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On Stability of Dynamic Equations on Time Scales Via Dichotomic Maps

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Abstract

Dichotomic maps are used to check the stability of ordinary differential equations and difference equations. In this paper, this method is extended to dynamic equations on time scales; the stability and asymptotic stability to the trivial solution of the first order system of dynamic equations are examined using dichotomic and strictly dichotomic maps. This method, in dynamic equations, also involves Lyapunov's direct method.

Keywords: Dichotomic Maps; Stability; Time Scales

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1. Introduction and Preliminaries

The extension of Lyapunov's direct method to discrete equations has been given by Carvalho and Ferreira (1988). Improvement of this method on ordinary differential equations have been made by Carvalho (1991), Carvalho and Cooke (1991). Many researches on this subject were done after those papers [Bená and Dos Reis (1998), Marconato (2005)]. Lyapunov stability theory on time scales was studied by Kaymakcalan (1992). Hoffacker and Tisdell (2005) also studied Lyapunov's direct method for the following dynamical equation.

In this study, we examine the stability of the trivial solution $x = 0$ to the first order system of dynamic equations (1):

$$x^{\Delta} = f\left(t,x\right), \ t \geq t_0, \quad x \in D \subset \mathbb{R}^n, \tag{1}$$

using dichotomic map, where $t, t_0 \in \mathbb{T}$, and D is a compact set. We also assume $f\left(t,0\right) = 0 \in D$, for all $t \in \mathbb{T}$, $t \geq t_0$, so that $x = 0$ is a solution to equation (1).

Bohner (2001) has defined the time scale $\mathbb{T}$ as a nonempty closed subset of the real numbers $\mathbb{R}$, and has given the definition of Δ- derivative as follows.

Definition 1.

Fix $t \in \mathbb{T}$ and let $x:\mathbb{T} \to \mathbb{R}^n$. Define $x^{\Delta}\left(t\right)$ to be the vector (if it exists) with the property that given any $\varepsilon > 0$, there is a neighbourhood U of t such that

$$\left\|\left[x_i\left(\sigma\left(t\right)\right) - x_i\left(s\right)\right] - x_i^{\Delta}\left(t\right)\left[\sigma\left(t\right) - s\right]\right\| \leq \varepsilon\left|\sigma\left(t\right) - s\right|,$$

for all $s \in U$ and each $i = 1,...,n$. The function $x^{\Delta}\left(t\right)$ is the delta derivative of x at t, and it is said that x is delta differentiable at t.

Definition 2.

A function $\phi:\left[0,r\right] \to \left[0,\infty\right)$ is of class $\mathcal{K}$ if it is well defined, continuous, and strictly increasing on $\left[0,r\right]$ with $\phi\left(0\right) = 0$.

Guseinov (2003) stated the following theorem.

Theorem 1. (Mean Value Theorem).

Let f be a continuous function on $[a,b]$, which is Δ-differentiable on $\left[a,b\right)$. Then, there exist $\xi, \tau \in \left[a,b\right)$ such that

$$f^{\Delta}\left(\tau\right) \leq \frac{f\left(b\right) - f\left(a\right)}{b-a} \leq f^{\Delta}\left(\xi\right).$$

Definition 3.

The system (1) is stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{T}$, there exists $\delta(\varepsilon, t_0) > 0$ such that $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < \varepsilon$ for all $t \geq t_0$. The system (1) is asymptotically stable; if it is stable and there exists a $\delta_0 > 0$ such that if $|x_0| < \delta_0$ then $\lim_{t \to \infty} x(t; t_0, x_0) = 0$.

Hoffacker and Tisdell (2005) has stated Theorem 2.

Theorem 2.

If there exists a continuously differentiable positive definite function V in a neighborhood of zero with $V^{\Delta}(x(t))$ negative semidefinite, then the trivial solution $x(t) = 0$ to equation (1) is stable.

2. Dichotomic Map on Time Scales

If $T \in \mathbb{T}$, $T > 0$ is a given constant and Ω is a neighborhood of the origin having $x(t, y)$ defined for $t \in [t_0, \infty)_{\mathbb{T}}$, $\eta(t) : [t_0, \infty)_{\mathbb{T}} \to [t_0, \infty)_{\mathbb{T}}$, $\eta(t) < t$ and $y \in \Omega$, then

$$\Omega_{-}(T) = \left\{ y \in \Omega : V(x(T, y)) < V(x(\eta(T), y)) \right\},$$
$$\Omega_{0}(T) = \left\{ y \in \Omega : V(x(T, y)) = V(x(\eta(T), y)) \right\},$$
$$\Omega_{+}^{\Delta} = \left\{ y \in \Omega : V^{\Delta}(x(., y)) > 0 \right\},$$
$$\Omega_{0}^{\Delta} = \left\{ y \in \Omega : V^{\Delta}(x(., y)) = 0 \right\},$$

and let $\Omega_{-}^{0}(T) = \Omega_{-}(T) \cap \Omega_{0}(T)$.

Definition 4.

We say a Δ- differentiable map $V : \mathbb{R}^n \to \mathbb{R}$ is dichotomic with respect to (1), if there are a constant $T > 0$ and a neighborhood Ω of the origin in $\mathbb{R}^n$ such that $\bar{\Omega}_{+}^{\Delta} \subset \Omega_{-}^{0}(T)$.

Definition 5.

We say that a given map V is strictly dichotomic with respect to (1), if it is dichotomic with respect to (1) and moreover, it satisfies the supplementary condition that $\left(\bar{\Omega}_{+}^{\Delta}\right)^{*} \subset \Omega_{-}(T)$ and $\Omega_{0}(T) \cap \Omega_{0}^{\Delta} = \{0\}$.

We see that Lyapunov functions are automatically dichotomic with respect to the given equation. Note that there may exist points $y \in \Omega_0^\Delta$ which do not belong to $\bar{\Omega}_0^\Delta$.

Now define, for a given $y \in \mathbb{R}^N$,

$$c_j = \max\left\{V\left(x(t,y)\right) : \phi_{j-1}(T) \le t \le \phi_j(T)\right\} \qquad j = 0,1,2,\ldots$$

and

$$t_j = \min\left\{t : \phi_{j-1}(T) \le t \le \phi_j(T), c_j = V\left(x(t,y)\right)\right\}.$$

where ϕ_j is a class $\mathcal{K}$ function for all $j = 0,1,2,\ldots$ Now, we state the following lemma.

Lemma 1.

If V is positive definite and $c_j = 0$ for some j, then $x(t,y) = 0$ for $t > t_j$.

Proof:

We have that $V\left(x\left(t_j, y\right)\right) = 0$ and, since $V(x) = 0$ implies $x = 0$, the result follows.

Lemma 2.

If V is dichotomic with respect to (1) and $x(t,y)$ is a solution of (1) defined for $0 \le t \le \phi_j(T)$, then $c_{j-1} \ge c_j$.

Proof:

If $t_j = \phi_{j-1}(T)$, then by the definition of c_{j-1}, we have $c_j = V\left(x\left(t_j, y\right)\right) \le c_{j-1}$. Now if $t_j > \phi_{j-1}(T)$, and note that $V\left(x(t,y)\right) < V\left(x\left(t_j,y\right)\right)$ for $\phi_{j-1}(T) \le t < \phi_j(T)$. By Mean Value Theorem there is $\tilde{t} \in \left(t, t_j\right)$ with $V^\Delta\left(x\left(\tilde{t},y\right)\right) = \left(V\left(x\left(t_j,y\right)\right) - V\left(x(t,y)\right)\right) \big/ \left(t_j - t\right)$, and this gives $V^\Delta\left(x\left(\tilde{t},y\right)\right) > 0$. Now $x\left(\tilde{t},y\right) \to x\left(t_j,y\right)$ as $t \to t_j$, which immediately means that $x\left(t_j,y\right) \in \bar{\Omega}_+^\Delta$. Since V is dichotomic $V\left(x\left(t_j,y\right)\right) \le V\left(x\left(\eta\left(t_j\right),y\right)\right)$, so that $c_j = V\left(x\left(t_j,y\right)\right) \le V\left(x\left(\eta\left(t_j\right),y\right)\right) \le c_{j-1}$.

Lemma 3.

If V is strictly dichotomic with respect to (1) and $x(t,y)$ is a solution of (1) defined for $0 \le t \le \phi_j(T)$ and $t_j > \phi_{j-1}(T)$, then $c_{j-1} > c_j$.

Proof:

Since $x(t_j, y) \in \Omega_+^\Delta$ implies $x(t_j, y) \in \Omega_-$, $c_j = V(x(t_j, y)) < V(x(\eta(t_j), y)) \le c_{j-1}$.

Theorem 3.

If V is strictly dichotomic with respect to (1) and $x(t,y)$ is a solution of (1) defined for $0 \le t \le \phi_j(T)$ and $c_{j-1} \ne 0$, then $c_{j-2} > c_j$.

Proof:

Under the above hypothesis, we know from Lemma 2 that $c_{j-2} \ge c_{j-1} \ge c_j$. Furthermore, from Lemma 3, we know that if $t_j > \phi_{j-1}(T)$, then $c_{j-1} > c_j$ so that $c_{j-2} > c_j$. In the case $t_j = \phi_{j-1}(T)$, we have either $c_{j-1} > c_j$ or $c_{j-1} = c_j$. In the first case we again have $c_{j-2} \ge c_{j-1} > c_j$. In the case that $c_{j-1} = c_j$, one notes that $\Omega_0(T) \cap \Omega_0^\Delta = \{0\}$ and $c_{j-1} \ne 0$, and sees that if one had $t_{j-1} = \phi_{j-2}(T)$, then one could not have $V^\Delta(x(t,y)) = 0$.

In fact, for $t_{j-1} = \phi_{j-2}(T)$, one could not have $V^\Delta(x(t,y)) < 0$, since that would immediately contradict with $t_{j-1} = \phi_{j-2}(T)$ Also, one could not have $V^\Delta(x(t,y)) > 0$, since that would contradict with $t_j = \phi_{j-1}(T)$. Then we conclude that $t_{j-1} > \phi_{j-2}(T)$ when $c_{j-1} = c_j$, so that $c_{j-2} > c_{j-1} = c_j$.

Theorem 4.

If V is a positive definite map that is dichotomic with respect to equation (1), then the trivial solution to system (1) is stable.

Proof:

Take $R > 0$ with $\Omega \subset \{|y| \le R\}$, and take any $r > 0$ with $\sup\{|x(t,y)| : |y| \le r, 0 \le t \le T\} \le R$. Set $\delta = \inf\{V(y) : r \le |y| \le R\}$, and note that $\delta > 0$. If $|y| \le R$ and $V(y) < \delta$, then $|y| < r$. Considering the continuity of V and the map $(t,y) \to x(t,y)$, we can take $d \in (0,r)$ such that for $0 \le t < T$ and $|y| \le d$, one has $V(x(t,y)) \le \delta/2$, $|x(t,y)| \le R$. For such t, y one has $|x(t,y)| \le r$.

Now take $j \ge 1$, and note that if $|x(t,y)| \le r$ for $\phi_{j-1}(T) \le t < \phi_j(T)$, then $|x(t,y)| \le R$ for $\phi_j(T) \le t < \phi_{j+1}(T)$, and since $c_1(y) < \delta$, we have $c_{j+1}(y) < \delta$. Thus $V(x(t,y)) \le \delta$ for $\phi_j(T) \le t < \phi_{j+1}(T)$, and we find that $|x(t,y)| \le r$ for $\phi_j(T) \le t < \phi_{j+1}(T)$. From this inductive argument on the index $j \ge 1$ we see that $|x(t,y)| \le r$ for all $t \ge 0$.

Theorem 5.

If V is a positive definite map that is strictly dichotomic with respect to equation (1), then the trivial solution to system (1) is asymptotically stable.

Proof:

Let V be a strictly dichotomic and take y having $|y| \le d$ with $d \in (0,r)$. By Theorem 4 we already know that $x(t,y)$ is stable and by Lemma 3 we have $c_j > c_{j+1}$. Thus we only need to show that $c_j \to 0$ as $j \to \infty$. Suppose there exists a positive constant $c < c_j$ such that $c_j \to c$ as $j \to \infty$. Since $c < d$ and the only point y with $|y| \le d$, where Ω_+^Δ and $\Omega_-(T)$ vanish simultaneously, is $y = 0$, the standing hypothesis guarantee the existence of a positive constant $h(c)$ such that either $\Omega_+^\Delta < -h(c)$ or $\Omega_-(T) < -h(c)$ for all y, $c < |y| < d$. As a consequence we must have

$$c_{j+1} - c_j < -h(c), \qquad j = 0,1,2,\ldots$$

which leads us to conclude that $c_j \to -\infty$, a contradiction. So, $c = 0$ and the theorem is proved.

It is clear from the Theorem 4 and Theorem 5 that we can obtain stability results for dynamic equations in the spirit of Lyapunov's direct method even when V^Δ is not negative semi-definite.

Example 1.

Consider the dynamic equation

$$x^{\Delta\Delta} + p^2 x = 0. \tag{2}$$

Rewrite equation (2) as,

$$x^\Delta = py$$

$$y^{\Delta} = -px$$

with $x(0) = x_0$ and $y(0) = y_0$. Take $V(x,y) = x^2 + \frac{1}{2}y^2$. Then V is positive definite. Calculating V^{Δ} by using product rule we get

$$\begin{aligned}
V^{\Delta}(t,x,y) &= \left(x^2 + \frac{1}{2}y^2\right)^{\Delta} \\
&= x(t)x^{\Delta}(t) + x^{\Delta}(t)x(\sigma(t)) + \frac{1}{2}\left(y^{\Delta}(t)y(\sigma(t)) + y^{\Delta}(t)y(t)\right) \\
&= x^{\Delta}(t)\left(2x(t) + \mu(t)x^{\Delta}(t)\right) + \frac{1}{2}y^{\Delta}(t)\left(2y(t) + \mu(t)y^{\Delta}(t)\right) \\
&= (py)\left(2x + \mu(t)(py)\right) + \frac{1}{2}(-px)\left(2y + \mu(t)(-px)\right) \\
&= 2pxy + \mu(t)\left(p^2y^2\right) - pxy + \mu(t)\left(\frac{1}{2}p^2x^2\right) \\
&= pxy + p^2\mu(t)\left(\frac{1}{2}x^2 + y^2\right).
\end{aligned}$$

Since $\mu(t) \geq 0$, $V^{\Delta}(t,x,y)$ is not negative semi definite, so V is not a Lyapunov function for this equation. The solution of the equation (2) is

$$x(t) = x_0 \cos_p(t,t_0) + y_0 \sin_p(t,t_0)$$

$$y(t) = -x_0 \sin_p(t,t_0) + y_0 \cos_p(t,t_0).$$

If we choose $T = a$ on time scale $\mathbb{T}$ with period $\sigma^2(a)$ *(i.e. let $T \in \mathbb{T} \cap [0,\infty)$ then a time scale $\mathbb{T}$ is periodic with period $\sigma^2(T)$ provided $t \in \mathbb{T}$ implies $t + \sigma^2(T) \in \mathbb{T}$)* then we obtain,

$$V(x(a), y(a)) - V(x_0, y_0) = 0.$$

So we have for $\Omega = B$, for instance,

$$\overline{\Omega}_{+}^{\Delta} = \{(x,y) \in \Omega : pxy \geq 0\}$$

$$\Omega_{-}^{0}(a) = B.$$

So that $\overline{\Omega}_{+}^{\Delta} \subset \Omega_{-}^{0}(a)$, and from Theorem 4, we can say that given equilibrium is stable.

3. Conclusion

Stability and instability for dynamic equations on time scales has been studied using Lyapunov functions. In addition to previous studies, using dichotomic and strictly dichotomic maps, we obtain stability results for dynamic equations on time scales in the spirit of Lyapunov's direct method even when V^{Δ} is not negative semi-definite. Investigating the stability of dynamic equations by using dichotomic maps is more useful for the choice of Lyapunov function V .

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REFERENCES

Bená, M.A. and Dos Reis, J.G. (1998). Some results on stability of retarded functional differential equations using dichotomic map techniques, *Positivity,* 2, N. 3, pp. 229-238.

Bohner, M. and Peterson, A. (2001). *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhauser, Boston.

Bohner, M. and Martynyuk, A.A. (2007). Elements of Lyapunov stability theory for dynamic equations on time scale, International Applied Mechanics, Vol. 43, No. 9, pp. 949-970.

Carvalho, L.A.V. (1991). *On the Stability of Discrete Equations and Ordinary Differential Equations*, In Delay Differential Equations and Dynamical Systems Lecture Notes in Mathematics, (S. Busenberg and M. Martelli, editors), Vol. 1475, pp. 88-97, Springer-Verlag, New York.

Carvalho, L.A.V. and Ferreira, R. R. (1988). On a new extension of Lyapunov's direct method to discrete equations, *Quart. Appl. Math.,* 66, pp. 779-788.

Carvalho, L.A.V. and Cooke, K.L. (1991). On dichotomic maps for a class of differential-difference equations, *Proc. Royal Soc. Edinburgh*, 117A, pp. 317-328.

Carvalho, L.A.V. and Marconato, S.A.S. (1997). On dichotomic maps for differential equations with piecewise continuous argument, *Communications in Applied Analysis,* 1, N.1, pp. 103-112.

Guseinov, G. Sh. (2003). Integration on time scales, *Journal of Mathematical Analysis and Applications,* 285, pp. 107–127.

Hoffacker, J. and Tisdell, C. C. (2005). Stability and instability for dynamic equations on time scales, Computers and Mathematics with Applications, 49, pp. 1327-1334.

Kaymakcalan, B. (1992). Lyapunov stability theory for dynamic systems on time scales, *J. Appl. Math. Stochastic. Anal.*, 5(5), pp. 275–281.

LaSalle, J.P. (1976). *Stability of Dynamical Systems*, SIAM, Philadelphia.

Marconato, S. A. S. (2005). The relationship between differential equations with piecewise constant argument and the associated discrete equations via dichotomic maps, Dynamics of Continuous Discrete and Impulsive Systems, 12, pp. 755-768.