Introducing an Efficient Modification of the Variational Iteration Method by Using Chebyshev Polynomials

M. M. Khader
Department of Mathematics
Benha University
Benha, Egypt
mohamedmbd@yahoo.com

Abstract

In this article an efficient modification of the variational iteration method (VIM) is presented using Chebyshev polynomials. Special attention is given to study the convergence of the proposed method. The new modification is tested for some examples to demonstrate reliability and efficiency of the proposed method. A comparison of our numerical results those of the conventional numerical method, the fourth-order Runge-Kutta method (RK4) are given. The comparison shows that the solution using our modification is fast-convergent and is in excellent conformance with the exact solution. Finally, we conclude that the proposed method can be applied to a large class of linear and non-linear differential equations.

1. Introduction

Over the last decade several analytical and approximate methods have been developed to solve the nonlinear differential equations. Among them is the variational iteration method proposed by He (1999) which is a modification of the general Lagrange multiplier method. This method is based on the use of restricted variations and correction functional which has found an extensive application in the solution of nonlinear differential equations (Abdou and Soliman (2005), He and Xu-Hong (2006), Sweilam and Khader (2010), Sweilam et al. (2007)). This method neither

requires the presence of small parameters in the differential equation, and nor the differentiability of the nonlinearities with respect to the dependent variable and its derivatives. The technique provides a sequence of functions which converges to the exact solution of the problem and the procedure represents a powerful tool for solving various kinds of problems; for example, delay differential equations in He (1997), the one dimensional system of nonlinear equations in thermo-elasticity Sweilam and Khader (2007) and the two dimensional Maxwell equations Sweilam et al. (2010).

This technique solves the problem without the need to discretize the variables, and therefore, in some problems, it is not affected by computation round off errors and does not require large computer memory and time. The proposed scheme provides the solution of the problem in a closed form while the mesh point techniques, such as the finite difference method Chang et al. (1999) provides the approximation at mesh points only.

The VIM, does have some drawbacks; for example, this method is invalid or slowly convergent, especially, in problems which are modeled by differential equations with non-homogeneous terms. So, the main aim of this paper is to introduce a new modification to the method. The proposed modification involves the use of Chebyshev polynomials. It overcomes the above drawbacks and increases the rate of convergence.

Our paper is organized as follows: section 2 is assigned to the analysis of the standard VIM. In section 3, the convergence study of the proposed method is given. In section 4, some test problems are solved by the modified VIM, to illustrate the efficiency of the proposed method. The conclusions will appear in section 5.

2.   Analysis of the Variational Iteration Method

To illustrate the analysis of VIM, consider the following nonlinear differential equation:

$$Lu + Ru + N(u) = 0, \tag{1}$$

with specified initial conditions, where L and R are linear bounded operators i.e., it is possible to find numbers $m_1, m_2 > 0$ such that $\|Lu\| \le m_1 \|u\|$, $\|Ru\| \le m_2 \|u\|$. The nonlinear term $N(u)$ is Lipschitz continuous with $|N(u) - N(v)| < m\,|u - v|, \forall\, t \in J = [\,0,T]$, for constant $m > 0$.

The VIM makes it possible to write the solution of Equation (1) in the following iteration formula:

$$u_p = u_{P\text{-}1} + \int_0^t \lambda(\tau)\,[\,Lu_{p\text{-}1} + R\tilde{u}_{p\text{-}1} + N(\tilde{u}_{p\text{-}1})]d\tau, \; p \ge 1. \tag{2}$$

It is obvious that the successive approximations u_p, $p \ge 1$ (the subscript p denotes the pth order approximation), can be established by determining the general Lagrange multiplier, λ, which

can be identified optimally via the variational theory. The function $\tilde{u}_p$ is a restricted variation, which means that $\delta \tilde{u}_p = 0$ Elsgolts (1997). Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations u_p, $p \geq 1$, of the solution u is readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. If λ determined, then several approximations u_p, $p \geq 1$, follow immediately. Consequently, the exact solution may be obtained by using:

$$u = \lim_{p \to \infty} u_p. \tag{3}$$

Now, to illustrate how to find the value of the Lagrange multiplier λ, we will consider the following case, which depends on the order of the operator L in Equation (1), we study the case of the operator $L = \dfrac{d}{dt}$ (without loose of generality).

Making the above correction functional stationary, and noticing that $\delta \tilde{u}_p = 0$, we obtain

$$\delta u_p = \delta u_{p-1} + \delta \int_0^t \lambda(\tau) \, [\, Lu_{p-1} + R\tilde{u}_{p-1} + N(\tilde{u}_{p-1}) \,] d\tau$$

$$= \delta u_{p-1} + [\lambda(\tau) \, u_{p-1}]_{\tau = t} - \int_0^t \dot{\lambda}(\tau) \, [\delta u_{p-1}] d\tau = 0,$$

where $\delta \tilde{u}_p$ is considered as a restricted variation i.e., $\delta \tilde{u}_p = 0$, yields the following stationary conditions:

$$\dot{\lambda}(\tau) = 0, \qquad 1 + \lambda(\tau)|_{\tau = t} = 0. \tag{4}$$

Equation (4) is called Lagrange-Euler equation with its boundary condition. The Lagrange multiplier can be identified by solving this equation as: $\lambda(\tau) = -1$.

Now, the following variational iteration formula can be obtained:

$$u_p = u_{p-1} - \int_0^t [\, Lu_{p-1} + Ru_{p-1} + N(u_{p-1})] d\tau, \qquad p \geq 1. \tag{5}$$

We start with an initial approximation, and by using the above iteration formula (5), we can obtain directly the other components of the solution.

3. Convergence Analysis of VIM

In this section, the sufficient conditions are presented to guarantee the convergence of VIM, when applied to solve the differential equations, where the main point is that we prove the convergence of the recurrence sequence, which is generated by using VIM.

Lemma 1:

Let $A:U \to V$ be a bounded linear operator and let $\{u_p\}$ be a convergent sequence in U with limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ in V.

Proof:

Since

$$\left\| Au_p - Au \right\|_V = \left\| A(u_p - u) \right\|_V \leq \left\| A \right\| \left\| u_p - u \right\|_U ,$$

hence,

$$\lim_{p \to \infty} \left\| Au_p - Au \right\|_V \leq \left\| A \right\| \lim_{p \to \infty} \left\| u_p - u \right\|_U = 0,$$

implies that $A(u_p) \to A(u)$.

Now, to prove the convergence of the variational iteration method, we will rewrite Equation (5) in the operator form as follows Elsgolts (1977)**:**

$$u_p = A \, [u_{p-1}] , \tag{6}$$

where the operator A takes the following form:

$$A \, [u] = -\int_0^t [\, Lu + Ru + N(u) \,] d\tau. \tag{7}$$

Theorem 1:

Assume that X be a Banach space and $A:X \to X$ is a nonlinear mapping, and suppose that:

$$\left\| A[u] - A[v] \right\| \leq \gamma \left\| u - v \right\|, \qquad \forall \, u,v \in X, \tag{8}$$

for some constant $0 < \gamma < 1$, where $\gamma = (m + m_1 + m_2)T$. Then, A has a unique fixed point. Furthermore, the sequence (6) using VIM with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A and:

$$\left\| u_p - u_q \right\| \leq [\frac{\gamma^q}{1-\gamma}] \left\| u_1 - u_0 \right\|. \tag{9}$$

Proof:

Denoting $(C[J], \|\cdot\|)$ the Banach space of all continuous functions on J with the norm defined by: $\|f(t)\| = \max\limits_{t \in J} |f(t)|$.

We are going to prove that the sequence $\{u_p\}$ is a Cauchy sequence in this Banach space:

$$\left\| u_p - u_q \right\| = \max_{t \in J} \left| u_p - u_q \right|$$

$$= \max_{t \in J} \left| -\int_0^t [L(u_{p-1} - u_{q-1}) + R(u_{p-1} - u_{q-1}) + N(u_{p-1}) - N(u_{q-1})] \, d\tau \right|$$

$$\leq \max_{t \in J} \int_0^t [\left| L(u_{p-1} - u_{q-1}) \right| + \left| R(u_{p-1} - u_{q-1}) \right| + \left| N(u_{p-1}) - N(u_{q-1}) \right|] \, d\tau$$

$$\leq \max_{t \in J} \int_0^t (m_1 + m_2 + m) \left| u_{p-1} - u_{q-1} \right| \, d\tau$$

$$\leq \gamma \left\| u_{p-1} - u_{q-1} \right\|.$$

Let, $p = q + 1$. Then,

$$\left\| u_{q+1} - u_q \right\| \leq \gamma \left\| u_q - u_{q-1} \right\| \leq \gamma^2 \left\| u_{q-1} - u_{q-2} \right\| \leq \ldots \leq \gamma^q \left\| u_1 - u_0 \right\|.$$

From the triangle inequality we have:

$$\left\| u_p - u_q \right\| \leq \left\| u_{q+1} - u_q \right\| + \left\| u_{q+2} - u_{q+1} \right\| + \ldots + \left\| u_p - u_{p-1} \right\|$$

$$\leq [\gamma^q + \gamma^{q+1} + \ldots + \gamma^{p-1}] \left\| u_1 - u_0 \right\|$$

$$\leq \gamma^q [1 + \gamma + \gamma^2 + \ldots + \gamma^{p-q-1}] \left\| u_1 - u_0 \right\|$$

$$\leq \gamma^q [\frac{1 - \gamma^{p-q-1}}{1-\gamma}] \left\| u_1 - u_0 \right\|.$$

Since $0 < \gamma < 1$ so, $1 - \gamma^{p-q} < 1$ then:

$$\left\| u_p - u_q \right\| \leq [\; \frac{\gamma^q}{1-\gamma} \;] \left\| u_1 - u_0 \right\|.$$

But $\left\| u_1 - u_0 \right\| < \infty$ so, as $q \to \infty$ then $\left\| u_p - u_q \right\| \to 0$. We conclude that $\{u_p\}$ is a Cauchy sequence in $C[J]$ so, the sequence converges and the proof is complete.

In the following theorem we introduce an estimation of the absolute error of the approximate solution of problem (1).

Theorem 2:

The maximum absolute error of the approximate solution u_p to problem (1) is estimated to be:

$$\max_{t \in J} \left| u_{exact} - u_p \right| < \beta, \tag{10}$$

where

$$\beta = \frac{\gamma^q T [(m_1 + m_2) \left\| u_0 \right\| + k]}{1 - \gamma}, \qquad k = \max_{t \in J} \left| N(u_0) \right|.$$

Proof:

From Theorem 1 inequality (9) we have:

$$\left\| u_p - u_q \right\| \leq \frac{\gamma^q}{1-\gamma} \left\| u_1 - u_0 \right\|,$$

as $p \to \infty$ then $u_p \to u_{exact}$ and:

$$\left\| u_1 - u_0 \right\| = \max_{t \in J} \left| - \int_0^t [Lu_0 + Ru_0 + N(u_0)] \, d\tau \right|$$

$$\leq \max_{t \in J} \int_0^t [\left| Lu_0 \right| + \left| Ru_0 \right| + \left| N(u_0) \right|] \, d\tau$$

$$\leq T [(m_1 + m_2) \left\| u_0 \right\| + k],$$

so, the maximum absolute error in the interval J is:

$$\left\| u_{exact} - u_p \right\| = \max_{t \in J} \left| u_{exact} - u_p \right| < \beta.$$

This completes the proof.

Our main goal in this paper is to concern with the implementation of VIM and its modification which have efficiently used to solve the ordinary differential equations Hosseini (2005). To achieve this goal, at the beginning of implementation of VIM, we use the orthogonal Chebyshev polynomials to expand the functions in the non-homogeneous term in the differential equation Bell (1967).

4.   Solution Procedure Using the Modified VIM

In this section, an efficient modification of VIM is presented by using Chebyshev polynomials. The well known Chebyshev polynomials Bell (1967) are defined on the interval *[-1,1]* and can be determined with the aid of the following recurrence formula:

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \quad n = 1,2,\dots .$$

The first three Chebyshev polynomials are $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$.

In this paper, we suggest that $f(x)$ be expressed in Chebyshev series:

$$f(x) = \sum_{k=0}^{\infty} c_k T_k(x). \tag{11}$$

Theorem 3: Chebyshev Truncation Theorem

The error in approximating $f(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If

$$f_m(x) \cong \sum_{k=0}^{m} c_k T_k(x), \tag{12}$$

then, for all $f(x)$, all m, and all $x \in [-1,1]$, we have

$$E_T(m) \equiv |f(x) - f_m(x)| \le \sum_{k=m+1}^{\infty} |c_k| \tag{13}$$

Proof:

The Chebyshev polynomials are bounded by one, that is, $|T_k(x)| \leq 1$ for all $x \in [-1,1]$ and for all k. This implies that the k-th term is bounded by $|c_k|$. Subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds gives the theorem.

Now, in order to use these polynomials on the interval $x \in [0,1]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = 2x - 1$. Let the shifted Chebyshev polynomials $T_n(2x - 1)$ be denoted by $P_n(x)$. Then $P_n(x)$ can be obtained as follows:

$$P_{n+1}(x) = 2(2x - 1)P_n(x) - P_{n-1}(x), \qquad n = 1,2,\dots . \tag{14}$$

Now, we use the shifted Chebyshev expansion to expand $f(x)$ in the following form:

$$f(x) \cong f_m(x) = \sum_{k=0}^{m} c_k P_k(x), \tag{15}$$

where the constants coefficients c_k, $k = 0,1,2,\dots,m$, by using the orthogonal property are defined by:

$$c_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(0.5x + 0.5)T_0(x)}{\sqrt{1 - x^2}} dx, \qquad c_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(0.5x + 0.5)T_k(x)}{\sqrt{1 - x^2}} dx. \tag{16}$$

Now, the proposed modification will implement to solve the following two initial nonlinear ordinary differential equations.

Model Problem 1:

Consider the following nonlinear ordinary differential equation:

$$u'' + xu' + x^2 u^3 = f(x) = (2 + 6x^2)e^{x^2} + x^2 e^{3x^2}, \qquad x \in [0,1], \tag{17}$$

subject to the following initial conditions:

$$u(0) = 1, \qquad u'(0) = 0, \tag{18}$$

with the exact solution $u(x) = e^{x^2}$.

The procedure of the solution follows the following two steps:

Step 1. Expand the function $f(x)$ using Chebyshev polynomials:

Using the above consideration, the function $f(x)$ can be approximated by eight terms ($m=8$) of the expansion (15) as follows:

$$f_C(x) \cong 2.00232 - 0.358488\,x + 18.0328\,x^2 - 86.4534\,x^3 + 416.556\,x^4 - 1042.66\,x^5$$
$$+ 1502.72x^6 - 1134.64x^7 + 366.624x^8.$$

Step 2. Implementation of VIM:

The VIM gives the possibility to write the solution of (17) with the aid of the correction functional:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\,[\,u_{n\tau\tau} + \tau\,\tilde{u}_{n\tau} + \tau^2 \tilde{u}_n^3 - f(\tau)]\,d\tau \quad n \geq 0, \tag{19}$$

where λ is the general Lagrange multiplier. Making the above correction functional stationary:

$$\delta\,u_{n+1}(x) = \delta\delta_n(x) + \delta \int_0^x \lambda(\tau)\,[\,u_{n\tau\tau} + \tau\,\tilde{u}_{n\tau} + \tau^2 \tilde{u}_n^3 - f(\tau)]\,d\tau$$
$$= \delta u_n(x) + [\lambda(\tau)\delta u'_n - \dot{\lambda}(\tau)\delta u_n]_{\tau=x} + \int_0^x \lambda''(\tau)\,[\,\delta u_n(\tau)]\,d\tau = 0, \tag{20}$$

where $\delta\,\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\,\tilde{u}_n = 0$, yields the following stationary conditions (by comparison the two sides in the above equation):

$$\lambda''(\tau) = 0, \qquad \lambda(\tau)|_{\tau=x} = 0, \qquad 1 - \lambda(\tau)|_{\tau=x} = 0. \tag{21}$$

The equations in (21) are called Lagrange-Euler equation and the natural boundary condition respectively, the Lagrange multiplier, therefore

$$\lambda(\tau) = \tau - x. \tag{22}$$

Now, by substituting from (22) in (19), the following variational iteration formula can be obtained:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\tau - x)\,[\,u_{n\tau\tau} + \tau\,u_{n\tau} + \tau^2 u_n^3 - f(\tau)]\,d\tau, \quad n \geq 0. \tag{23}$$

We start with initial approximation $u_0(x)$, and by using the above iteration formula (23), we can directly obtain the components of the solution.

Now, the first four components of the solution $u(x)$ by using (23) of Equation (17) are:

$u_0(x) = 1,$

$u_1(x) = 1 + 1.0011x^2 - 0.0597x^3 + 1.4194x^4 - 4.3227x^5 + 13.8852x^6 - 24.8252x^7$

$\qquad + 26.8343x^8 - 15.7589x^9 + 4.0736x^{10} + ...,$

$u_2(x) = 1 + 1.0012x^2 - 0.0597x^3 + 1.2525x^4 - 4.3137x^5 + 13.5958x^6 - 24.3064x^7$

$\qquad + 25.2169x^8 - 13.1602x^9 + 1.1195x^{10} + ...,$

$u_3(x) = 1 + 1.0012x^2 - 0.0597x^3 + 1.2525x^4 - 4.3137x^5 + 13.6181x^6 - 24.3074x^7$

$\qquad + 25.2568x^8 - 13.211x^9 + 1.2840x^{10} + 1.9809x^{11} - 1.3526x^{12} + ...,$

$u_4(x) = 1 + 1.0012x^2 - 0.0597x^3 + 1.2525x^4 - 4.3137x^5 + 13.6181x^6 - 24.3074x^7$

$\qquad + 25.2544x^8 - 13.2109x^9 + 1.2798x^{10} + 1.9851x^{11} - 1.367x^{12} + ...$

Now, also to perform VIM, we can expand the function $f(x)$ using Taylor series at the point $x = a$:

$$f(x) \cong \sum_{k=0}^{m} \frac{f^{(k)}(a)}{k!}(x-a)^k, \qquad\qquad (24)$$

for an arbitrary natural number m.

If we expand the function $f(x)$ by the Taylor series (24) about the point $x = 0$ with eight terms, we have:

$$f_T(x) \cong 2 + 9x^2 + 10x^4 + 7.83x^6 + 5.58333x^8 + O(x^9).$$

Now, the first four components of the solution $u(x)$ of Eq.(17) by using (23) are:

$u_0(x) = 1,$

$u_1(x) = 1 + 1.0011x^2 - 0.0597x^3 + 1.4194x^4 - 4.3227x^5 + 13.8852x^6 - 24.8252x^7$

$\qquad + 26.8343x^8 - 15.7589x^9 + 4.0736x^{10} + ...,$

$u_2(x) = 1 + 1.0012x^2 - 0.0597x^3 + 1.2525x^4 - 4.3137x^5 + 13.5958x^6 - 24.3064x^7$

$\qquad + 25.2169x^8 - 13.1602x^9 + 1.1195x^{10} + ...,$

$u_3(x) = 1 + 1.0012x^2 - 0.0597x^3 + 1.2525x^4 - 4.3137x^5 + 13.6181x^6 - 24.3074x^7$

$\qquad + 25.2568x^8 - 13.211x^9 + 1.2840x^{10} + 1.9809x^{11} - 1.3526x^{12} + ...,$

$$u_4(x) = 1 + 1.0012x^2 - 0.0597x^3 + 1.2525x^4 - 4.3137x^5 + 13.6181x^6 - 24.3074x^7$$

$$+ 25.2568x^8 - 13.2109x^9 + 1.2798x^{10} + 1.9851x^{11} - 1.367x^{12} + ...,$$

Also, to solve Equation (17) by the numerical method, fourth-order Runge-Kutta method, we reduce this equation to the following system of ordinary differential equations:

$$u'(x) = v(x), \tag{25}$$

$$v'(x) = -xv(x) - x^2 u^3(x) + f(x), \tag{26}$$

subject to the following initial conditions:

$$u(0) = 1, \qquad v(0) = 0. \tag{27}$$

The absolute error between the function $f(x)$ and its approximation by using the Taylor expansion and the Chebyshev expansion are presented in figure 1.
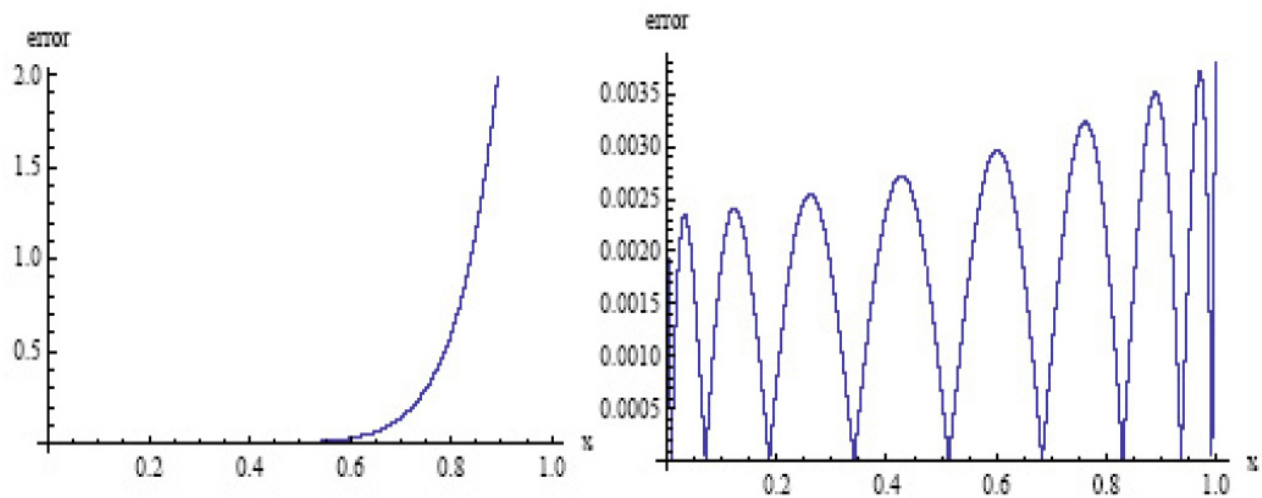
Figure 1: The absolute error $|f(x)-f_T(x)|$ (Left) and $|f(x)-f_C(x)|$ (Right).

The absolute error between the exact solution $u(x)$ and the approximate solution $u_c(x) = u_4(x)$ (after four iterations) and using the shifted Chebyshev expansion for $f(x)$ with $m=8$ is presented in figure 2(Right). Also, the absolute error between the exact solution $u(x)$ and the approximate solution $u_T(x) = u_4(x)$ (after four iterations) using the Taylor expansion for $f(x)$ with eight terms is presented in figure 2(Left).
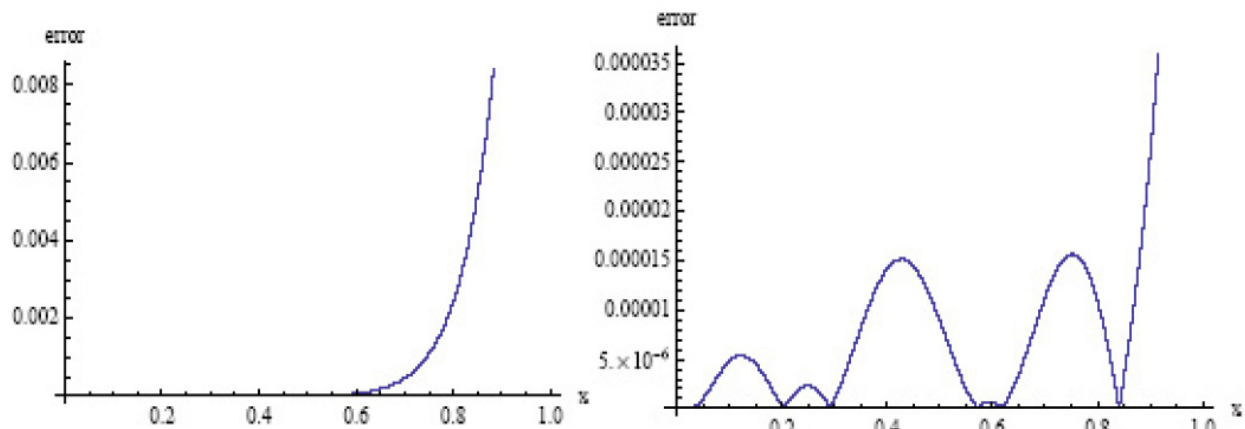
Figure 2. The absolute error $|u(x) - u_T(x)|$ (Left) and $|u(x) - u_C(x)|$ (Right).

But, the figure 3 presents a comparison between the exact solution $u(x)$, with the numerical method, fourth-order Runge-Kutta u_{RK4} and the approximate solution of our proposed method $u_C(x)$. From this figure, we can see that the two methods are in excellent agreement with the exact solution.
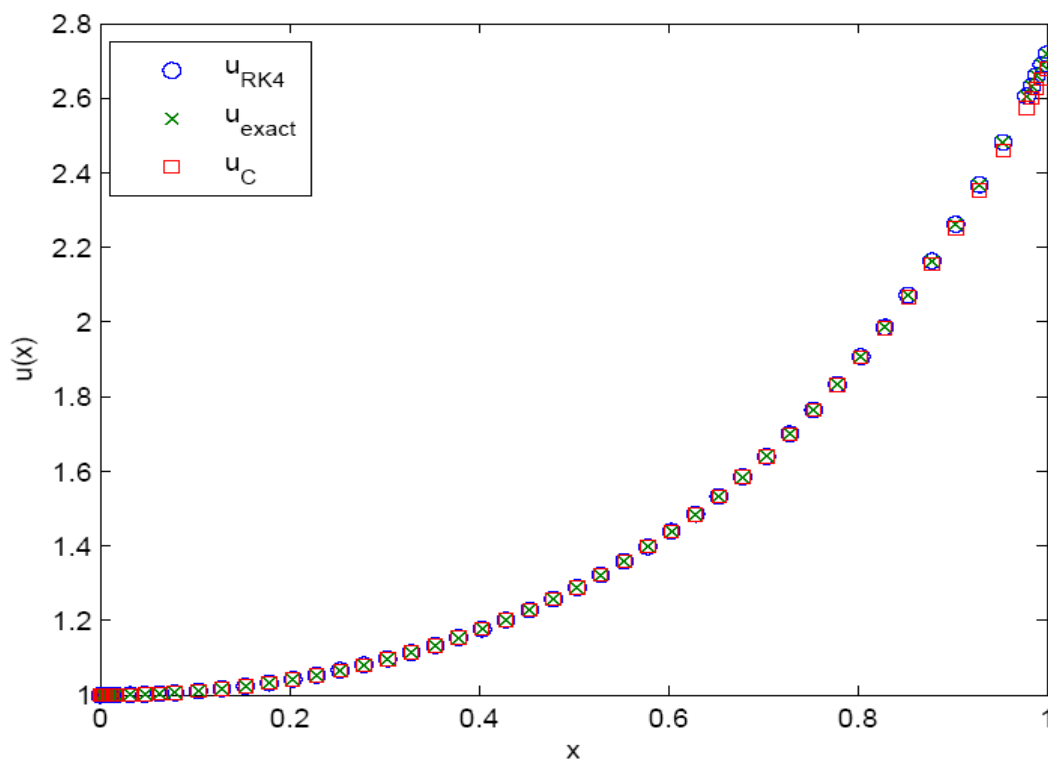

Figure 3. Comparison between the exact solution u_{exact}, u_{RK4} and the solution of our proposed method $u_C(x)$.

Model Problem 2.

Consider the following nonlinear ordinary differential equation:

$$u'' + u\,u' = f(x) = x\,\sin(2x^2) - 4x^2\sin(x^2) + 2\,\cos(x^2), \qquad x \in [0,\,1], \qquad (28)$$

subject to the following initial conditions:

$$u(0) = 0, \qquad u'(0) = 0, \qquad (29)$$

with the exact solution $u(x) = \sin(x^2)$.

The procedure of the solution follows the following two steps:

Step 1.

Expand the function $f(x)$ using Chebyshev polynomials:
Using the above consideration, the function $f(x)$ can be approximated by eight terms ($m = 8$) of the expansion (15) as follows:

$$f_C(x) \cong 2 - 0.0003x + 0.008x^2 + 1.892x^3 - 4.308x^4 - 2.3986x^5 + 4.6816x^6$$
$$- 6.276x^7 + 3.025x^8.$$

Step 2.

Implementation of VIM:

According to VIM, and the same procedure in section 2, we can construct the following iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\tau - x)\,[\,u_{n\tau\tau} + u_n u_{n\tau} - f(\tau)]d\tau, \qquad n \geq 0. \qquad (30)$$

Therefore, the first four components of the solution $u(x)$ of Eq.(28) by using (30) are:

$$u_0(x) = 0,$$

$$u_1(x) = x^2 + 0.1x^5 - 0.16667x^6 - 0.0185185x^9 + 0.0083333x^{10} + ...,$$

$$u_2(x) = x^2 - 0.16667x^6 - 0.012x^8 + 0.008333x^{10} - 0.0004545x^{11} + 0.002932x^{12} + ...,$$

$$u_3(x) = x^2 - 0.1667x^6 + 0.0083x^{10} + 0.0011x^{11} - 0.0017x^{13} + 0.00003x^{14} - 0.0003x^{15} + ...,$$

$$u_4(x) = x^2 - 0.1667x^6 + 0.0083x^{10} - 0.0017x^{13} - 0.00008x^{14} + 0.0001x^{16} + 0.00008x^{17} + ...$$

Now, if we expand the function $f(x)$ by the Taylor series (24), with eight terms we have:

$$f_T(x) \cong 2 + 2x^3 - 5x^4 - 1.33333x^7 + 0.75x^8 + O(x^9).$$

So, the first four components of the solution $u(x)$ of Eq.(28) by using (30) are:

$$u_0(x) = 0,$$

$$u_1(x) = x^2 - 0.00004x^3 + 0.0007x^4 - 0.0946x^5 - 0.1436x^6 - 0.0571x^7 + 0.00836x^8 + ...,$$

$$u_2(x) = x^2 - 0.00004x^3 + 0.0007x^4 - 0.0054x^5 - 0.1436x^6 - 0.0572x^7 + 0.0718x^8 + ...,$$

$$u_3(x) = x^2 - 0.00004x^3 + 0.0007x^4 - 0.0054x^5 - 0.1436x^6 - 0.0572x^7 + 0.0843x^8 + ...,$$

$$u_4(x) = x^2 - 0.00004x^3 + 0.0007x^4 - 0.0055x^5 - 0.1436x^6 - 0.0572x^7 + 0.0843x^8 + ...,$$

To solve Equation (28) by the numerical method, fourth-order Runge-Kutta method, we reduce this equation to the following system of ODEs:

$$u'(x) = v(x), \tag{31}$$

$$v'(x) = -u(x)v(x) + f(x), \tag{32}$$

subject to the following initial conditions:

$$u(0) = 0, \qquad v(0) = 0. \tag{33}$$

Figure 4 presents the absolute error between the function $f(x)$ and its approximation by using the Taylor expansion (Left) and the Chebyshev expansion (Right).
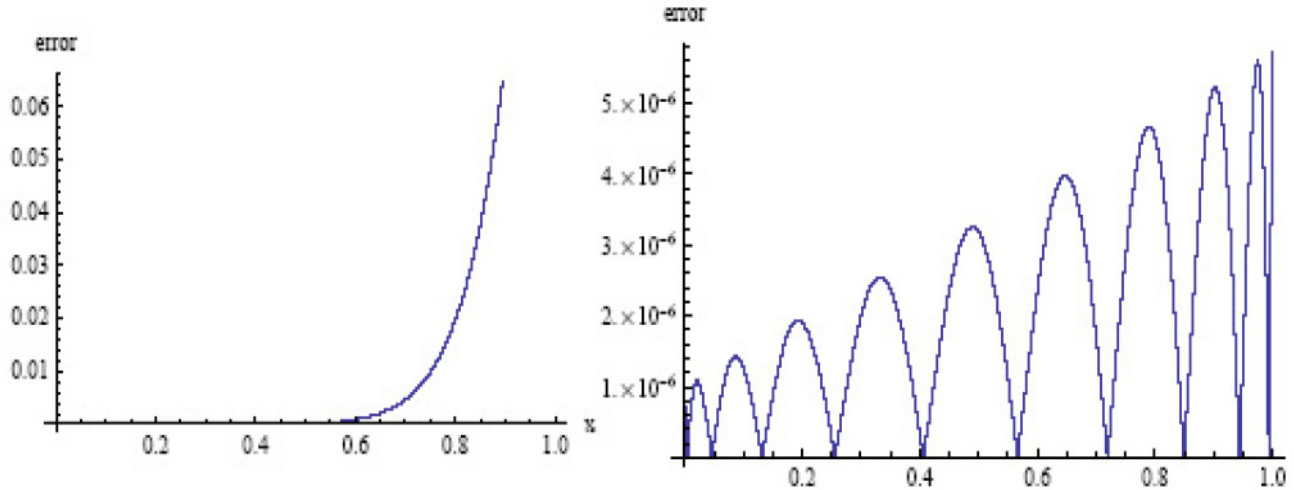
Figure 4. The absolute error: $\left|f(x)\text{-}f_T(x)\right|$ (Left) and $\left|f(x)\text{-}f_C(x)\right|$ (Right).
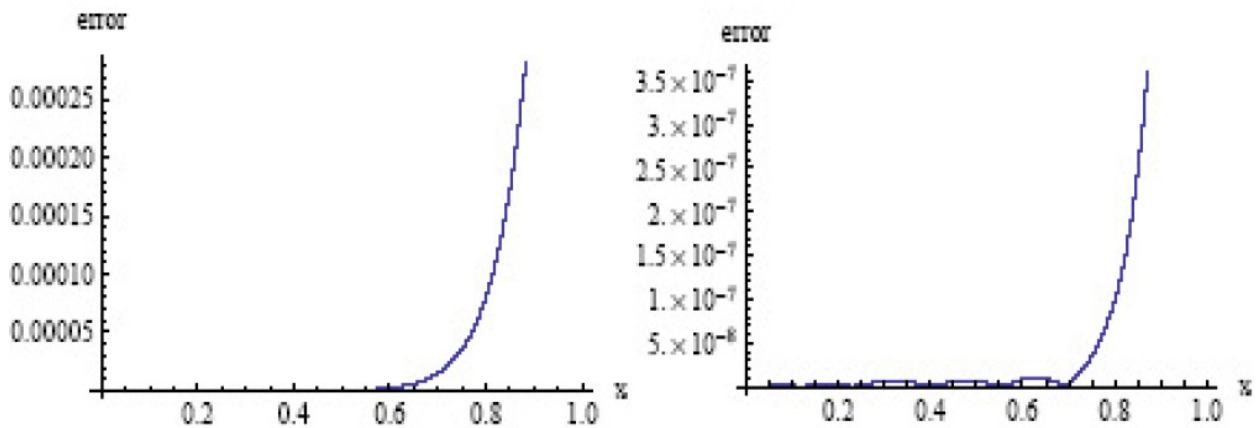
Figure 5. The absolute error: $\left|u(x)\text{-}u_T(x)\right|$ (Left) and $\left|u(x)\text{-}u_C(x)\right|$ (Right).

The absolute error between the exact solution $u(x)$ and the approximate solution $u_C(x) = u_4(x)$ (after four iterations) and using the Chebyshev expansion for $f(x)$ with $m=8$ is presented in figure 5 (Right). Also, the absolute error between the exact solution $u(x)$ and the approximate solution $u_T(x) = u_4(x)$ (after four iterations) using the Taylor expansion for $f(x)$ with eight terms is presented in figure 5(Left).

But, the figure 6 presents a comparison between exact solution u_{exact}, with the numerical method, fourth-order Runge-Kutta u_{RK4} and the approximate solution of our proposed method $u_C(x)$. From this figure, we can conclude that the proposed method is in excellent agreement with the exact solution.
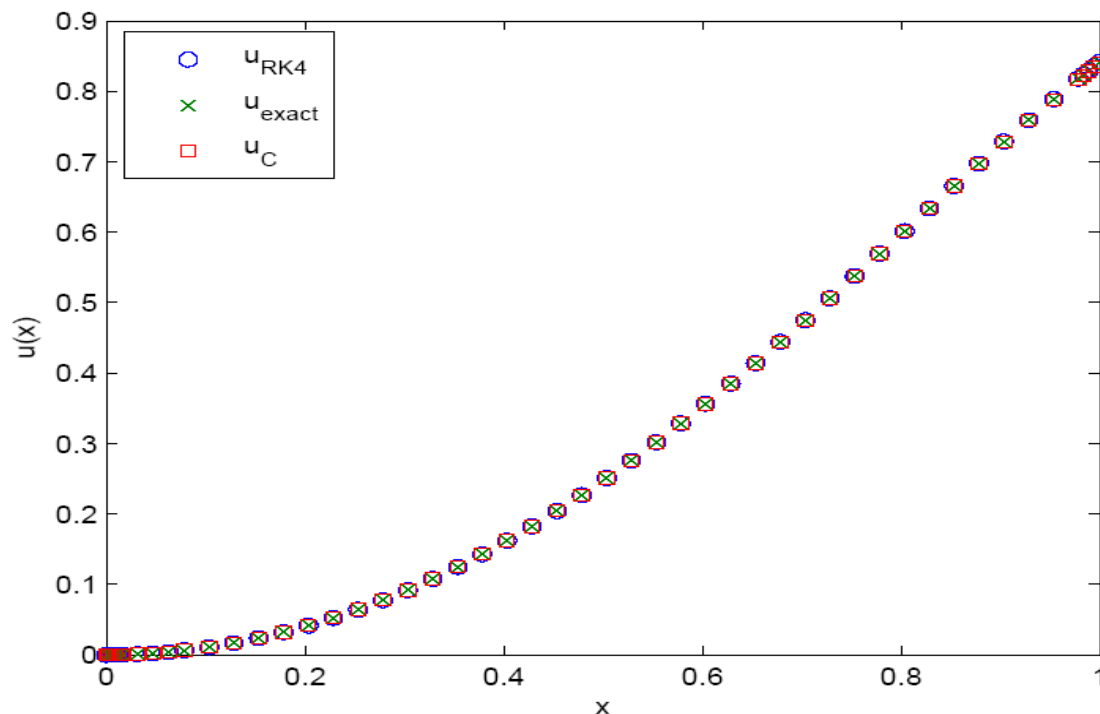
Figure 6. Comparison between the exact solution u_{exact}, u_{RK4} and the solution of our
proposed method $u_C(x)$.

5. Conclusion

An efficient modification of VIM is presented using Chebyshev polynomials. Parallel solutions
with the proposed method and by the numerical method, the fourth-order Runge-Kutta method
(RK4) have been presented. We choose the conventional RK4 as our benchmark, as it is widely
accepted and used. These show that the proposed method can be applied to linear and non-linear
models that are represented by differential equations. Likewise, the obtained results demonstrate
the reliability and efficiency of the proposed method and the faster convergence of the method.
From the resulting numerical solution we also see that the results from the proposed method are
in an excellent agreement with the exact solution. An interesting point about VIM is that only
little iteration or, even in some special cases, one iteration leads to exact solutions or solutions
with high accuracy. Finally, all the numerical results were obtained by using Mathematica
programming version 6.

REFERENCES

Abdou, M. A. and Soliman, A. A. (2005). New applications of variational iteration method,
     Physica D, Vol. **211**, pp. 1-8.
Bell, W. W. (1967). Special Functions for Scientists and Engineers, New York Toronto
     Melbourne.
Chang, Q., Erhui, J. and Sun, W. (1999). Difference schemes for solving the generalized non-
     linear Schrödinger equation, J. Comp. Physics, Vol. **148**, pp. 397-415.

Elsgolts, L. (1977). Differential Equations and the Calculus of Variations, translated from the Russian by G. Yankovsky, Mir, Moscow.

He, J. H. (1999). Variational iteration method-a kind of non-linear analytical technique: some examples, International Journal of Non-Linear Mechanics, Vol, **34**, pp. 699-708.

He, J. H. and Xu-Hong W. (2006). Construction of solitary solution and compacton-like solution by variational iteration method, Chaos, Solitons and Fractals, Vol. **29**, pp. 108-113.

He, J. H. (1997). Variational iteration method for delay differential equations, Comm. Non-Linear Sci. Numer, Simulation, Vol. **2**, pp. 235-236.

Hosseini, M. M. (2005). Numerical solution of ordinary differential equations with impulse solution, Appl. Math. Comput., Vol. **163**, pp.373-381.

Sweilam, N. H. and Khader, M. M. (2007). Variational iteration method for one dimensional nonlinear thermo-elasticity, Chaos, Solitons and Fractals, Vol. **32**, pp.145-149.

Sweilam, N. H. and Khader, M. M. (2010). On the convergence of VIM for nonlinear coupled system of partial differential equations, Int. J. of Computer Maths., Vol. **87(5)**, pp. 1120-1130.

Sweilam, N. H., Khader, M. M. and Al-Bar, R. F. (2007). Numerical studies for a multi-order fractional differential equation, Physics Letters A, Vol. **371**, pp. 26-33.

Sweilam, N. H., Khader, M. M. and Mohamed, F. T. (2010). On the numerical solutions of two dimensional Maxwell's equations, Studies in Nonlinear Sciences, Vol. **1(3)**, pp. 82-88.