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A Constructive Proof of Fundamental Theory for Fuzzy Variable Linear Programming Problems

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Abstract

Two existing methods for solving fuzzy variable linear programming problems based on ranking functions are the fuzzy primal simplex method proposed by Mahdavi-Amiri et al. (2009) and the fuzzy dual simplex method proposed by Mahdavi-Amiri and Nasseri (2007). In this paper, we prove that in the absence of degeneracy these fuzzy methods stop in a finite number of iterations. Moreover, we generalize the fundamental theorem of linear programming in a crisp environment to a fuzzy one. Finally, we illustrate our proof using a numerical example.

Keywords: Fuzzy variable linear programming, fuzzy primal simplex algorithm, fuzzy dual simplex algorithm, ranking function, trapezoidal fuzzy number

MSC 2010 No.: 90C05, 90C70, 03E72

1.Introduction

In the classical linear programming problems, the coefficients of the problems are assumed to be exactly known. However in practice this assumption is seldom satisfied by a great majority of real-life problems. The modeling of input data inaccuracy can be done by means of fuzzy set theory [Pop and Stancu-Minasian (2008)]. A number of researchers have, therefore, shown an interest in the area of fuzzy linear programming (FLP) problems with various attempts made to study the solution of fuzzy linear programming problems [Allahviranloo et al. (2008),

Ebrahimnejad and Nasseri (2009, 2012), Hosseinzadeh Lotfi et al. (2009), Kumar and Kaur (2010, 2011), Kumar et al. (2011) and Nasseri et al. (2010)].

Since the fuzziness may appear in a linear programming problem in many ways, the definition of FLP problem is not unique. In this paper we try to study a class of FLP problems in which the right-hand-side vectors and the decision variables are represented by fuzzy numbers while the rest of the parameters are represented by real numbers. This class of FLP problems is known as the fuzzy variable linear programming (FVLP) problem. The FVLP problems have been explored in Zimmermann's discussion (1985) of the so-called non-symmetric flexible linear programming (NFLP) problems, where the problem data are considered to be crisp but certain constraints are considered to be fuzzy inequality constraints; see also Lai and Hwang (1992). Maleki et al. (2000) used the crisp solution of linear programming with fuzzy cost coefficients as an auxiliary problem for finding the fuzzy solution of FVLP problem. Mahdavi-Amiri and Nasseri (2007) showed that this auxiliary problem is indeed the dual of the FVLP problem. They then stated and proved duality results obtained by a natural extension of the results in crisp linear programming. Using the obtained results, they developed a dual simplex algorithm for solving the FVLP problem directly, without any need of an auxiliary problem. Ebrahimnejad et al. (2010) described another method for solving FVLP problems called the primal-dual algorithm, which is similar to the dual simplex method.

An important difference between the dual simplex method and the primal-dual method is that the primal-dual algorithm does not require a dual feasible solution to be basic. Here we show that in the absence of primal and dual degeneracy, these fuzzy methods stop in a finite number of iterations. In addition, the key in these fuzzy algorithms is that the fuzzy optimal solution is obtained at a basic solution. Thus we prove that if there is a fuzzy optimal feasible solution for the FVLP problem then there is a fuzzy optimal basic feasible solution.

The rest of the paper is organized as follows: in Section 2, we discuss some necessary concepts and backgrounds of fuzzy arithmetic. In Section 3, the fuzzy simplex methods for solving one FVLP problems are reviewed. Finite convergence of the existing methods and the generalization of the fundamental theorem of linear programming for FVLP problems are proved in Section 4. Conclusions are made in Section 5.

2. Preliminaries

The purpose of this section is to recall some concepts of fuzzy set theory which will be needed in the sequel, taken from Ebrahimnejad and Nasseri (2010) and Ebrahimnejad et al. (2011).

Definition 2.1.

The characteristic function μ_A of a crisp set A assigns a value of either one or zero to each individual in the universal set X. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the values assigned to the element of the universal set X fall within a specified range i.e.,

$\mu_{\tilde{A}} : X \to [0,1]$. The assigned value indicates the membership grade of the element in the set μ_A. Larger values denote higher degrees of set membership.

The function $\mu_{\tilde{A}}$ is called a membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ defined by $\mu_{\tilde{A}}$ for each $x \in X$ is called a fuzzy set.

Definition 2.2.

A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be a fuzzy number if its membership function has the following characteristics:

i) $\tilde{A}$ is convex, i.e. $\forall x, y \in R, \forall \lambda \in [0,1], \mu_{\tilde{A}}(\lambda x + (1-\lambda) y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$,

ii) $\tilde{A}$ is normal, i.e., $\exists \bar{x} \in R;\ \mu_{\tilde{A}}(\bar{x}) = 1$,

iii) $\mu_{\tilde{A}}$ is piecewise continues.

Definition 2.3.

A fuzzy number $\tilde{A} = (m, n, \alpha, \beta)$ is said to be a trapezoidal fuzzy number if its membership function is given by (see Figure 1)

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-(m-\alpha)}{\alpha} & m-\alpha \leq x \leq m \\ 1 & m \leq x \leq n \\ \dfrac{(n+\beta)-x}{\beta} & n \leq x \leq n+\beta \\ 0 & else \end{cases}$$

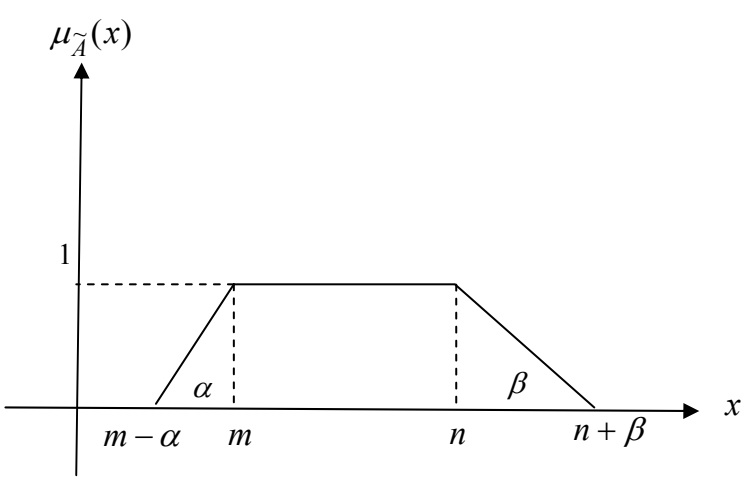

Figure 1: A trapezoidal fuzzy number $\tilde{A} = (m, n, \alpha, \beta)$

Now, we define the arithmetic operations of trapezoidal fuzzy numbers.

Definition 2.3.

Let $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)$ be two trapezoidal fuzzy numbers. Then, the arithmetic operations on $\tilde{A}_1$ and $\tilde{A}_2$ are given by:

$$\tilde{A}_1 + \tilde{A}_2 = (m_1 - n_2, n_1 - m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

$$\tilde{A}_1 - \tilde{A}_2 = (m_1 - n_2, n_1 - m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

$$k > 0, k \in R,\ k\tilde{A}_1 = (k\, m_1, k\, n_1, k\alpha_1, k\beta_1)$$

$$k < 0, k \in R,\ k\tilde{A}_1 = (k\, n_1, k\, m_1, -k\beta_1, -k\alpha_1)$$

Ranking procedures are useful in various applications and one of them will be in the study of fuzzy mathematical programming in later sections. There are numerous methods proposed in the literature for the ranking of fuzzy numbers. Here, we describe only a simple method for the ordering of fuzzy numbers.

An efficient approach to the ordering of fuzzy numbers is based on the concept of comparison of fuzzy numbers by the use of ranking functions, in which a ranking function $\Re : F(R) \to R$ that maps each fuzzy number into the real line is defined for ranking the elements of $F(R)$. Thus, using the natural order of the real numbers we can compare fuzzy numbers easily as follows:

$$\tilde{A}_1 \tilde{\geq} \tilde{A}_2 \iff \Re(\tilde{A}_1) \geq \Re(\tilde{A}_2)$$

$$\tilde{A}_1 \tilde{\leq} \tilde{A}_2 \iff \Re(\tilde{A}_1) \leq \Re(\tilde{A}_2)$$

$$\tilde{A}_1 \cong \tilde{A}_2 \iff \Re(\tilde{A}_1) = \Re(\tilde{A}_2)$$

Several ranking functions have been proposed by researchers to suit their requirements of the problems under consideration. We restrict our attention to linear ranking functions, that is, a ranking function $\Re$ such that $\Re(k\tilde{A}_1 + \tilde{A}_2) = k\Re(\tilde{A}_1) + \Re(\tilde{A}_2)$ for any $\tilde{A}_1$ and $\tilde{A}_2$ belonging to $F(R)$ and any $k \in R$. For a trapezoidal fuzzy number $\tilde{A} = (m, n, \alpha, \beta)$, one of the most linear ranking functions introduced by Yager's (1981) is as follows:

$$Y_2(\tilde{A}) = \frac{1}{2}\left[m + n + \frac{\beta - \alpha}{2} \right]$$

3. Fuzzy Variable Linear Programming Problems

In this section two existing methods for solving fuzzy variable linear programming problems are reviewed [Mahdavi-Amiri and Nasseri (2007) and Nasseri et al. (2009)].

An FVLP problem is defined as follows:

$$\begin{aligned} \min\ & \tilde{z} \approx c\tilde{x} \\ s.t.\ & A\tilde{x} \tilde{\geq} \tilde{b} \\ & \tilde{x} \tilde{\geq} \tilde{0} \end{aligned} \tag{1}$$

Definition 3.1.

Any fuzzy vector $\tilde{x} \in F(R^n)$ which satisfies the constraints and nonnegative restrictions of (1) is said to be a fuzzy feasible solution.

Definition 3.2.

Let X be the set of all fuzzy feasible solutions of (1). Any fuzzy vector $\tilde{x}_* \in X$ is said to be a fuzzy optimum solution to (1) if $c\tilde{x}_* \tilde{\leq} c\tilde{x}$ for all $\tilde{x} \in X$, where $c = (c_1, c_2, \ldots, c_n)$ and $c\tilde{x} = c_1\tilde{x}_1 + c_2\tilde{x}_2 + \cdots + c_n\tilde{x}_n$.

Definition 3.3. (Fuzzy basic solution)

Suppose $\tilde{\bar{x}} = (\tilde{\bar{x}}_1, \tilde{\bar{x}}_2, \ldots, \tilde{\bar{x}}_n)$ solves $A\tilde{x} \cong \tilde{b}$. If all $\tilde{\bar{x}}_j = (m_j, n_j, \alpha_j, \beta_j)$ for some $m_j, n_j, \beta_j \geq \alpha_j \geq 0$, such that $\frac{1}{2}\left[m_j + n_j + \frac{\beta_j - \alpha_j}{2}\right] = 0$, then $\tilde{\bar{x}}$ is said to be a fuzzy basic solution. Otherwise, $\tilde{\bar{x}}$ has some non-zero components, say $\tilde{\bar{x}}_1, \tilde{\bar{x}}_2, \ldots, \tilde{\bar{x}}_k, 1 \leq k \leq m$. Then, $A\tilde{x} \cong \tilde{b}$ can be written as:

$$a_1\tilde{\bar{x}}_1 + a_2\tilde{\bar{x}}_2, \ldots + a_k\tilde{\bar{x}}_k + a_{k+1}(m_{k+1}, n_{k+1}, \alpha_{k+1}, \beta_{k+1}) + \cdots + a_n(m_n, n_n, \alpha_n, \beta_n) \cong \tilde{b}$$

If the columns $a_1, a_2, \ldots, a_k$ corresponding to non-zero components $\tilde{\bar{x}}_1, \tilde{\bar{x}}_2, \ldots, \tilde{\bar{x}}_k$ are linear independent, then $\tilde{\bar{x}}$ is said to be fuzzy basic solution.

Remark 3.1.

Consider the fuzzy system of constraints (1) where A is a matrix of order $(m \times n)$ and $rank(A) = m$. Any $(m \times m)$ matrix B formed by m linearly independent columns of A is known as a basis for this fuzzy system. The column vectors of A and the fuzzy variables in the problem, can be partitioned into the basic and the nonbasic part with respect to this basis B. Each column vector

of A, which is in the basis B, is known as a basic column vector. All the remaining column vectors of A are called the nonbasic column vectors.

Remark 3.2.

Let $\tilde{x}_B$ be the vector of the variables associated with the basic column vectors. The variables in $\tilde{x}_B$ are known as the fuzzy basic variables with respect to basis B, and $\tilde{x}_B$ is the fuzzy basic vector. Also, let $\tilde{x}_N$ and N be the vector and the matrix of the remaining variables and columns, which are called the fuzzy nonbasic variables and nonbasic matrix, respectively. In this case, $\tilde{x} = (\tilde{x}_B, \tilde{x}_N) = (B^{-1}\tilde{b}, \tilde{0})$ is also a fuzzy basic solution.

Definition 3.4.

Suppose $\tilde{\tilde{x}}$ is a fuzzy basic feasible solution of fuzzy system $A\tilde{x} \cong \tilde{b}, \tilde{x} \tilde{\geq} \tilde{0}$. If the number of fuzzy positive variables $\tilde{\tilde{x}}$ is exactly m, then it is called a non-degenerate fuzzy basic feasible solution, i.e. $\tilde{\tilde{x}} = (\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2, \ldots, \tilde{\tilde{x}}_m) \succ (\tilde{0}, \tilde{0}, \cdots, \tilde{0})$. If the number of positive $\tilde{\tilde{x}}$ is less than m, then $\tilde{\tilde{x}}$ is called a degenerate fuzzy basic feasible solution.

Suppose $\tilde{\tilde{x}}$ is a fuzzy basic feasible solution of (1). Let y_k and w be the solutions to $By_k = a_k$ and $wB = c_B$, respectively and define $z_j = wa_j$. Mahdavi-Amiri and Nasseri (2007) proved some important theorems of FVLP problems concerning the improvement of a fuzzy feasible solution, unbounded criteria and the optimality conditions and then proposed a new algorithm for solving FVLP problems. Here, we give a summary of their method in tableau format [Mahdavi-Amiri et al. (2009)].

<u>Algorithm 3.1 A fuzzy primal simplex for FVLP problem</u>

Initialization Step

Suppose a fuzzy basic feasible solution with basic B is at hand. Form the initial tableau as Table 1.

Table 1: The initial FVLP simplex tableau

Basis	$\tilde{x}_B$	$\tilde{x}_N$	R.H.S	R
$\tilde{z}$	0	$z_N - c_N = c_B Y_N - c_N$	$\tilde{\tilde{z}} = c_B B^{-1}\tilde{b}$	$\Re(\tilde{\tilde{z}}) = \Re(c_B B^{-1}\tilde{b})$
$\tilde{x}_B$	I	Y_N	$\tilde{\tilde{b}} = B^{-1}\tilde{b}$	$\Re(\tilde{\tilde{b}})$

Main Step

(1) Calculate $z_j - c_j$ for all nonbasic variables. Let $z_k - c_k = \max\{z_j - c_j, j \in T\}$ in which T is the index set of the current nonbasic variables.

If $z_k - c_k \le 0$, then stop; the current solution is fuzzy optimal. Otherwise go to step (2).

(2) Let $y_k = B^{-1}a_k$. If $y_k \le 0$, then stop; the problem is unbounded. Otherwise, determine the index of variable $\tilde{x}_{B_r}$ leaving the basic as follows:

$$\frac{1}{2y_{rk}}\left[m_r + n_r + \frac{\beta_r - \alpha_r}{2}\right] = \min_{1\le i\le m}\left\{\frac{1}{2y_{ik}}\left[m_i + n_i + \frac{\beta_i - \alpha_i}{2}\right] \middle| \, y_{ik} > 0\right\}$$

(3) Update the tableau by pivoting at y_{rk} Update the fuzzy basic and fuzzy nabasic variable where $\tilde{x}_k$ enters the basic and $\tilde{x}_{B_r}$ leaves the basic and go to (1).

Remark 3.3.

In the step 3 of the above mentioned method, a new fuzzy basic solution and a new fuzzy objective are obtained as follows, respectively:

$$\tilde{\hat{x}}_{B_i} = \left(\tilde{\bar{x}}_{B_i} - \frac{\tilde{\bar{x}}_{B_r}}{y_{rk}} y_{rj}\right),\ i \ne r,\ \tilde{\hat{x}}_{B_k} = \tilde{\bar{x}}_{B_r} = \frac{\tilde{\bar{x}}_{B_r}}{y_{rk}} \tag{2}$$

$$\tilde{\hat{z}} = \tilde{\bar{z}} - \frac{\tilde{\bar{x}}_{B_r}}{y_{rk}}(z_k - c_k)\,. \tag{3}$$

Mahdavi-Amiri and Nasseri (2009) defined the duality in FVLP problem using linear ranking functions leading to a standard primal-dual linear programming pair.

The dual of FVLP problem (1) is defined as follows:

$$\begin{aligned} \max\ & \tilde{y} \approx w\tilde{b} \\ s.t.\ & wA \le c \\ & w \ge 0 \end{aligned} \tag{4}$$

In addition, Mahdavi-Amiri and Nasseri (2007) proved some important results concerning the FVLP problem and its dual problem and also based on these results proposed a new algorithm for solving FVLP problems. Here, we give a summary of their method in tableau format.

<u>Algorithm 3.2 A Fuzzy Dual Simplex for FVLP Problem</u>

Initialization Step

Suppose that basic B be dual feasible for the FVLP problem (1) in its standard form, i.e., $z_j - c_j \le 0, j = 1, 2, \cdots, n$. Form the Table 1 as an initial dual simplex tableau.

Main Step

(1) Suppose $\tilde{\bar{b}} = (\bar{m}, \bar{n}, \bar{\alpha}, \bar{\beta}) = B^{-1}\tilde{b}$. If $\frac{1}{2}\left[\bar{m} + \bar{n} + \frac{\bar{\beta} - \bar{\alpha}}{2}\right] \geq 0$, then stop; the current solution is fuzzy optimal. Else select the pivot row *r*, with $\frac{1}{2}\left[\bar{m}_r + \bar{n}_r + \frac{\bar{\beta}_r - \bar{\alpha}_r}{2}\right] < 0$.

(2) If $y_{rj} \geq 0$ for all j, then stop; the problem is infeasible. Else select the pivot column *k* by the following test:

$$\frac{z_k - c_k}{y_{rk}} = \min_{1 \leq j \leq m}\left\{\frac{z_j - c_j}{y_{rj}} \middle| \, y_{ik} < 0\right\}$$

(3) Update the tableau by pivoting at y_{rk}. Update the fuzzy basic and fuzzy nabasic variable where $\tilde{x}_k$ enters the basic and $\tilde{x}_{B_r}$ leaves the basic and go to (1).

Remark 3.4.

In the step 3 of the above mentioned method a new fuzzy objective value is obtained as follows:

$$\tilde{\hat{z}} = \tilde{\bar{z}} - \frac{\tilde{\bar{b}}_r}{y_{rk}}(z_k - c_k) \tag{5}$$

Definition 3.5.

A basis B for the FVLP problem (1) is said to be dual degenerate if for at least one nonbasic variable, say $\tilde{x}_j$, we have $z_j - c_j = 0$. Otherwise, it said to be dual non-degenerate. The FVLP problem (1) is said to be totally dual non-degenerate, if for all fuzzy nonbasic variables in the dual simplex tableau with respect to any basis for (1), we have $z_j - c_j \leq 0$.

4. Main Results

In this section, we generalize some important results in crisp linear programming problems to fuzzy variable linear programming problems.

It needs pointing out that the fuzzy primal method for FVLP problems, starting a fuzzy basic feasible solution moves to another fuzzy basic solution with a better (at least not worse) objective value until it finds a fuzzy optimal basic feasible solution after a finite number of steps. Here, we prove that in the absence of degeneracy, the primal method stops in a finite number of iterations.

Theorem 4.1.

In the absence of primal degeneracy, the fuzzy primal simplex algorithm stops in a finite number of iterations, either with a fuzzy optimal basic feasible solution or with the conclusion that the optimal value is unbounded.

Proof:

In the absence of primal degeneracy, every fuzzy basic feasible solution has exactly m positive components and has a unique associated basis. Also, at each iteration, one of the following three actions is executed at each iteration:

(1) It may stop with a fuzzy optimal basic solution if $z_j - c_j \le 0$;
(2) It may stop with an unbounded solution if $z_k - c_k > 0$ and $y_k \le 0$;
(3) It gives a new fuzzy basic feasible solution if $z_k - c_k > 0$ and $y_k > 0$.

In the absence of degeneracy, $\tilde{\bar{b}}_r \succ \tilde{0}$, i.e. $\tilde{x}_{B_r} = \tilde{\bar{b}}_r \succ \tilde{0}$ and hence $\frac{\tilde{\bar{x}}_{B_r}}{y_{rk}} \succ \tilde{0}$. By (3), the difference between the fuzzy objective values at the previous iteration and the current iteration is $\frac{\tilde{\bar{x}}_{B_r}}{y_{rk}}(z_k - c_k)$. Thus, the fuzzy objective value decreases strictly in all iterations. Hence a basis that appears once in the course of method can never reappear. Also the total number of bases for (1) is less than or equal to $\binom{n}{m}$ Hence, the method would stop in a finite number of steps with a finite fuzzy optimal basic solution or with an unbounded optimal solution.

We note that the fuzzy dual simplex algorithm for FVLP problems starts with a dual fuzzy basic feasible solution, but primal basic infeasible solution and walks to a fuzzy optimal solution by moving among adjacent dual fuzzy basic feasible solutions. Now, we show that in the absence of dual degeneracy, this fuzzy dual method stops in a finite number of iterations.

Theorem 4.2.

In the absence of dual degeneracy, the dual primal simplex algorithm stops in a finite number of iterations, either with a fuzzy optimal basic feasible solution or with the conclusion that the problem is infeasible.

Proof:

We know that the fuzzy dual simplex method moves among dual feasible bases. Also, at each iteration of the method, one of the following three actions is executed. It may be stop with an optimal basic solution if $\tilde{\bar{b}}_r \succ \tilde{0}$; it may stop with the conclusion that the problem is infeasible, if $\tilde{\bar{b}}_r \prec \tilde{0}$ and $y_{rj} \ge 0$ for all j ; or else it gives a new fuzzy basic feasible solution if $\tilde{\bar{b}}_r \prec \tilde{0}$

and $\exists j;\ y_{rj} < 0$. In addition, the difference in the dual fuzzy objective values between two successive iterations is $-\frac{\tilde{\bar{b}}_r}{y_{rk}}(z_k - c_k)$. Note that $\tilde{\bar{b}}_r \prec \tilde{0}$ and $y_{rk} < 0$. Moreover, by Definition 3.5 in the absence of dual degeneracy we have $z_k - c_k < 0$. Thus, the fuzzy objective value increases strictly in each iteration and hence no basis can be repeated and algorithm must converge in a finite number of steps.

It is important to note that the key in the fuzzy simplex methods is that the optimal solution is obtained at a basic solution. Thus we prove that if a FVLP problem has an optimal solution, then it also has a basic optimal solution.

Theorem 4.3. (Generalized fuzzy fundamental theorem)

If FVLP (1) in the standard from has a fuzzy optimum feasible solution, then it has a fuzzy basic feasible solution that is optimal.

Proof:

Suppose $\tilde{\bar{x}} = (\tilde{\bar{x}}_1, \tilde{\bar{x}}_2, \ldots, \tilde{\bar{x}}_n)$ be a fuzzy optimum feasible solution in which $\tilde{\bar{x}}_j = (\bar{m}_j, \bar{n}_j, \bar{\alpha}_j, \bar{\beta}_j)$. Let $\left\{ a_j \left| \frac{1}{2}\left[\bar{m}_j + \bar{n}_j + \frac{\bar{\beta}_j - \bar{\alpha}_j}{2} \right] > 0 \right. \right\} = \{a_1, a_2, \cdots, a_k\}$. So we have

$$a_1 \tilde{\bar{x}}_1 + a_2 \tilde{\bar{x}}_2, \ldots + a_k \tilde{\bar{x}}_k + a_{k+1}(\bar{m}_{k+1}, \bar{n}_{k+1}, \bar{\alpha}_{k+1}, \bar{\beta}_{k+1}) + \cdots + a_n(\bar{m}_n, \bar{n}_n, \bar{\alpha}_n, \bar{\beta}_n) \cong \tilde{\bar{b}} \tag{6}$$

If $\{a_1, a_2, \cdots, a_k\}$ is linearly independent, then $\tilde{\bar{x}}$ is a fuzzy basic feasible solution of (1) and we are done. Suppose this set is linearly dependent. So, there exists $y = (y_1, y_2, \cdots, y_k) \neq (0, 0, \cdots, 0)$ such that

$$a_1 y_1 + a_2 y_2 + a_k y_k = 0 . \tag{7}$$

Let $\tilde{y}_j = (y_j, y_j, 0, 0)$ for $j = 1, 2, \cdots, k$. Thus, from we have

$$a_1 \tilde{y}_1 + a_2 \tilde{y}_2 + a_k \tilde{y}_k = \tilde{0} . \tag{8}$$

Using Equations (6) and (8), we get the following relation in which θ is a real number.

$$a_1(\tilde{\bar{x}}_1 + \theta \tilde{y}_1) + a_2(\tilde{\bar{x}}_2 + \theta \tilde{y}_2) + \ldots + a_k(\tilde{\bar{x}}_k + \theta \tilde{y}_k) + a_{k+1}(\bar{m}_{k+1}, \bar{n}_{k+1}, \bar{\alpha}_{k+1}, \bar{\beta}_{k+1}) + \cdots + a_n(\bar{m}_n, \bar{n}_n, \bar{\alpha}_n, \bar{\beta}_n) \cong \tilde{\bar{b}} . \tag{9}$$

Define the fuzzy vector $\tilde{\bar{x}}(\theta) = (\tilde{\bar{x}}_1(\theta), \tilde{\bar{x}}_2(\theta), \ldots, \tilde{\bar{x}}_n(\theta))$), where

$$\tilde{\bar{x}}_j(\theta) = \begin{cases} \tilde{\bar{x}}_j + \theta \tilde{y}_j & j = 1, 2, \cdots, k \\ 0 & j = k+1, \cdots, n \end{cases}. \tag{10}$$

Clearly, $\tilde{\bar{x}}(\theta)$ satisfies $A\tilde{x} \cong \tilde{b}$. Define

$$\theta_1 = \max_{1 \le j \le n} \left\{ \frac{1}{2} \left[\bar{m}_j + \bar{n}_j + \frac{\bar{\beta}_j - \bar{\alpha} j}{2} \right] \middle| \, y_j > 0 \right\}. \tag{11}$$

$$\theta_2 = \min_{1 \le j \le n} \left\{ \frac{1}{2} \left[\bar{m}_j + \bar{n}_j + \frac{\bar{\beta}_j - \bar{\alpha} j}{2} \right] \middle| \, y_j < 0 \right\}. \tag{12}$$

Since $y = (y_1, y_2, \cdots, y_k) \neq (0, 0, \cdots, 0)$, at least one θ_1 or θ_2 of the must be finite. It is clear that $\theta_1 < 0$ and $\theta_2 > 0$. Let $0 < \varepsilon \le \min\{|\theta_1|, \theta_2\}$. Therefore, $\tilde{\bar{x}}(\theta)$ is a fuzzy feasible solution for all satisfing $-\varepsilon \le \theta \le \varepsilon$. In addition, we have,

$$\tilde{z}\left(\tilde{\bar{x}}(\theta)\right) = \tilde{z}(\tilde{\bar{x}}) + \theta(c_1\tilde{y}_1 + c_2\tilde{y}_2 + c_k\tilde{y}_k). \tag{13}$$

We now show that the assumption $\tilde{\bar{x}}$ is fuzzy optimal implies that any $(y_1, y_2, \cdots, y_k)$ satisfying (8) must also satisfy

$$c_1\tilde{y}_1 + c_2\tilde{y}_2 + c_k\tilde{y}_k \cong \tilde{0}. \tag{14}$$

Suppose not. If $c_1\tilde{y}_1 + c_2\tilde{y}_2 + c_k\tilde{y}_k \succ \tilde{0}$, let $\pi = -\varepsilon$ and if $c_1\tilde{y}_1 + c_2\tilde{y}_2 + c_k\tilde{y}_k \prec \tilde{0}$, let $\pi = \varepsilon$. Then, $\tilde{\bar{x}}(\pi)$ is a fuzzy solution of (1) and $\tilde{z}\left(\tilde{\bar{x}}(\pi)\right) \prec \tilde{z}(\tilde{\bar{x}})$, which contradicts the assumption that $\tilde{\bar{x}}$ is an optimal fuzzy solution of (1). This means that the fuzzy feasible solution $\tilde{\bar{x}}(\pi)$ is a fuzzy optimal feasible solution of (1) which the number of its positive fuzzy variables is at least one less than the number of positive fuzzy variables of $\tilde{\bar{x}}$. In a similar way, it is possible to obtain another fuzzy feasible solution $\tilde{\bar{\bar{x}}}(\pi)$ in which the number of positive variables is at least one less than the number of positive variables of $\tilde{\bar{x}}(\pi)$. By (14), any such fuzzy feasible $\tilde{\bar{\bar{x}}}(\pi)$ that we obtain must also satisfy $\tilde{z}\left(\tilde{\bar{\bar{x}}}(\pi)\right) \cong \tilde{z}(\tilde{\bar{x}})$. Hence, when this procedure is applied repeatedly, an optimal fuzzy basic feasible solution of (1) will be obtained after at most $(k-1)$) applications of the procedure.

Here, for an illustration of the above theorem we consider the following fuzzy variable linear programming problem.

Example 4.1.

A company produces two products P1 and P2. These products on two different machines M1 and M2. The time required manufacturing one unit of each products and the daily capacity of the machines are given in Table 2:

Table 2. Daily capacity of the machines

Machines	Time per unit (minutes) P1	P2	Machine capacity (min/day)
M1	1	1	10
M2	-	1	8

Note that the time availability can vary from day to day due to break down of machines, overtime work etc. At the same time the company wants to keep the profit somewhat close to 2 dollars for P1 and 2 dollars for P2. The company wants to determine the range of each product to be produced per day to maximize its profit. It is assumed that all the amounts produced are consumed in the market.

Since the time availability on each machine is uncertain, the number of units to be produced on each product will also be uncertain. So we will model the problem as a fuzzy variable linear programming problem. We use trapezoidal fuzzy numbers for each uncertain value which are especially useful in solving FVLP problems and have the most importance among the various types of fuzzy numbers.

Times availability for M1 and M2 which are close to 10 and 8 respectively, are modeled as (6, 14, 5, 5) and (4, 12, 4, 4). So the problem is formulated as follows:

$$\begin{aligned} \max\ & \tilde{z} \cong 2\tilde{x}_1 + 2\tilde{x}_2 \\ s.t.\ & \tilde{x}_1 + \tilde{x}_2 \tilde{\leq} (6,14,5,5) \\ & 2\tilde{x}_1 + \tilde{x}_2 \tilde{\leq} (4,12,4,4) \\ & \tilde{x}_1, \tilde{x}_2 \tilde{\geq} \tilde{0} \end{aligned}$$

Now the standard form of the fuzzy linear programming problem becomes as follows where $\tilde{x}_3$ and $\tilde{x}_4$ are the slack fuzzy variables:

$$\begin{aligned} \max\ & \tilde{z} \cong 2\tilde{x}_1 + 2\tilde{x}_2 \\ s.t.\ & \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \cong (6,14,5,5) \\ & 2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_4\ (4,12,4,4) \\ & \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \tilde{\geq} \tilde{0} \end{aligned}$$

It should be note that

$$\tilde{\bar{x}} = \begin{bmatrix} (4,8,3,3) \\ (2,6,2,2) \\ (0,0,0,0) \\ (2,6,2,2) \end{bmatrix}$$

is a fuzzy optimal solution for the above problem with optimal objective value $\tilde{z}^*(\tilde{\bar{x}}) = (12,28,10,10)$ and $\Re(\tilde{z}^*(\tilde{\bar{x}})) = 20$. It is clear that

$$\left\{ a_j \left| \frac{1}{2}\left[\bar{m}_j + \bar{n}_j + \frac{\bar{\beta}_j - \bar{\alpha}_j}{2} \right] > 0 \right. \right\} = \left\{ a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

are linear dependence. So, there exists $y = (y_1, y_2, y_4) \neq (0, 0, 0)$ such that

$$y_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let $y = (y_1, y_2, y_4) = (1, -1, 1)$. Now, define $\tilde{y}_1 = (1,1,0,0), \tilde{y}_2 = (-1,-1,0,0)$ and $\tilde{y}_4 = (1,1,0,0)$. By definition of θ_1 and θ_2 we have $\theta_1 = -4$ and $\theta_2 = 4$. Thus, we get $\varepsilon = 4$. In this case, if we have $\pi = \varepsilon = 4$, we obtain a new fuzzy optimal solution as

$$\tilde{\bar{x}}(\pi) = \tilde{\bar{x}} + \pi\,\tilde{y} = \begin{bmatrix} (8,12,3,3) \\ (-2,2,2,2) \\ (0,0,0,0) \\ (6,10,2,2) \end{bmatrix}.$$

For this new fuzzy solution, we have $\tilde{z}^*\left(\tilde{\bar{x}}(\pi)\right) = (16,24,6,6)$ and $\Re\left(\tilde{z}^*\left(\tilde{\bar{x}}(\pi)\right)\right) = 20$. This shows that the new fuzzy solution is optimal too. In addition, for this fuzzy solution we have

$$\left\{ a_j \left| \frac{1}{2}\left[\bar{m}_j + \bar{n}_j + \frac{\bar{\beta}_j - \bar{\alpha}_j}{2} \right] > 0 \right. \right\} = \left\{ a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

which is linear independence. Thus, the new fuzzy optimal solution is a fuzzy optimal basis feasible solution. On the other hand, if we let $\pi = \varepsilon = -4$, we obtain another new fuzzy optimal solution as

$$\tilde{\bar{x}}(\pi) = \tilde{\bar{x}} - 4\,\tilde{y} = \begin{bmatrix} (0,4,3,3) \\ (6,10,2,2) \\ (0,0,0,0) \\ (-2,2,2,2) \end{bmatrix}.$$

Also, for this new fuzzy solution, we have

$$\tilde{z}^*\left(\tilde{\tilde{x}}(\pi)\right)=(12,28,10,10) \text{ and } \Re\left(\tilde{z}^*\left(\tilde{\tilde{x}}(\pi)\right)\right)=20.$$

This shows that it is an alternative fuzzy optimal solution. In addition, for this fuzzy solution we have

$$\left\{a_j \left| \frac{1}{2}\left[\overline{m}_j+\overline{n}_j+\frac{\overline{\beta}_j-\overline{\alpha}_j}{2}\right]>0\right.\right\}=\left\{a_1=\begin{bmatrix}1\\0\end{bmatrix}, a_2=\begin{bmatrix}1\\1\end{bmatrix}\right\},$$

which is linear independence. Thus, this fuzzy optimal solution is another fuzzy optimal basis feasible solution. In addition the fuzzy amount of each product corresponding to both optimal solutions that company should produce per day to maximize its profit is summarized in Table3.

Table 3. The Fuzzy Optimal Basic Solutions

	Product P1	Product P2	Maximum Profit
Optimal solution $\tilde{\tilde{x}}(\pi)$	$(8,12,3,3)$	-----	$(16,24,6,6)$
Optimal solution $\tilde{\tilde{\tilde{x}}}(\pi)$	$(0,4,3,3)$	$(6,10,2,2)$	$(12,28,10,10)$

Based on this table, the company has two strategies to obtain its maximum profit. In the first strategy, it can only produce (8, 12, 3, 3) amount of product P1. In the second strategy, it can produce (0, 4, 3, 3) amount of product P1 and (6, 10, 2, 2) amount of product P2.

Remark 4.1.

We realize that the results obtained here are independent of the type of the fuzzy numbers. In other words, we can use any other types of fuzzy numbers, and although the solution obtained may be different, the results remain valid for the new solution. As for the types of the uncertain data in the model and the assumption of fuzziness in the variables, the type of fuzzy numbers for FVLP problems should be the decision maker main concerns. For trapezoidal fuzzy numbers and variables, the linear ranking function used here is deemed to be appropriate. In fact, the decision maker can choose any type of fuzzy numbers to suit the requirements of the problem under consideration. So, this will be a strengthening point of the proposed method. In what follows, we explore that the proposed method can be generalized for each *LR* flat fuzzy number defined as follows:

Definition 4.1.

A fuzzy number $\tilde{A}$ defined on the universal set of real numbers denoted as $\tilde{A}=(m,n,\alpha,\beta)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\tilde{A}}(x)$ is given by:

$$\mu_{\widetilde{A}}(x)=\begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & m-\alpha\le x\le m \\ 1 & m\le x\le n \\ R\left(\dfrac{x-n}{\beta}\right) & n\le x\le n+\beta \\ 0 & else \end{cases}$$

where the symmetric non-increasing function $L:[0,\infty)\to[0,1]$ is the left shape function, that $L(0)=1$. Also, a right shape function $R(.)$ is similarly defined as $L(.)$.

Yager (1981) proposed a procedure for ordering fuzzy sets in which a ranking index $\Re(\widetilde{A})$ is calculated for the fuzzy number $\widetilde{A}=(m,n,\alpha,\beta)_{LR}$ according to the following formula:

$$\Re(\widetilde{A})=\frac{1}{2}\int_0^1\left[(m-\alpha)L^{-1}(\lambda)+(n+\beta)R^{-1}(\lambda)\right]d\lambda$$

Now it is possible to generalize the fundamental theorem in crisp environment to fuzzy one same as the proposed approach in this paper.

5. Conclusions

These days a number of researchers have shown interest in the area of fuzzy linear programming problems and various attempts have been made to study the solution of fuzzy linear programming problems. In this paper we studied a class of fuzzy linear programming problems known as fuzzy variable linear programming problems in which the right-hand-side vectors and the decision variables are represented by fuzzy numbers while the rest of the parameters are represented by real numbers. We showed that in the absence of primal and dual degeneracy, the existing fuzzy simplex methods stop in a finite number of iterations. Finally, we proved that if an FVLP problem has a fuzzy optimal solution, then it also has a fuzzy basic optimal solution.

In our opinion, we feel that there are many other points of research that should be explored later on. Some of these points are discussed below.

In this paper, we generalized the fundamental theorem of linear programming in a crisp environment to a fuzzy one. Development of the Representation Theorem in fuzzy environment will be interesting.

In this paper we obtain some new results for FVLP problems where the right-hand-side vectors and the decision variables are represented by fuzzy numbers and the rest of the parameters are represented by real numbers. However, the method is not very efficient when the cost coefficients and decision variables are trapezoidal fuzzy numbers. This will be an interesting research work in the future.

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