Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 7, Issue 2 (December 2012), pp. 595 - 608

Applications and Applied Mathematics:
An International Journal
(AAM)

Numerical Studies for Solving Fractional Riccati Differential Equation

N. H. Sweilam[1], M. M. Khader[2] and A. M. S. Mahdy[3]

Department of Mathematics
[1]Cairo University, Giza, Egypt
[2]Benha University, Benha, Egypt
[3]Zagazig University, Zagazig, Egypt
n_sweilam@yahoo.com; mohamedmbd@yahoo.com; amr_mahdy85@yahoo.com

Received: December 6, 2011; Accepted: September 10, 2012

Abstract

In this paper, finite difference method (FDM) and Pade'-variational iteration method (Pade'-VIM) are successfully implemented for solving the nonlinear fractional Riccati differential equation. The fractional derivative is described in the Caputo sense. The existence and the uniqueness of the proposed problem are given. The resulting nonlinear system of algebraic equations from FDM is solved by using Newton iteration method; moreover the condition of convergence is verified. The convergence's domain of the solution is improved and enlarged by Pade'-VIM technique. The results obtained by using FDM is compared with Pade'-VIM. It should be noted that the Pade'-VIM is preferable because it always converges to the solution even for large domain.

Keywords: Fractional Riccati differential equation; Caputo fractional derivative; Finite difference method; Variational iteration method; Pade' approximation

MSC 2010 No.: 65K10, 65G99, 35E99, 68U20

1. Introduction

Ordinary and partial fractional differential equations (FDEs) have the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering Bagley et al. (1984). Fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary non-integer order. Many physical processes appear to exhibit fractional order behavior that may vary with time or space Podlubny (1999). Most FDEs do not have exact solutions, so approximate and numerical techniques [El-Sayed et al. (2007), Sweilam et al. (2007), Sweilam et al. (2011)] must be used. Several numerical and approximate methods to solve the FDEs have been given such as variational iteration method He (1999), homotopy perturbation method Sweilam et al. (2007), Adomian's decomposition method Wazwaz (1998), homotopy analysis method Tan and Abbasbandy (2008), collocation method [Khader (2011)-Kahder et al. (2012), Sweilam et al. (2012c)] and finite difference method [Sweilam et al. (2012a), Sweilam et al. (2012b)].

The Riccati differential equation is named after the Italian Nobleman Count Jacopo Francesco Riccati (1676-1754). The book of Reid (1972) contains the fundamental theories of Riccati equation, with applications to random processes, optimal control and diffusion problems. Besides important engineering science applications that today is considered classical, such as stochastic realization theory, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics Lasiecka and Triggiani (1991). The solution of this equation can be reached using classical numerical methods such as the forward Euler method and Runge-Kutta method. An unconditionally stable scheme was presented by Dubois and Saidi (2000). Bahnasawi et al. (2004) presented the usage of Adomian's decomposition method to solve the nonlinear Riccati differential equation in an analytic form. Tan and Abbasbandy (2008) employed the analytic technique called homotopy analysis method to solve the quadratic Riccati equation.

The fractional Riccati differential equation is studied by many authors and using different numerical methods. Namely, Khan et al. (2011), solved this problem using by the homotopy analysis method, Jafari and Tajadodi (2010) solved the same problem by the variational iteration method and Momani and Shawagfeh (2006) solved it using the Adomian's decomposition method.

We present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

Definition 1.

The Caputo fractional derivative D^{α} of order α is defined in the following form

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \qquad \alpha > 0, \quad x > 0,$$

where $m-1 < \alpha \le m, m \in N$ and $\Gamma(.)$ is the Gamma function.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation $D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x)$, where λ and μ are constants.

Recall that for $\alpha \in N$, the Caputo differential operator coincides with the usual differential operator of integer order.

Definition 2.

The fractional integral of order $\alpha \in \Re^{+}$ of the function $f(x), x > 0$ is defined as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-s)^{\alpha-1} f(s) ds.$$

For more details on fractional derivatives definitions and its properties see [Podlubny (1999), Samko et al. (1993)].

In this article, we consider the fractional Riccati differential equation (FRDE) of the form

$$D^{\alpha} u(t) + u^{2}(t) - 1 = 0, \qquad t > 0, \qquad 0 < \alpha \leq 1, \tag{1}$$

the parameter α refers to the fractional order of the time derivative.

We also assume an initial condition

$$u(0) = u_{0}. \tag{2}$$

For $\alpha = 1$ Equation (1) is the standard Riccati differential equation

$$\frac{du(t)}{dt} + u^{2}(t) - 1 = 0.$$

The exact solution to this equation is

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

Our paper is organized as follows: In section 2, we study the existence and the uniqueness of the FRDE. In section 3, we introduce an approximate formula of the fractional derivative. In section 4, we present the FDM scheme of the FRDE. In section 5, we give the procedure of the solution using Pade'-VIM. Finally in section 6, the paper ends with a brief conclusion.

2. Existence and Uniqueness

Let $J = [0, T], T < \infty$ and $C(J)$ be the class of all continuous functions defined on J, with the norm

$$\| u \| = \sup_{t \in J} | e^{-Nt} u(t) |, \quad N > 0,$$

which is equivalent to the sup-norm $\| u \| = \sup_{t \in J} | u(t) |$.

To study the existence and the uniqueness of the initial value problem of the fractional Riccati differential equation (1), we suppose that the solution $u(t)$ belongs to the space

$$B = \{u \in \mathrm{R} : |u| \leq b, \text{ for any constant b}\}.$$

Definition 3.

The space of integrable functions $L_1[0,T]$ in the interval $[0,T]$ is defined as

$$L_1[0,T] = \{f(t) : \int_0^T |f(t)|\, dt < \infty\}.$$

Theorem 1.

The initial value problem (1) has a unique solution

$$u \in C(J), u' \in X = \{u \in L_1[0,T], \ \|u\| = \|e^{-Nt}u(t)\|_{L_1}\}.$$

Proof:

From the properties of fractional calculus, the fractional differential equation (1) can be written as [El-Sayed et al. (2007)]

$$I^{1-\alpha}\frac{du(t)}{dt} = 1 - u^2(t).$$

Operating with I^{α} we obtain

$$u(t) = I^{\alpha}(1 - u^2(t)). \tag{3}$$

Now, let us define the operator $F : C(J) \to C(J)$ by

$$Fu(t) = I^{\alpha}(1 - u^2(t)), \tag{4}$$

then,

$$\begin{aligned} e^{-Nt}(Fu - Fv) &= e^{-Nt}I^{\alpha}[(1-u^2(t)) - (1-v^2(t))] \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)}(v(s)-u(s))(v(s)+u(s))e^{-Ns}ds \\ &\leq \|v-u\| \int_0^t \frac{s^{\alpha-1}e^{-Ns}}{\Gamma(\alpha)}ds, \end{aligned}$$

therefore, we obtain

$$\| Fu - Fv \| < \| u - v \|,$$

and the operator F given by Equation(4) has a unique fixed point. Consequently the integral Equation (3) has a unique solution $u \in C(J)$. Also, we can deduce that [El-Sayed et al. (2007)]

$$I^{\alpha}(1-u^2(t))|_{t=0} = 0.$$

Now from Equation (3), we formally have

$$u(t) = [\frac{t^{\alpha}}{\Gamma(\alpha+1)}(1-u_0^2) + I^{\alpha+1}(0-2u(t)u'(t))],$$

and

$$\frac{du}{dt} = [\frac{t^{\alpha-1}}{\Gamma(\alpha)}(1-u_0^2) + I^{\alpha}(-2u(t)u'(t))],$$

$$e^{-Nt}u'(t) = e^{-Nt}[\frac{t^{\alpha-1}}{\Gamma(\alpha)}(1-u_0^2) + I^{\alpha}(-2u(t)u'(t))],$$

from which we can deduce that $u' \in C(J)$ and $u' \in X$. Now from Equation (3), we get

$$\frac{du}{dt} = \frac{d}{dt} I^{\alpha}[1-u^2(t)],$$

$$I^{1-\alpha}\frac{du}{dt} = I^{1-\alpha}\frac{d}{dt} I^{\alpha}[1-u^2(t)] = \frac{d}{dt} I^{1-\alpha} I^{\alpha}[1-u^2(t)],$$

$$D^{\alpha}u(t) = \frac{d}{dt} I[1-u^2(t)] = 1-u^2(t),$$

and

$$u(0) = I^{\alpha}[1-u^2(t)]_{t=0} = 0.$$

Then the integral Equation(3) is equivalent to the initial value problem (1) and the theorem is proved.

3. Approximate Formula of Fractional Derivatives

In this section, we present a discrete approximation to the fractional derivative $D^{\alpha}u(t)$. For any positive integer M, the grid in time for the finite difference algorithm is defined by $k = \frac{T_f}{M}$. The grid points in the time interval $[0, T_f]$ are labeled $t_n = nk,\ n = 0,1,2\ \ldots, M$.

Now, the discrete approximation of the fractional derivative $D^{\alpha}u(t)$ can be obtained by a simple quadrature formula as follows [Burden and Faires (1993), Diethelm (1997), Richtmyer and Morton (1967)]

$$\begin{aligned}
\frac{d^{\alpha}}{dt^{\alpha}}u(t_n) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\frac{d}{dt}u(s)(t_n-s)^{-\alpha}\,ds\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{n}\int_{(j-1)k}^{jk}\left[\frac{u_j-u_{j-1}}{k}+o(k)\right](nk-s)^{-\alpha}\,ds\\
&= \frac{1}{\Gamma(1-\alpha)(1-\alpha)}\sum_{j=1}^{n}\left[\frac{u_j-u_{j-1}}{k}+o(k)\right][(n-j+1)^{1-\alpha}-(n-j)^{1-\alpha}][k^{1-\alpha}]\\
&= \frac{1}{\Gamma(2-\alpha)k^{\alpha}}\sum_{j=1}^{n}(u_j-u_{j-1})\left[(n-j+1)^{1-\alpha}-(n-j)^{1-\alpha}\right]\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{n}\left[(n-j+1)^{1-\alpha}-(n-j)^{1-\alpha}\right]o(k^{2-\alpha}).
\end{aligned}$$

Setting and shifting indices, we have

$$\sigma_{\alpha,k}=\frac{1}{\Gamma(2-\alpha)k^{\alpha}} \quad \text{and} \quad \omega_j^{(\alpha)}=j^{1-\alpha}-(j-1)^{1-\alpha}, \tag{5}$$

and

$$\begin{aligned}
\frac{d^{\alpha}u(t_n)}{dt^{\alpha}} &= \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_{n-j+1}-u_{n-j})+\frac{1}{\Gamma(2-\alpha)}n^{1-\alpha}o(k^{2-\alpha})\\
&= \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_{n-j+1}-u_{n-j})+o(k).
\end{aligned}$$

Here

$$\frac{d^{\alpha}u(t_n)}{dt^{\alpha}}=D^{\alpha}u_n+o(k),$$

and the first-order approximation method for the computation of Caputo's fractional derivative is given by

$$D^{\alpha}u_n \cong \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_{n-j+1}-u_{n-j}), \qquad n=1,2,\ldots,M. \tag{6}$$

4. Discretization for Fractional Riccati Differential Equation Using FDM

In this section, finite difference method with the discrete formula (6) is used to estimate the time α-order fractional derivative to solve numerically the FRDE (1). Using (6) the restriction of the exact solution to the grid points centered at $t_n = nk$, in Equation (1)

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_{n-j+1}-u_{n-j})+o(k)=1-u_n^2, \qquad n=1,2,...,M.$$

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_{n-j+1}-u_{n-j})=1-u_n^2+T(t), \tag{7}$$

where $T(t)$ is the truncation term. Thus, according to Equation(7), the numerical scheme is consistent, first order correct in time. The resulting finite difference equations are defined by

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_{n-j+1}-u_{n-j})=1-u_n^2, \qquad n=1,2,...,M. \tag{8}$$

This scheme presents a nonlinear system of algebraic equations. In our calculation, we use the Newton iteration method to solve this system.

In view of Newton iteration method, we can write the nonlinear system (8) in the following iteration formula

$$U^{m+1}=U^m-J^{-1}(U^m)F(U^m), \tag{9}$$

where $F(U^m)$ is a vector represents the nonlinear equations and $J^{-1}(U^m)$ is the inverse of the Jacobian matrix.

To clear the structure of the Jacobian matrix, we take $M=5$, therefore we have

$$J(U^m)=$$

$$\begin{pmatrix}
\sigma_{\alpha,k}w_1^{(\alpha)}+2u_1^m & 0 & 0 & 0 & 0 \\
\sigma_{\alpha,k}(w_1^{(\alpha)}+w_2^{(\alpha)}) & \sigma_{\alpha,k}w_1^{(\alpha)}+2u_2^m & 0 & 0 & 0 \\
\sigma_{\alpha,k}(w_2^{(\alpha)}+w_3^{(\alpha)}) & \sigma_{\alpha,k}(w_1^{(\alpha)}+w_2^{(\alpha)}) & \sigma_{\alpha,k}w_1^{(\alpha)}+2u_3^m & 0 & 0 \\
\sigma_{\alpha,k}(w_3^{(\alpha)}+w_4^{(\alpha)}) & \sigma_{\alpha,k}(w_2^{(\alpha)}+w_3^{(\alpha)}) & \sigma_{\alpha,k}(w_1^{(\alpha)}+w_2^{(\alpha)}) & \sigma_{\alpha,k}w_1^{(\alpha)}+2u_4^m & 0 \\
\sigma_{\alpha,k}(w_4^{(\alpha)}+w_5^{(\alpha)}) & \sigma_{\alpha,k}(w_3^{(\alpha)}+w_4^{(\alpha)}) & \sigma_{\alpha,k}(w_2^{(\alpha)}+w_3^{(\alpha)}) & \sigma_{\alpha,k}(w_1^{(\alpha)}+w_2^{(\alpha)}) & \sigma_{\alpha,k}w_1^{(\alpha)}+2u_5^m
\end{pmatrix}$$

and $\quad |J(U^m)|=\prod_{i=1}^{M}(\sigma_{\alpha,k}\,\omega_1^{(\alpha)}+2u_i^{(m)}).$

Since, it is easy to see that the quantity $\prod_{i=1}^{M}(\sigma_{\alpha,k}\,\omega_1^{(\alpha)}+2u_i^{(m)})\neq 0$, we can deduce that the resulting Jacobian matrix from Newton's formula (9) is nonsingular, therefore, we can show that the numerical scheme (8) is convergent.

5. Procedure Solution Using Pade'-VIM

In spite of the advantages of VIM [Sweilam and Khader (2007), Sweilam and Khader (2010)], it has some drawbacks. By using VIM we get a series, in particular a truncated sequence solution. The series often coincides with the Taylor expansion of the true solution at point $t = 0$, in the initial value case. Although the series can be rapidly convergence in a very small region, it has very slow convergence rate in the wider region we examine and the truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method. All the truncated series solutions have the same problem. Many examples given can be used to support this assertion [Wazwaz (1998), Wazwaz (1999)].

In this section, we present a modification of VIM by using the Pade' approximation and then apply this modification to solve the fractional Riccati differential equations. When we obtain the truncated series solution of order at least $L+M$ in t by VIM, we will use it to obtain Pade' approximation $PA[L/M](t)$ for the solution $u(t)$.

Definition 4.

The Pade' approximation is a particular type of rational fraction approximation to the value of the function (Baker (1975), Boyd (1997)). The idea is to match the Taylor series expansion as far as possible.

We define $PA[L/M](t)$ to $R(t) = \sum_{i=0}^{\infty} a_i t^i$ by

$$PA[L/M](t) = \frac{P_L(t)}{Q_M(t)}, \tag{10}$$

where $P_L(t)$ is a polynomial of degree at most L and $Q_M(t)$ is a polynomial of degree at most M

$$P_L(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_L t^L, \quad Q_L(t) = 1 + q_1 t + q_2 t^2 + \ldots + q_M t^M. \tag{11}$$

The suggested modification of VIM can be done by using the following algorithm.

Algorithm

Step 1.

Application of VIM to solve Equation (1):

The VIM gives the possibility to write the solution of Equation (1), $(0 < \alpha \le 1)$ with the aid of the correction functionals in the form

$$u_{p+1}(t) = u_p(t) + \int_0^t \lambda(\tau)[\frac{du_p}{d\tau} - 1 + \tilde{u}_p^2]d\tau, \tag{12}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory He (1999). The function $\tilde{u}_p$ is a restricted variation, which means that $\delta\tilde{u}_p = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_p, p \geq 1$, of the solution u will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. With λ determined, then several approximations $u_p, p \geq 1$, follow immediately.

Making the above correction functional stationary

$$\begin{aligned}\delta u_{p+1}(t) &= \delta u_p(t) + \delta\int_0^t \lambda(\tau)[\frac{du_p}{d\tau} - 1 + \tilde{u}_p^2]d\tau \\ &= \delta u_p(t) + \int_0^t \delta\lambda(\tau)[\frac{du_p}{d\tau}]d\tau \\ &= \delta u_p(t) + [\lambda(\tau)\delta u_p(\tau)]_{\tau=t} - \int_0^t \delta u_p \lambda'(\tau)d\tau = 0,\end{aligned}$$

where $\delta\tilde{u}_p$ is considered as a restricted variation, i.e., $\delta\tilde{u}_p = 0$, yields the following stationary conditions (by comparison the two sides in the above equation)

$$\lambda'(\tau) = 0, \quad 1 + \lambda(\tau)|_{\tau=t} = 0. \tag{13}$$

The equations in (13) are called Lagrange-Euler equation and the natural boundary conditions, respectively. The solution of this equation gives the Lagrange multiplier $\lambda(\tau) = -1$.

Now, by substituting in Equation(12), the following variational iteration formula can be obtained

$$u_{p+1}(t) = u_p(t) - \int_0^t [\frac{d^\alpha u_p}{d\tau^\alpha} - 1 + u_p^2]d\tau, \qquad p \geq 0. \tag{14}$$

We start with initial approximation $u_0(t) = u_0 = 0$, and by using the above iteration formula (14), we can directly obtain the components of the solution. Now, the first three components of the solution $u(t)$ by using (14) of the fractional Riccati equation are given by

$$\begin{aligned}&u_0(t) = 0, \\ &u_1(t) = t, \\ &u_2(t) = t - \left(\frac{t}{3}(-3 + t^2) + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\right),\end{aligned}$$

and so on, other components can be obtained in a like manner. Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{p \to \infty} u_p(t). \quad (15)$$

Step 2.

Truncate the obtained sequence solution by using VIM:
Therefore, the approximate solution can be readily obtained by $u(t) \cong u_4(t)$, (at $\alpha = 1$) coincide until fifth term with the partial sum of the Taylor series of the solution $u(t)$ at $t = 0$. Figure 1, shows the error between the exact solution $u(t)$ and the approximate solution using VIM, $u_4(t)$. From this figure we can conclude that the error at $t \in [0,1]$ is nearly to 0, but at $t \in [1,3]$, the error takes large values.

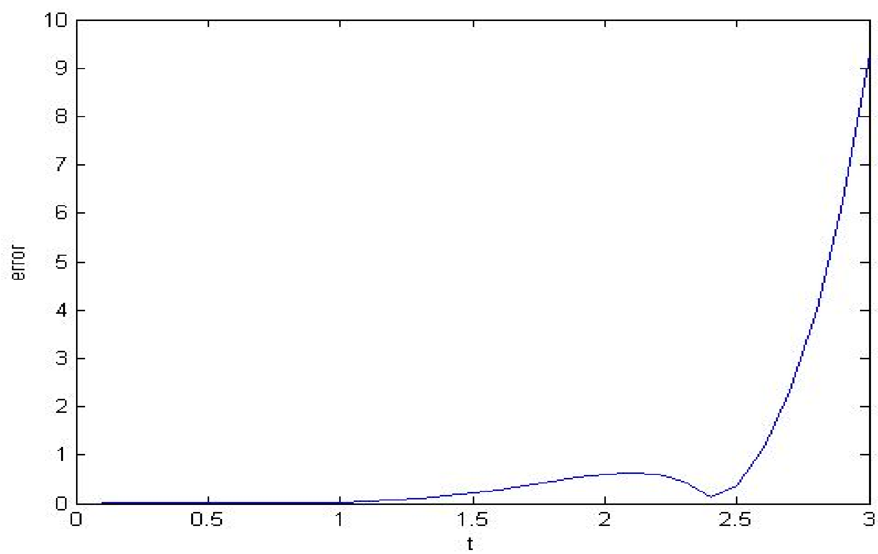

Figure 1. The behavior of the error between $u(t)$ and $u_4(t)$.

Step 3.

Find the Pade' approximation of the previous step: we find the Pade' approximation $PA[4/4](t)$ of the solution $u_4(t)$ at $\alpha = 1$ as follows

$$PA[4/4](t) = \frac{4t^2 - 4.27029t^3 + 1.85930t^4}{1 + 0.06080t + 0.03343t^2 + 0.08255t^3 + 1.24767t^4}.$$

The approximate solution using Pade'-VIM, $u_{Pad'e-VIM}$, with $n = 4$, and the numerical solution using FDM, u_{FDM}, with $k = 0.1$ for different values of α are presented in figures 2-4 in the interval [0, 3].

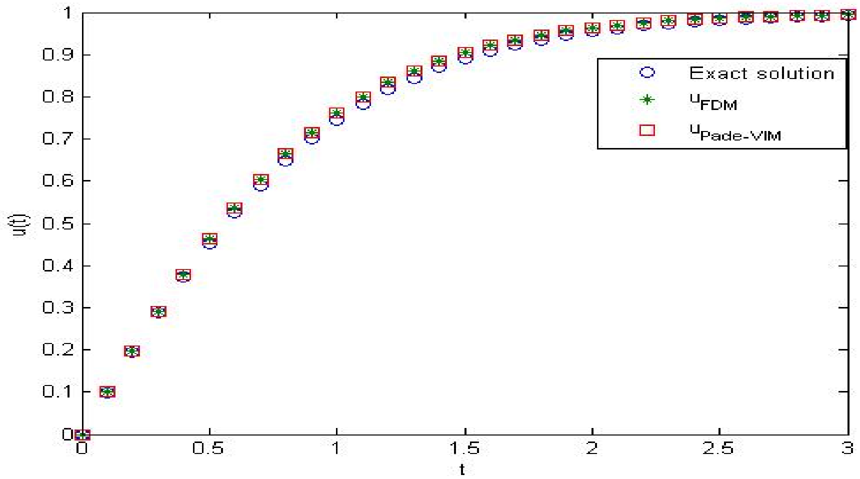

Figure 2: The behavior of the approximate solution using Pade'-VIM, the numerical solution using FDM at $k = 0.1$ and the exact solution at $\alpha = 1$.

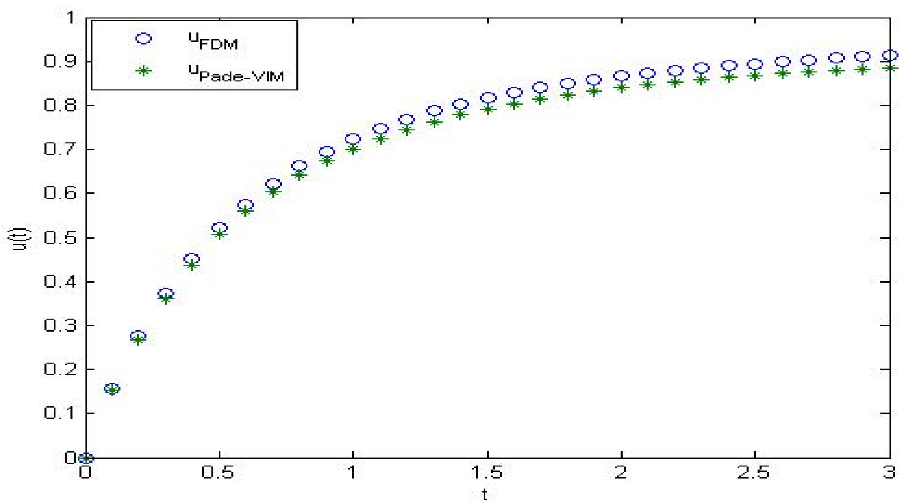

Figure 3: The behavior of u_{FDM} and $u_{Pad\grave{e}-VIM}$ at $\alpha = 0.75$.

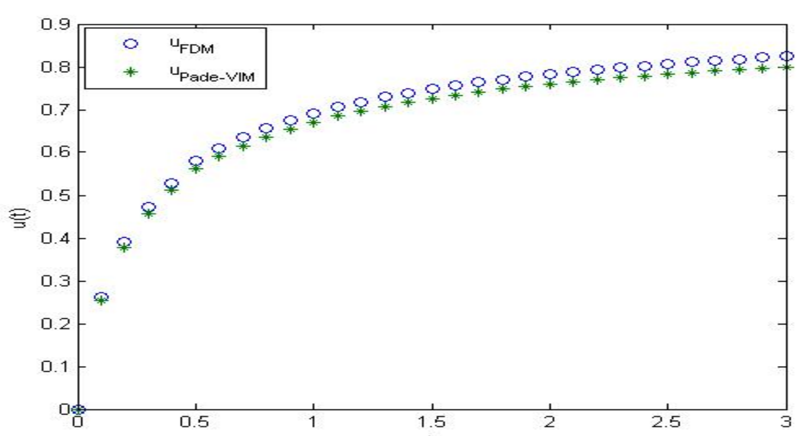

Figure 4. The behavior of u_{FDM} and $u_{Pad\grave{e}-VIM}$ at $\alpha = 0.5$.

6. Conclusion

In this paper, we used two computational methods, FDM and Pade'-VIM for solving the fractional Riccati differential equation. Special attention is given to study the existence and the uniqueness of the proposed problem. We derived an approximate formula of the fractional derivative. The presented formula reduced the fractional Riccati differential equation to a nonlinear system of algebraic equations which solved by the Newton iteration method. The convergence of the resulting nonlinear system of algebraic equations is discussed. From the obtained numerical results we can conclude that these two methods give results with an excellent agreement with the exact solution. Also, it is evident that the overall errors can be made smaller by adding new terms of the sequence which obtained from VIM or using the Pade' approximation. All numerical results are obtained using the Matlab program.

References

Bagley, R. L. and Torvik, P. J. (1984). On the appearance of the fractional derivative in the behavior of real materials, Journal of Applied Mechanics, Vol. **51**, pp. 294-298.

Bahnasawi, A. A., El-Tawil, M. A. and Abdel-Naby, A. (2004). Solving Riccati differential equation using Adomian's decomposition method, Applied Mathematics and Computation, Vol. **157**, pp. 503-514.

Baker, G. A. (1975). Essentials of Pad$\grave{e}$ Approximants, Academic Press, London.

Boyd, J. (1997). Pad$\grave{e}$ approximants algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain, Journal Computers in Physics, Vol. **11(3)**, pp. 299-303.

Burden, R. L. and Faires, J. D. (1993). Numerical Analysis, PWS, Boston.

Diethelm, K. (1997). An algorithm for the numerical solution of differential equatios of fractional order, Electron Transact. Numer. Anal., Vol. **5**, pp. 1-6.

Dubois, F. and Saidi, A. (2000). Unconditionally stable scheme for Riccati equation, ESAIM: Proceedings, Vol. **8**, pp. 39-52.

El-Sayed, A. M. A., El-Mesiry, A. E. M. and El-Saka, H. A. A. (2007). On the fractional-order logistic equation, Applied Mathematics Letters, Vol. **20(7)**, pp. 817-823.

He, J. H. (1999). Variational iteration method-a kind of non-linear analytical technique: some examples, International Journal of Non-Linear Mechanics, Vol. **34**, pp. 699-708.

Jafari, H. and Tajadodi, H. (2010). He's variational iteration method for solving fractional Riccati differential equation, International Journal of Differential Equations, Vol. **2010**, pp. 1-8.

Khader, M. M. (2011). On the numerical solutions for the fractional diffusion equation, Communications in Nonlinear Science and Numerical Simulation, Vol. **16**, pp. 2535-2542.

Khader, M. M., Sweilam, N. H. and Mahdy, A. M. S. (2011). An efficient numerical method for solving the fractional diffusion equation, Journal of Applied Mathematics and Bioinformatics, Vol. **1**, pp. 1-12.

Khader, M. M. (2012). Introducing an efficient modification of the variational iteration method by using Chebyshev polynomials, Application and Applied Mathematics: An International Journal, Vol. **7(1)**, pp. 283-299.

Khader M. M. and Hendy, A. S. (2012). The approximate and exact solutions of the fractional-order delay differential equations using Legendre pseudospectral method, International Journal of Pure and Applied Mathematics, Vol. **74(3)**, pp. 287-297.

Khader, M. M. and Hendy, A. S. (2012). A numerical technique for solving fractional variational problems, Accepted in Mathematics Methods in Applied Sciences.

Khader, M. M., EL Danaf, T. S. and Hendy, A. S. (2012). Efficient spectral collocation method for solving multi-term fractional differential equations based on the generalized Laguerre polynomials, *Journal of Fractional Calculus and Applications*, **3(13)**, pp. 1-14.

Khan, N. A., Ara, A. and Jamil, M. (2011). An efficient approach for solving the Riccati equation with fractional orders, Computers and Mathematics with Applications, Vol. **61**, pp. 2683-2689.

Lasiecka, L. and Triggiani, R. (1991). Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory, Lecture notes in control and information sciences, Berlin: Springer, Vol. **164**.

Momani, S. and Shawagfeh, N. (2006). Decomposition method for solving fractional Riccati differential equations, Applied Mathematics and Computation, Vol. **182**, pp. 1083-1092.

Podlubny, I. (1999). Fractional Differential Equations, Academic Press, New York, 1999.

Reid, W. T. (1972). Riccati differential equations, Mathematics in Science and Engineering, New York, Academic Press, Vol. **86**.

Richtmyer, R. D. and Morton, K. W. (1697). Difference Methods for Initial-Value Problems, Inter. Science Publishers, New York.

Samko, S. Kilbas, A. and Marichev, O. (1993). Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, London.

Sweilam N. H. and Khader, M. M. (2007). Variational iteration method for one dimensional nonlinear thermoelasticity, Chaos, Solitons and Fractals, Vol. **32**, pp. 145-149.

Sweilam N. H. and Khader, M. M. (2010). On the convergence of VIM for nonlinear coupled system of partial differential equations, International Journal of Computer Mathematics, Vol. **87(5)**, pp. 1120-1130.

Sweilam, N. H., Khader, M. M. and Al-Bar, R. F. (2007). Numerical studies for a multi-order fractional differential equation, Physics Letters A, Vol. **371**, pp. 26-33.

Sweilam, N. H., Khader, M. M. and Nagy, A. M. (2011). Numerical solution of two-sided space-fractional wave equation using finite difference method, Journal of Computional and Applied Mathematics, Vol. **235**, pp. 2832-2841.

Sweilam, N. H., Khader M. M. and Adel, M. (2012). On the stability analysis of weighted average finite difference methods for fractional wave equation, Accepted in Fractional Differential Calculus, Vol. **2**(1), pp. 17-29.

Sweilam, N. H., Khader, M. M. and Mahdy, A. M. S. (2012). Crank-Nicolson finite difference method for solving time-fractional diffusion equation, Journal of Fractional Calculus and Applications, Vol. **2(2)**, pp. 1-9.

Sweilam, N. H., Khader, M. M. and Mahdy, A. M. S. (2012). Numerical studies for solving fractional-order Logistic equation, International Journal of Pure and Applied Mathematics, Vol. **78(8)**, pp. 1199-1210.

Sweilam, N. H., Khader, M. M. and Mahdy, A. M. S. (2012). Numerical studies for fractional-order Logistic differential equation with two different delays, Accepted in Journal of Applied Mathematics.

Tan, Y. and Abbasbandy, S. (2008). Homotopy analysis method for quadratic Riccati differential equation, Communications in Nonlinear Science and Numerical Simulation, Vol. **13(3)**, pp.

539-546.

Wazwaz, A. M. (1998). A comparison between Adomian decomposition method and Taylor series method in the series solution, Applied Mathematics and Computation, Vol. **97**, pp. 37-44.

Wazwaz, A. M. (1999). Analytical approximations and Pade' approximants for Volterra's population model, Applied Mathematics and Computation, Vol. **100**, pp. 13-25.