ or both. Also either $x_{i^*} \in \Omega^N_{\varepsilon_1,\varepsilon_2}$ or $x_{i^*} = x_{N/2}$.

Case (i): $x_{i^*} \in \Omega^N_{\varepsilon_1,\varepsilon_2}$ and $Z_1(x_{i^*}) + \xi\, S_1(x_{i^*}) = 0.$ Then for $x_{i^*} < x_{N/2}$, we have

$$L_1^N(\overline{Z}(x_{i^*}) + \xi\,\overline{S}(x_{i^*})) = -\varepsilon_1\delta^2(Z_1(x_{i^*}) + \xi\, S_1(x_{i^*})) + a_1(x_{i^*})D^-(Z_1(x_{i^*}) + \xi\, S_1(x_{i^*}))$$

$$+b_{11}(x_{i^*})(Z_1(x_{i^*})+\xi\, S_1(x_{i^*}))+b_{12}(x_{i^*})(Z_2(x_{i^*})+\xi\, S_2(x_{i^*}))\le 0,$$

a contradiction. Also for $x_{i^*} > x_{N/2}$, we have

$$L_1^N(\bar{Z}(x_{i^*})+\xi\bar{S}(x_{i^*})) = -\varepsilon_1\delta^2(Z_1(x_{i^*})+\xi S_1(x_{i^*}))+a_1(x_{i^*})D^+(Z_1(x_{i^*})+\xi S_1(x_{i^*}))$$
$$+b_{11}(x_{i^*})(Z_1(x_{i^*})+\xi\, S_1(x_{i^*}))+b_{12}(x_{i^*})(Z_2(x_{i^*})+\xi\, S_2(x_{i^*}))\le 0,$$

a contradiction.

Case (ii): Similarly, we can consider the case $x_{i^*}\in\Omega^N_{\varepsilon_1,\varepsilon_2}$ and $Z_2(x_{i^*})+\xi\, S_2(x_{i^*})=0$, and arrive at a contradiction.

Case (iii): $x_{i^*}=x_{N/2}$. Then,

$$D^+(Z_1(x_{i^*})+S_1(x_{i^*}))-D^-(Z_1(x_{i^*})+S_1(x_{i^*}))\ge 0,\quad \text{if}\quad Z_1(x_{i^*})+\xi\, S_1(x_{i^*})=0\,,$$

or

$$D^+(Z_2(x_{i^*})+S_2(x_{i^*}))-D^-(Z_2(x_{i^*})+S_2(x_{i^*}))\ge 0,\quad \text{if}\quad Z_2(x_{i^*})+\xi\, S_2(x_{i^*})=0,$$

which is a contradiction. Hence, we get the desired result.

Theorem 3.2 .

If $Y_j(x_i)$ is the solution of the problem (7), then

$$|Y_j(x_i)|\le C,\qquad j=1,2,\qquad x_i\in\overline{\Omega}^N_{\varepsilon_1,\varepsilon_2}.$$

To bound the nodal error $|(\bar{Y}-\bar{y})(x_i)|$, we define mesh functions $\bar{V}_L$ and $\bar{V}_R$, which approximate $\bar{v}$ respectively to the left and right of the point of discontinuity $x=d$. Then, we construct mesh functions $\bar{W}_L$ and $\bar{W}_R$, so that the amplitude of the jump $\bar{W}_R(d)-\bar{W}_L(d)$ is determined by the size of the jump $|[\bar{v}](d)|$. Also $\bar{W}_L$ and $\bar{W}_R$, are sufficiently small away from the interior layer region. Using these mesh functions the nodal error $|(\bar{Y}-\bar{y})(x_i)|$ is then bounded separately outside and inside the layer. Define the mesh functions $\bar{V}_L$ and $\bar{V}_R$ to be the solutions of the following discrete problems respectively :

$$L^N\bar{V}_L(x_i)=\bar{f}(x_i),\quad \text{for}\quad i=1,\dots,N/2-1, \tag{9}$$

$$\bar{V}_L(x_0)=\bar{v}(0),\qquad \bar{V}_L(x_{N/2})=\bar{v}(d-), \tag{10}$$

and

$$L^N \overline{V}_R(x_i) = \overline{f}(x_i), \quad \text{for } i = N/2+1, \ldots, N-1, \tag{11}$$

$$\overline{V}_R(x_{N/2}) = \overline{v}(d+), \qquad \overline{V}_R(x_N) = \overline{v}(1). \tag{12}$$

Now, we define the mesh functions $\overline{W_L}$ and $\overline{W_R}$ to be the solutions of the following system of finite difference equations

$$L^N \overline{W}_L(x_i) = \overline{0}, \quad \text{for} \quad i = 1, \ldots, N/2-1, \tag{13}$$

$$L^N \overline{W}_R(x_i) = \overline{0}, \quad \text{for} \quad i = N/2+1, \ldots, N-1, \tag{14}$$

$$\overline{W}_L(x_0) = \overline{0}, \quad \overline{W}_R(x_N) = \overline{0}, \tag{15}$$

$$\overline{W_R}(x_{N/2}) + \overline{V}_R(x_{N/2}) = \overline{W_L}(x_{N/2}) + \overline{V}_L(x_{N/2}), \tag{16}$$

$$D^+\overline{W_R}(x_{N/2}) + D^+\overline{V}_R(x_{N/2}) = D^-\overline{W_L}(x_{N/2}) + D^-\overline{V}_L(x_{N/2}). \tag{17}$$

Now, we can define $\overline{Y}(x_i)$ to be

$$\overline{Y}(x_i) = \begin{cases} \overline{V}_L(x_i) + \overline{W}_L(x_i), & \text{for } i = 1, \ldots, N/2-1, \\ \overline{W}_R(x_i) + \overline{V}_R(x_i), & \text{for } i = N/2, \\ \overline{V}_L(x_i) + \overline{W}_L(x_i) = \overline{V}_R(x_i) + \overline{W}_R(x_i), & \text{for } i = N/2+1, \ldots, N-1. \end{cases} \tag{18}$$

Since $|\overline{Y}(x_{N/2})| \le \begin{pmatrix} C \\ C \end{pmatrix}$, we have $|\overline{W_L}(x_{N/2})| \le \begin{pmatrix} C \\ C \end{pmatrix}$ and $|\overline{W_R}(x_{N/2})| \le \begin{pmatrix} C \\ C \end{pmatrix}$. Using the arguments in Cen (2005), for $i \le N/4$, we have

$$|\overline{W_L}(x_i)| \le |\overline{W_L}(x_{N/2})|\, N^{-1} \le \begin{pmatrix} C\, N^{-1} \\ C\, N^{-1} \end{pmatrix},$$

and

$$|(\overline{W_L} - \overline{w})(x_i)| \le |\overline{W_L}(x_i)| + |\overline{w}(x_i)| \le \begin{pmatrix} C\, N^{-1} \\ C\, N^{-1} \end{pmatrix}. \tag{19}$$

Similarly, for $i \ge 3N/4$, we have

$$|\overline{W_R}(x_i)| \le |\overline{W_R}(x_{N/2})|\, N^{-1} \le \begin{pmatrix} C\, N^{-1} \\ C\, N^{-1} \end{pmatrix}$$

and

$$|(\overline{W}_R - \overline{w})(x_i)| \le |\overline{W}_R(x_i)| + |\overline{w}(x_i)| \le \begin{pmatrix} C\ N^{-1} \\ C\ N^{-1} \end{pmatrix}. \tag{20}$$

Lemma 3.3.

At each mesh point $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$, the regular component of the error satisfies the estimate for $j = 1, 2$,

$$|(V_j - v_j)(x_i)| \le \begin{cases} C\ x_i\ N^{-1}, & \text{for} \quad i = 1, \cdots, N/2-1, \\ C\ (1-x_i)\ N^{-1}, & \text{for} \quad i = N/2+1, \cdots, N-1, \end{cases}$$

where $\overline{v}$ and $\overline{V}$ are the solutions of (3) - (4) and (9) - (12) respectively.

Proof:

Consider the local truncation error, $|L^N(\overline{V} - \overline{v})(x_i)| \le \begin{pmatrix} C\ N^{-1} \\ C\ N^{-1} \end{pmatrix}$. Using the two mesh functions

$\overline{\Psi}^{\pm}(x_i) = \overline{\phi}(x_i) \pm (\overline{V} - \overline{v})(x_i),$ where for $j = 1,2$,

$$\varphi_j(x_i) = \begin{cases} C\ x_i\ N^{-1}/d, \text{for} \quad i = 1, \cdots, N/2-1, \\ C\ (1-x_i)\ N^{-1}/(1-d), \text{for} \quad i = N/2+1, \cdots, N-1, \end{cases}$$

on the appropriate sides of the discontinuity, and applying discrete maximum principle, we get $\overline{\Psi}^{\pm}(x_i) \ge \overline{0}, \quad \forall \quad x_i \in \overline{\Omega}^N_{\varepsilon_1,\varepsilon_2}$, which completes the proof.

Lemma 3.4.

At each mesh point $x_i \in \overline{\Omega}^N_{\varepsilon}$, the singular component of the error satisfies the estimate

$$|(\overline{W} - \overline{w})(x_i)| \le \begin{pmatrix} C\ N^{-1}(\ln N)^2 \\ C\ N^{-1}(\ln N)^2 \end{pmatrix},$$

where $\overline{w}$ and $\overline{W}$ are the solutions of (5) - (6) and (13) - (17) , respectively.

Proof:

First we consider the case $\sigma^-_{\varepsilon_2} = \sigma^+_{\varepsilon_2} = \dfrac{2\varepsilon_2}{\alpha}\ln N$ and $\sigma^-_{\varepsilon_1} = \sigma^+_{\varepsilon_1} = \dfrac{2\varepsilon_1}{\alpha}\ln N$. From (19) and (20), it follows that

$$|(W_L - w)(x_{N/4})| \le CN^{-1} \quad \text{and} \quad |(W_R - w)(x_{3N/4})| \le CN^{-1}.$$

Adopting the procedure in Farrel et al. (2004a), Cen (2005, Theorem 3); using the inequality $\varepsilon_1^{-k} e^{-\alpha t/\varepsilon_1} \le \varepsilon_2^{-k} e^{-\alpha t/\varepsilon_2}$ for $t > k\varepsilon_1/\alpha, \quad k = 1,2,\cdots,$ and Lemma 2.4, we have

for $j = 1, 2; \quad i = N/4+1, \cdots, 3N/8,$

$$\begin{aligned} |(L_j^N \overline{W_L} - L_j \overline{w})(x_i)| &\le \varepsilon_j (x_{i+1} - x_{i-1}) |w_j^{(3)}(x_i)| + a_j(x_i)(x_i - x_{i-1}) |w_j^{(2)}(x_i)| \\ &\le C\, N^{-1}(\sigma_{\varepsilon_2}^- - \sigma_{\varepsilon_1}^-) \quad \varepsilon_2^{-2}, \end{aligned}$$

and

for $j = 1, 2; \quad i = 3N/8+1, \cdots, N/2-1,$ we have

$$\begin{aligned} |(L_j^N \overline{W_L} - L_j \overline{w})(x_i)| &\le \varepsilon_j (x_{i+1} - x_{i-1}) |w_j^{(3)}(x_i)| + a_j(x_i)(x_i - x_{i-1}) |w_j^{(2)}(x_i)| \\ &\le C\, N^{-1}\sigma_{\varepsilon_1}^- (\varepsilon_1^{-2} + \varepsilon_2^{-2}). \end{aligned}$$

Similarly for $j = 1, 2; \quad i = N/2+1, \cdots, 5N/8-1,$ we have

$$\begin{aligned} |(L_j^N \overline{W_R} - L_j \overline{w})(x_i)| &\le \varepsilon_j (x_{i+1} - x_{i-1}) |w_j^{(3)}(x_i)| + a_j(x_i)(x_{i+1} - x_i) |w_j^{(2)}(x_i)| \\ &\le C\, N^{-1}\sigma_{\varepsilon_1}^+ (\varepsilon_1^{-2} + \varepsilon_2^{-2}), \end{aligned}$$

and for $j = 1,2; \quad i = 5N/8, \cdots, 6N/8-1,$ we have

$$\begin{aligned} |(L_j^N \overline{W_R} - L_j \overline{w})(x_i)| &\le \varepsilon_j (x_{i+1} - x_{i-1}) |w_j^{(3)}(x_i)| + a_j(x_i)(x_{i+1} - x_i) |w_j^{(2)}(x_i)| \\ &\le C\, N^{-1}(\sigma_{\varepsilon_2}^+ - \sigma_{\varepsilon_1}^+)\, \varepsilon_2^{-2}. \end{aligned}$$

At the mesh point $x_{N/2} = d,$ since $(D^+ - D^-)\overline{W}(x_{N/2}) = \overline{0}$ we have

$$(D^+ - D^-)\overline{W}(x_{N/2}) - (D^+ - D^-)\overline{w}(x_{N/2}) = (D^+ - D^-)\overline{w}(x_{N/2}).$$

Let H_3 and H_4 be the mesh interval size on either side of $x_{N/2}$. Thus,

$$\begin{aligned} |(D^+ - D^-)(\overline{W} - \overline{w})(x_{N/2})| &= |(D^+ - D^-)\overline{w}(x_{N/2})| \\ &\le |(D^+ - \frac{d}{dx})\overline{w}(d)| + |(D^- - \frac{d}{dx})\overline{w}(d)| \\ &\le \frac{1}{2} H_4 |\overline{w}''(x_{N/2})| + \frac{1}{2} H_3 |\overline{w}''(x_{N/2})| \\ &\le \begin{pmatrix} C\, N^{-1}(\sigma_{\varepsilon_1}^- + \sigma_{\varepsilon_1}^+)(\varepsilon_1^{-2} + \varepsilon_2^{-2}) \\ C\, N^{-1}(\sigma_{\varepsilon_1}^- + \sigma_{\varepsilon_1}^+)(\varepsilon_1^{-2} + \varepsilon_2^{-2}) \end{pmatrix}. \end{aligned}$$

Consider the mesh functions $\overline{\Psi}^{\pm}(x_i) = \overline{\phi}(x_i) \pm (\overline{W} - \overline{w})(x_i),$ where for $j = 1, 2$,

$$\varphi_j(x_i) = C\,N^{-1} + C\,N^{-1}\begin{cases} (\sigma^-_{\varepsilon_2} - \sigma^-_{\varepsilon_1})\,\varepsilon_2^{-2}\,(x_i - (d - \sigma^-_{\varepsilon_2})), \text{for } x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d - \sigma^-_{\varepsilon_2}, d - \sigma^-_{\varepsilon_1}), \\ \sigma^-_{\varepsilon_1}(\varepsilon_1^{-2} + \varepsilon_2^{-2})\,(x_i - (d - \sigma^-_{\varepsilon_1})), \text{ for } x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d - \sigma^-_{\varepsilon_1}, d), \\ \sigma^+_{\varepsilon_1}(\varepsilon_1^{-2} + \varepsilon_2^{-2})\,(d + \sigma^+_{\varepsilon_1} - x_i), \quad \text{for } x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d, d + \sigma^+_{\varepsilon_1}), \\ (\sigma^+_{\varepsilon_2} - \sigma^+_{\varepsilon_1})\,\varepsilon_2^{-2}\,(d + \sigma^+_{\varepsilon_2} - x_i), \quad \text{for } x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d + \sigma^+_{\varepsilon_1}, d + \sigma^+_{\varepsilon_2}). \end{cases}$$

Applying the discrete maximum principle to $\Psi^{\pm}_j(x_i),$ for $j = 1,2$ over the interval $[d - \sigma^-_{\varepsilon_2}, d + \sigma^+_{\varepsilon_2}],$ we get

$$\begin{aligned} |(W_j - w_j)(x_i)| &\le C\,N^{-1} + C\,N^{-1}\begin{cases} (\sigma^-_{\varepsilon_2} - \sigma^-_{\varepsilon_1})^2\,\varepsilon_2^{-2}, \text{for} \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d - \sigma^-_{\varepsilon_2}, d - \sigma^-_{\varepsilon_1}), \\ (\sigma^-_{\varepsilon_1})^2(\varepsilon_1^{-2} + \varepsilon_2^{-2}), \text{for} \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d - \sigma^-_{\varepsilon_1}, d), \\ (\sigma^+_{\varepsilon_1})^2(\varepsilon_1^{-2} + \varepsilon_2^{-2}), \text{for} \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d, d + \sigma^+_{\varepsilon_1}), \\ (\sigma^+_{\varepsilon_2} - \sigma^+_{\varepsilon_1})^2\,\varepsilon_2^{-2}, \text{for} \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d + \sigma^+_{\varepsilon_1}, d - \sigma^+_{\varepsilon_2}). \end{cases} \\ &\le C\,N^{-1}(\ln N)^2, \quad \text{for} \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cap (d - \sigma^-_{\varepsilon_2}, d + \sigma^+_{\varepsilon_2}), \end{aligned}$$

which is the required result.

Now we complete the proof by considering the case where at least one of the four transition points take the value $\sigma^-_{\varepsilon_1} = \dfrac{d}{4}, \quad \sigma^-_{\varepsilon_2} = \dfrac{d}{2}, \quad \sigma^+_{\varepsilon_1} = \dfrac{1-d}{4}$ and $\sigma^+_{\varepsilon_2} = \dfrac{1-d}{2}.$

In all such cases $\varepsilon_1^{-1} \le C \ln N$ and $\varepsilon_2^{-1} \le C \ln N.$ We have for $j = 1, 2$,

$$|(L^N_j \overline{W} - L_j \overline{w})(x_i)| \le \varepsilon_j (x_{i+1} - x_{i-1})\,|w'''_j(x_i)| + a_j(x_i)\,(x_{i+1} - x_i)\,|w''_j(x_i)| \le CN^{-1}(\ln N)^2$$

and

$$|(D^+ - D^-)(W_j - w_j)(x_{N/2})| = |(D^+ - D^-) w_j(x_{N/2})| \le C\,N^{-1}(\ln N)^2.$$

Using the mesh functions

$$\Psi^{\pm}_j(x_i) = C\,N^{-1}(\ln N)^2\begin{cases} (1-d)\,x_i, \quad \text{for} \quad x_i \in \Omega_{\varepsilon_1,\varepsilon_2} \cap (0, d) \\ d\,(1 - x_i), \quad \text{for} \quad x_i \in \Omega_{\varepsilon_1,\varepsilon_2} \cap (d, 1) \end{cases}$$

and applying the discrete maximum principle across the entire domain $\overline{\Omega}^N_{\varepsilon_1,\varepsilon_2}$, we get the required result.

Theorem 3.5.

Let $\bar{y}(x) = (y_1(x), y_2(x))^T$ be the solution of (1) - (2) and let $\bar{Y}(x_i) = (Y_1(x_i), Y_2(x_i))^T$ be the corresponding numerical solution of (7) - (8). Then, we have

$$\| Y_1 - y_1 \|_{\bar{\Omega}^N_{\varepsilon_1,\varepsilon_2}} \le C\, N^{-1} (\ln N)^2 \quad \text{and} \quad \| Y_2 - y_2 \|_{\bar{\Omega}^N_{\varepsilon_1,\varepsilon_2}} \le C\, N^{-1} (\ln N)^2.$$

Proof:

Proof follows immediately, if one applies the above Lemmas 3.3 and 3.4 to $\bar{Y} - \bar{y} = \bar{V} - \bar{v} + \bar{W} - \bar{w}.$

4. Nonlinear Problems

Consider the nonlinear BVP

$$\begin{pmatrix} -\varepsilon \dfrac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \dfrac{d^2}{dx^2} \end{pmatrix} \bar{y}(x) = \bar{F}(x, \bar{y}, \bar{y}'), \quad x \in \Omega^- \cup \Omega^+, \tag{21}$$

$$\bar{y}(0) = \begin{pmatrix} p \\ r \end{pmatrix}, \quad \bar{y}(1) = \begin{pmatrix} q \\ s \end{pmatrix}, \tag{22}$$

where $\bar{F}(x, \bar{y}, \bar{y}')$ is a function such that

$$\begin{aligned}
&\alpha_1^* > F_{1_{y_{1'}}}(x, \bar{y}, \bar{y}') > \alpha_1 > 0, \quad x < d, \quad -\alpha_2^* < F_{1_{y_{1'}}}(x, \bar{y}, \bar{y}') < -\alpha_2 < 0, \quad d < x, \\
&\alpha_1^* > F_{2_{y_{2'}}}(x, \bar{y}, \bar{y}') > \alpha_1 > 0, \quad x < d, \quad -\alpha_2^* < F_{2_{y_{2'}}}(x, \bar{y}, \bar{y}') < -\alpha_2 < 0, \quad d < x, \\
&\quad \{F_{1_{y_1}}(x, \bar{y}, \bar{y}') + F_{1_{y_2}}(x, \bar{y}, \bar{y}')\} \ge 0, \quad \{F_{2_{y_1}}(x, \bar{y}, \bar{y}') + F_{2_{y_2}}(x, \bar{y}, \bar{y}')\} \ge 0, \\
&\quad F_{1_{y_2}}(x, \bar{y}, \bar{y}') \le 0, \quad F_{2_{y_1}}(x, \bar{y}, \bar{y}') \le 0, \quad x \in \bar{\Omega}.
\end{aligned}$$

Assume that the BVP (21) - (22) has a unique solution. In order to obtain a numerical solution of (21) - (22), the Newton's method of quasilineraisation is applied. Consequently, with a proper choice of initial approximation $\bar{y}^{[0]}$ ($\bar{y}^{[0]} = \bar{y}(0) + (\bar{y}(1) - \bar{y}(0))x$ may be a proper initial guess), we get a sequence of $\{\bar{y}^{[m]}\}_0^\infty$ of successive approximations. In fact, we define $\bar{y}^{[m+1]}$ for each fixed non-negative integer m to be the solution of the linear problem:

$$L^m \bar{y}^{[m+1]} \equiv \begin{pmatrix} -\varepsilon \dfrac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \dfrac{d^2}{dx^2} \end{pmatrix} \bar{y}^{[m+1]} + \begin{pmatrix} a_1^m \dfrac{d}{dx} & 0 \\ 0 & a_2^m \dfrac{d}{dx} \end{pmatrix} \bar{y}^{[m+1]} + B^m \bar{y}^{[m+1]} = \bar{f}^{\,m}, \tag{23}$$

$$\bar{y}^{[m+1]}(0) = \begin{pmatrix} p \\ r \end{pmatrix}, \quad \bar{y}^{[m+1]}(1) = \begin{pmatrix} q \\ s \end{pmatrix}, \tag{24}$$

where

$$a_1^m(x) = F_{1_{y_{1'}}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}), \quad a_2^m(x) = F_{2_{y_{2'}}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}),$$

$$B^m(x) = \begin{pmatrix} b_{11}^m(x) & b_{12}^m(x) \\ b_{21}^m(x) & b_{22}^m(x) \end{pmatrix} = \begin{pmatrix} F_{1_{y_1}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) & F_{1_{y_2}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) \\ F_{2_{y_1}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) & F_{2_{y_2}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) \end{pmatrix},$$

and

$$\bar{f}^m(x) = \bar{F}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) - \begin{pmatrix} y_{1'}^{[m]} F_{1_{y_{1'}}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) \\ y_{2'}^{[m]} F_{2_{y_{2'}}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) \end{pmatrix}$$
$$- \begin{pmatrix} y_1^{[m]} F_{1_{y_1}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) + y_2^{[m]} F_{1_{y_2}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) \\ y_1^{[m]} F_{2_{y_1}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) + y_2^{[m]} F_{2_{y_2}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) \end{pmatrix}.$$

From the assumption on $\bar{F}(x, \bar{y}, \bar{y}')$ it follows that each m, we have

$$\alpha_1^* > a_1^m(x) = F_{1_{y_{1'}}}(x, \bar{y}, \bar{y}') > \alpha_1 > 0, \qquad x < d,$$
$$-\alpha_2^* < a_1^m(x) = F_{1_{y_{1'}}}(x, \bar{y}, \bar{y}') < -\alpha_2 < 0, \quad d < x,$$
$$\alpha_1^* > a_2^m(x) = F_{2_{y_{2'}}}(x, \bar{y}, \bar{y}') > \alpha_1 > 0, \qquad x < d,$$
$$-\alpha_2^* < a_2^m(x) = F_{2_{y_{2'}}}(x, \bar{y}, \bar{y}') < -\alpha_2 < 0, \quad d < x,$$
$$\{b_{11}^m(x) + b_{12}^m(x)\} = \{F_{1_{y_1}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) + F_{1_{y_2}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]})\} \geq 0,$$
$$\{b_{21}^m(x) + b_{22}^m(x)\} = \{F_{2_{y_1}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) + F_{2_{y_2}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]})\} \geq 0,$$
$$b_{12}^m(x) = F_{1_{y_2}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) \leq 0, \quad b_{21}^m(x) = F_{2_{y_1}}(x, \bar{y}^{[m]}, \bar{y}'^{[m]}) \leq 0.$$

The problem (23) - (24) for each fixed m, is a linear BVP and is of the form (1) - (2). Hence, it can be solved by the our method. We can prove that if the initial approximation $\bar{y}(0)$ is sufficiently close to $\bar{y}(x)$, the Newton sequence $\{\bar{y}^m(x)\}_{m=0}^{\infty}$ converge to $\bar{y}(x)$. Infact, in Doolan et al. (1980) the author's has proved a similar result for a single nonlinear equation. For the above Newton's quasilinearisation process the following convergence criterion can be used:

$$|(\bar{y}^{[m+1]} - \bar{y}^{[m]})(x_j)| \le \lambda, \quad x_j \in \overline{\Omega}, \quad m \ge 0,$$

where λ is the prescribed tolerance bound.

5. Numerical Experiments

In this section, two examples are given to illustrate the numerical method discussed in this paper for the BVP (1)-(2).

Example 5.1.

$$a_1(x) = \begin{cases} 1, & x < 0.5, \\ -1, & x \ge 0.5, \end{cases} \qquad a_2(x) = \begin{cases} 1, & x < 0.5, \\ -1, & x \ge 0.5, \end{cases}$$

$$B(x) = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \qquad f_1(x) = \begin{cases} 2, & x < 0.5, \\ 3, & x \ge 0.5, \end{cases} \quad f_2(x) = \begin{cases} 4, & x < 0.5, \\ 2, & x \ge 0.5, \end{cases}$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example 5.2.

$$a_1(x) = \begin{cases} 1+x, & x < 0.5, \\ -(1+x), & x \ge 0.5, \end{cases} \quad a_2(x) = \begin{cases} 1+x, & x < 0.5, \\ -(1+x), & x \ge 0.5, \end{cases}$$

$$B(x) = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad f_1(x) = \begin{cases} 2+x, & x < 0.5, \\ 3+x, & x \ge 0.5, \end{cases} \quad f_2(x) = \begin{cases} 4+x, & x < 0.5, \\ 2+x, & x \ge 0.5, \end{cases}$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let Y_j^N be a numerical approximation for the exact solution y_j on the mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$ and N be the number of mesh points. The exact solutions to the test problems are not available. For a finite set of values of ε_1 and ε_2 we compute the maximum pointwise errors for $j = 1, 2$,

$$D^N_{\varepsilon_1,\varepsilon_2,j} = \max |Y_j^N - \tilde{Y}_j^{2048}|_{\Omega^N_{\varepsilon_1,\varepsilon_2}}, \quad D_j^N = \max_{\varepsilon_1} \max_{\varepsilon_2} D^N_{\varepsilon_1,\varepsilon_2,j},$$

where $\tilde{Y}^{2048}$ is the piecewise linear interpolant of the mesh function Y_j^{2048} onto [0,1]. The range of singular perturbation parameter $\varepsilon_1 = \{2^{-8}, 2^{-9}, 2^{-10}, \cdots, 2^{-33}\}$. From these quantities the parameter-uniform order of convergence p_j^N are computed from

$$p_j^N = \log_2\left(\frac{D_j^N}{D_j^{2N}}\right), \quad j = 1, 2.$$

The computed errors $D_j^N (j = 1, 2)$ and the computed order of convergence $p_j^N (j = 1, 2)$ are tabulated (Tables 1-3). The nodal errors are plotted as graphs (Figures 1-2).

Table 1. Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components Y_1 and Y_2 respectively for the Example 5.1 with $\varepsilon_2 = 2^{-7}$

	Number of mesh points N				
	32	64	128	256	512
D_1^N	4.0340e-2	2.3020e-2	1.2591e-2	6.5353e-3	3.0404e-3
p_1^N	8.0932e-1	8.7049e-1	9.4607e-1	1.1040	-
D_2^N	7.4571e-2	4.5565e-2	2.6495e-2	1.4335e-2	6.8671e-3
p_2^N	7.1069e-1	7.8221e-1	8.8618e-1	1.0618	-

Table 2. Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components Y_1 and Y_2 respectively for the Example 5.1 with $\varepsilon_2 = 2\varepsilon_1$

	Number of mesh points N				
	32	64	128	256	512
D_1^N	4.4617e-2	2.5294e-2	1.3969e-2	7.2117e-3	3.3525e-3
p_1^N	8.1880e-1	8.5657e-1	9.5382e-1	1.1051	-
D_2^N	6.4090e-2	3.6512e-2	1.9915e-2	1.0213e-2	4.7275e-3
p_2^N	8.1173e-1	8.7452e-1	9.6345e-1	1.1113	-

Table 3. Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components Y_1 and Y_2 respectively for the Example 5.2 with $\varepsilon_2 = 2^{-7}$

	Number of mesh points N				
	32	64	128	256	512
D_1^N	2.4117e-2	1.5288e-2	8.7279e-3	4.5904e-3	2.1649e-3
p_1^N	6.5765e-1	8.0869e-1	9.2701e-1	1.0843	-
D_2^N	8.0944e-2	5.6433e-2	3.3275e-2	1.9077e-2	9.4286e-3
p_2^N	5.2039e-1	7.6210e-1	8.0260e-1	1.0167	-

Table 4. Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components Y_1 and Y_2 respectively for the Example 5.2 with $\varepsilon_2 = 2\varepsilon_1$

	Number of mesh points N				
	32	64	128	256	512
D_1^N	2.6653e-2	1.6374e-2	9.1051e-3	4.8012e-3	2.2536e-3
p_1^N	7.0289e-1	8.4666e-1	9.2328e-1	1.0912	-
D_2^N	7.5325e-2	4.3871e-2	2.4740e-2	1.3022e-2	6.1480e-3
p_2^N	7.7986e-1	8.2642e-1	9.2601e-1	1.0827	-

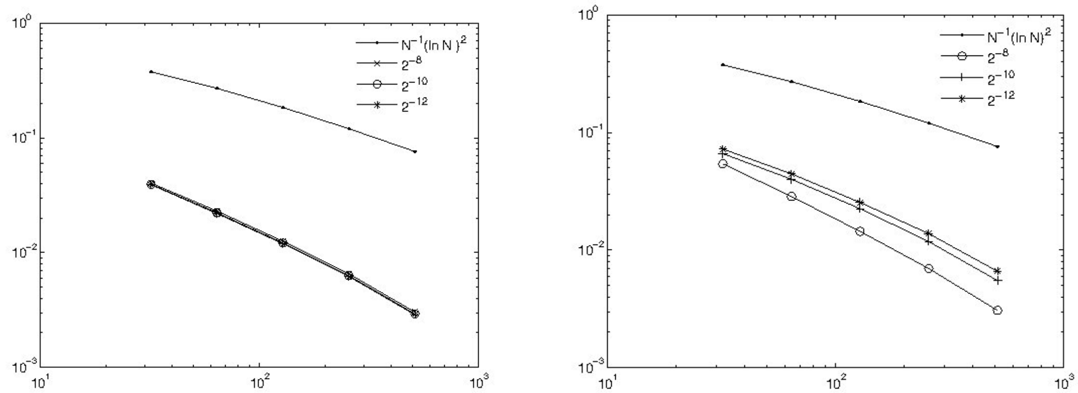

Figure 1. Nodal error for the component Y_1 and Y_2 of the Example 5.1

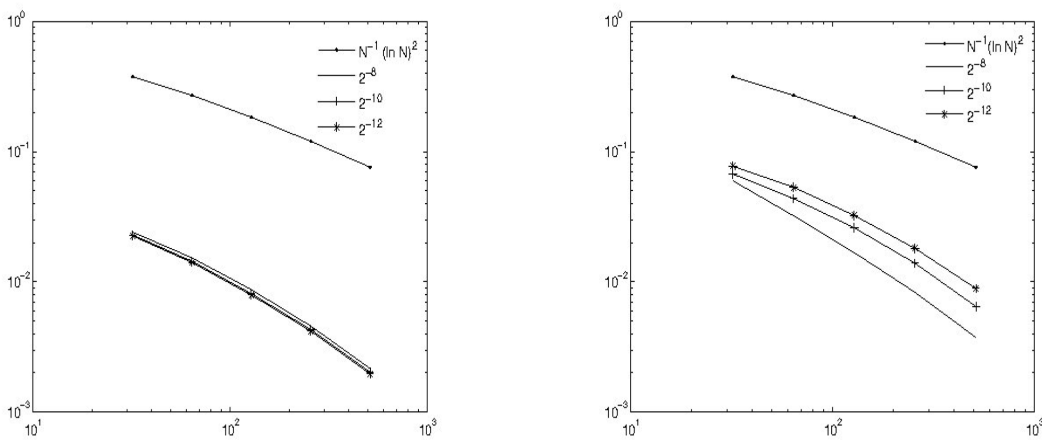

Figure 2. Nodal error for the component Y_1 and Y_2 of the Example 5.2

6. Conclusion

A finite difference method is derived for a system of singularly perturbed convection-diffusion equations with discontinuous convection coefficients and source terms. The distinct singular perturbation parameters and the discontinuity in the interior domain lead to the overlap and interact interior layer in the solution. The numerical method uses a piecewise uniform mesh, which is fitted to the interior layer and the upwind finite difference operator on this mesh.

Tables and figures show that the numerical results agree with the theoretical claims. The graphs plotted in the figures 1-2 are convergent curves in the maximum norm at nodal points for the different values of ε_1 and $\varepsilon_2 = 2^{-7}$ for Examples 5.1-5.2. These graphs clearly indicate that the optimal error bound is of order $O(N^{-1}(\ln N)^2)$ as predicted by Theorem 3.5.

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