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Applying Differential Transform Method to Nonlinear Partial Differential Equations: A Modified Approach

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Abstract

This paper proposes another use of the Differential transform method (DTM) in obtaining approximate solutions to nonlinear partial differential equations (PDEs). The idea here is that a PDE can be converted to an ordinary differential equation (ODE) upon using a wave variable, then applying the DTM to the resulting ODE. Three equations, namely, Benjamin-Bona-Mahony (BBM), Cahn-Hilliard equation and Gardner equation are considered in this study. The proposed method reduces the size of the numerical computations and use less rules than the usual DTM method used for multi-dimensional PDEs. The results show that this new approach gives very accurate solutions.

Keywords: Wave variables, Differential transform method, Nonlinear PDEs

MSC 2010: 34A 45, 65L15, 65L 20

1. Introduction

In this paper, we study the solutions of the nonlinear Benjamin-Bona-Mahony:

$$u_t = u_{xxt} - u_x - uu_x, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = \operatorname{sech}^2(\frac{x}{4}). \tag{1.2}$$

The Cahn-Hilliard equation:

$$u_t = u_{xx} - u^3 + u, \tag{1.3}$$

subject to the initial condition

$$u(x,0) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}. \tag{1.4}$$

The Gardner equation:

$$u_t = 6u^2 u_x + 6u + u_{xxx}, \tag{1.5}$$

subject to the initial condition

$$u(x,0) = -\frac{1}{2}\left(1 - \tanh(\frac{x}{2})\right). \tag{1.6}$$

Wazwaz (2005 a,b) derived a variety of exact traveling wave solutions of distinct physical structures for the BBM equation, where the Tanh and the sine-cosine methods are used. Also Wazwaz (2005 c) is devoted to analyzing the physical structures of the nonlinear dispersive variants of the BBM equation, where new exact solutions with compact and noncompact structures for BBM are derived. Al-Khaled (2005) applied the decomposition method to obtain explicit and numerical solutions of different types of generalized BBM. Finally, Alquran and Al-Khaled (2011) obtained Sinc and solitary wave solutions to the generalized BBMB equation by the Sinc-Galerkin approximations and Tanh method.

Furihata (2001) applied the finite difference method to obtain a numerical solution of the Cahn-Hilliard equation. Ugurlu and Kaya (2008) solved the Cahn-Hilliard equation by the Tanh function method. Many articles have investigated this equation analytically and numerically, see Kim 2007, Wells 2006. Also, Kourosh (2011) used the Exp-function method to obtain exact solutions of the Cahn-Hilliard.

Many powerful methods, such as the inverse scattering method, the Backlund transformation, the Wadati trace method, Hirota bilinear forms (Jaradat et. al.), the pseudo spectral method, the tanh-sech method (Mafliet (a,b) 1996), the sine-cosine method (Alquran 2012), and the Riccati equation expansion method were used to investigate the solution of the Gardner equation. Wazwaz (2007) employed the tanh method and obtained kink solitons solution to the Gardner equation.

2. The Differential Transform Method (DTM)

The goal of this section is to recall notations, definitions and some theorems of the DTM that will be used in this paper. These are discussed in Alquran and Al-Khaled 2010 (a,b), Alquran and Al-Khamaiseh 2010, Odibat 2008, and Ayas 2004. Also, we will highlight the main steps for implementing the DTM in solving PDEs.

The differential transform of the k^{th} derivative of a function $u(x)$ is defined to be

$$U(k)=\frac{1}{k!}\left(\frac{d^k}{dx^k}u(x)\right)|_{x=x_0} \tag{2.1}$$

and the inverse transform of $U(k)$ is defined as

$$u(x)=\sum_{k=0}^{\infty}U(k)(x-x_0)^k . \tag{2.2}$$

Equation (2.2) is known as the Taylor series expansion of $u(x)$ around $x=x_0$. Also, we need the following theorems. If $G(k)$ is the differential transform of $g(x)$, then:

Theorem 2.1. If $f(x)=\frac{d^n g(x)}{dx^n}$, then $F(k)=\frac{(k+n)!}{k!}G(k+n)$.

Theorem 2.2. If $f(x)=g^2(x)$, then $F(k)=\sum_{i=0}^{k}G(i)\,G(k-i)$.

Theorem 2.3. If $f(x)=g^3(x)$, then $F(k)=\sum_{i=0}^{k}\sum_{j=0}^{k-i}G(i)\,G(j)\,G(k-i-j)$.

Now, we first consider a general form of nonlinear equation

$$P(u,u_t,u_x,u_{xx},u_{xt},u_{xxx},u_{xxt},...)=0. \tag{2.3}$$

Second, we introduce the wave variable

$$\zeta=\lambda(x-ct), \tag{2.4}$$

so that

$$u(x,t)=U(\zeta), \tag{2.5}$$

where the localized wave solution $U(\zeta)$ travels with speed c. Based on this, the PDE (2.3) converted to an ODE

$$P(U,c\lambda U',\lambda U',\lambda^2U'',c\lambda^2U'',\lambda^3U''',c\lambda^3U''',...)=0. \tag{2.6}$$

Third, we apply the DTM to Equation (2.6) and get

$$U(\zeta)=\sum_{k=0}^{\infty}F(k)\zeta^{k}, \tag{2.7}$$

where $F(\zeta)$ is the differential transform of $U(\zeta)$. Accordingly,

$$u(x,t)=\sum_{k=0}^{\infty}F(k)\big(\lambda(x-ct)\big)^{k}. \tag{2.8}$$

Finally, the approximate solution is

$$u_{appr}(x,t)=\sum_{k=0}^{N}F(k)\big(\lambda(x-ct)\big)^{k}. \tag{2.9}$$

3. Applications of the DTM

In this section, we apply the new approach to study three examples

Example 3.1. We consider the BBM equation

$$u_t=u_{xxt}-u_x-uu_x, \tag{3.1}$$

subject to the initial condition

$$u(x,0)=\operatorname{sech}^2(\frac{x}{4}), \tag{3.2}$$

where the exact solution is

$$u(x,t)=\operatorname{sech}^2(\frac{x}{4}-\frac{t}{3}). \tag{3.3}$$

Now using the wave variable $\zeta=\lambda(x-ct)$, (3.1 - 3.2) converted to the ODE

$$(1-c)U+c\lambda^2U''+\frac{1}{2}U^2=0, \tag{3.4}$$

subject to the initial condition

$$U(0)=1. \tag{3.5}$$

Applying the Differential Transform to (3.4-3.5) and by means of Theorems (2.1-2.3), we obtain the following recursive formula

$$F(k+2)=\frac{1}{c\lambda^2(k+1)(k+2)}\left((c-1)F(k)-\frac{1}{2}\sum_{i=0}^{k}F(i)F(k-i)\right) \tag{3.6}$$

and

$$F(0) = U(0), \qquad F(1) = U'(0) = a, \tag{3.7}$$

where a is a constant to be determined. Refereing to Equation (2.9), the approximate solution is

$$u_{appr}(x,t) = \sum_{k=0}^{8} F(k)(\lambda(x-ct))^k. \tag{3.8}$$

Using the initial condition (3.2), and by the aid of Mathematica software, we have

$$\begin{aligned} u_{appr}(x,t) = {} & 1 + 0.0000492189(x-1.33322t) - 0.062548(x-1.33322t)^2 \\ & -4.10262*10^{-6}(x-1.33322t)^3 + 0.00260683(x-1.33322t)^4 \\ & +2.18047*10^{-7}(x-1.33322t)^5 - 0.0000923655(x-1.33322t)^6 \\ & -9.47054*10^{-9}(x-1.33322t)^7 + 3.00882*10^{-6}(x-1.33322t)^8. \end{aligned}$$

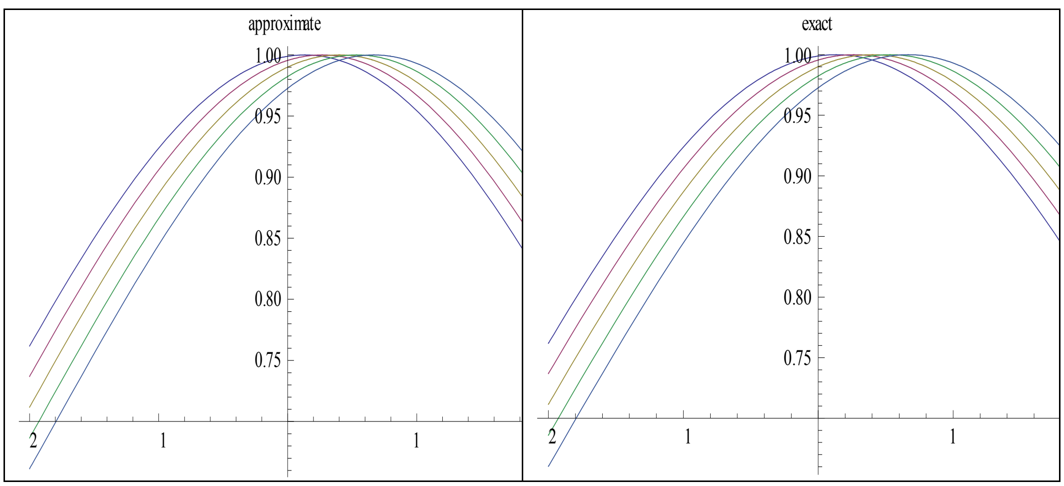

Figure 1: The approximate and exact solutions for Example 3.1 when $-2 < x < 2$ and $t = 0.1, 0.2, 0.3, 0.4, 0.5$.

Example 3.2. We consider the Cahn-Hilliard equation

$$u_t = u_{xx} - u^3 + u, \tag{3.9}$$

subject to the initial condition

$$u(x,0) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}, \tag{3.10}$$

where the exact solution is

$$u(x,t) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}-\frac{3t}{2}}}. \tag{3.11}$$

The associated ODE of (3.9 - 3.10) is

$$-c\,\lambda U' - \lambda^2 U'' + \frac{1}{2}U^3 - U = 0, \tag{3.12}$$

subject to the initial condition

$$U(0) = 0.5\,. \tag{3.13}$$

Applying the Differential Transform to (3.12-3.13), we get the recursive formula

$$F(k+2) = -\frac{cF(k+1)}{\lambda(k+2)} - \frac{F(k)}{\lambda^2(k+1)(k+2)} + \frac{1}{\lambda^2(k+1)(k+2)}\sum_{i=0}^{k}\sum_{j=0}^{k-i}F(i)F(j)F(k-i-j). \tag{3.14}$$

Refereing to Equation (2.9), the approximate solution is

$$u_{appr}(x,t) = \sum_{k=0}^{8}F(k)(\lambda(x-ct))^k\,. \tag{3.15}$$

Using the initial condition (3.10), and by the aid of Mathematica software, we have

$$\begin{aligned} u_{appr}(x,t) = {} & 0.5 - 0.176469(x-2.11966\,t) - 0.00047338(x-2.11966\,t)^2 \\ & +0.00768733(x-2.11966\,t)^3 - 0.000171119(x-2.11966\,t)^4 \\ & -0.00028579(x-2.11966\,t)^5 - 0.0000347312(x-2.11966\,t)^6 \\ & +0.0000312117(x-2.11966\,t)^7 - 4.04237*10^{-6}(x-2.11966\,t)^8. \end{aligned}$$

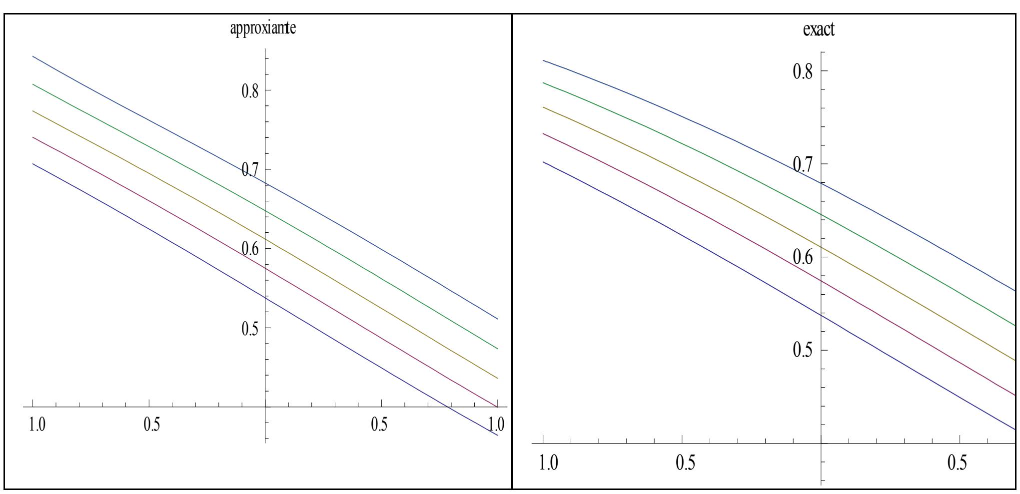

Figure 2: The approximate and exact solutions for Example 3.2 when $-1 < x < 1$ and $t = 0.1, 0.2, 0.3, 0.4, 0.5$.

Example 3.3. We consider the Gardner equation

$$u_t = 6u^2 u_x + 6u + u_{xxx}, \tag{3.16}$$

subject to the initial condition

$$u(x,0) = -\frac{1}{2}\left(1 - \tanh(\frac{x}{2})\right), \tag{3.17}$$

where the exact solution is

$$u(x,0) = -\frac{1}{2}\left(1 - \tanh(\frac{x-t}{2})\right) \tag{3.18}$$

The associated ODE of (3.16, 3.17) based on the wave variable is

$$cU + 3U^2 + U^3 + \lambda^2 U'' = 0, \tag{3.19}$$

subject to the initial condition

$$U(0) = -0.5. \tag{3.20}$$

Applying the Differential Transform to (3.19-3.20) we get the recursive formula

$$F(k+2) = -\frac{cF(k)}{\lambda^2(k+1)(k+2)} - \frac{3\sum_{i=0}^{k} F(i)F(k-i)}{\lambda^2(k+1)(k+2)} - \frac{2\sum_{i=0}^{k}\sum_{j=0}^{k-i} F(i)F(j)F(k-i-j)}{\lambda^2(k+1)(k+2)}. \tag{3.21}$$

Refereing to Equation (2.9), the approximate solution is

$$u_{appr}(x,t) = \sum_{k=0}^{8} F(k)(\lambda(x-ct))^k. \tag{3.22}$$

Using the initial condition (3.17), and by the aid of Mathematica software, we have

$$\begin{aligned} u_{appr}(x,t) = &-0.5 - 0.278222(x-1.22853t) + 0.0571319(x-1.22853t)^2 \\ &-0.0125883(x-1.22853t)^3 + 0.00129248(x-1.22853t)^4 \\ &+0.00198278(x-1.22853t)^5 - 0.000872791(x-1.22853t)^6 \\ &+0.000281753(x-1.22853t)^7 - 0.0000644879(x-1.22853t)^8. \end{aligned}$$

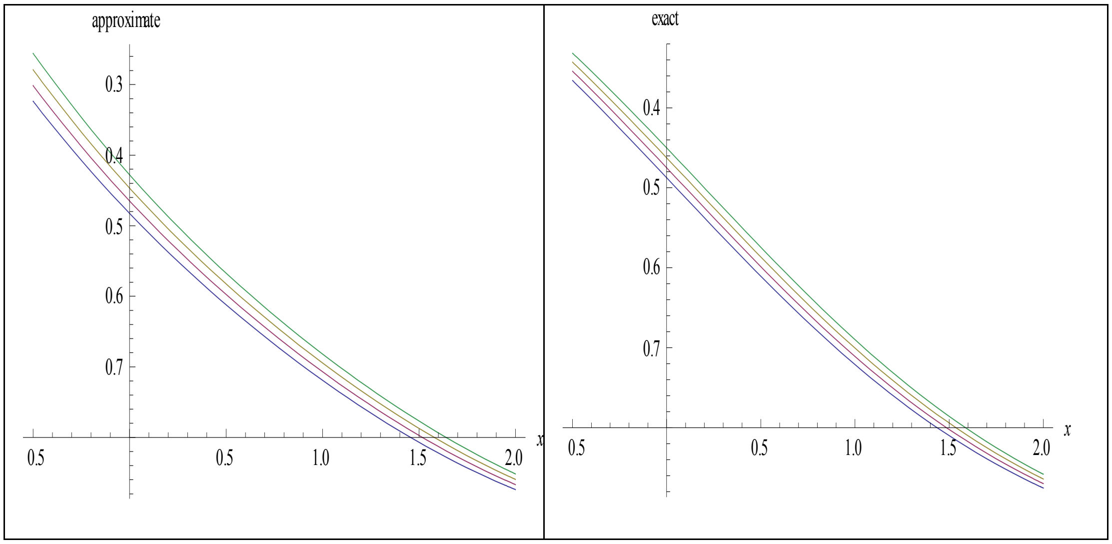

Figure 3: The approximate and exact solutions for Example 3.3 when $-0.5 < x < 2$ and $t = 0.05, 0.1, 0.15, 0.20, 0.25$.

4. Conclusion

The DTM has been successfully implemented to find an approximate solution for nonlinear PDEs by considering a change of variables to a new wave variable. Three different physical models were tested and the results were in excellent agreement with the exact solution by considering only the first nine terms of the Differential Transform series. Computations of this paper have been carried out using Mathematica 7.

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