



Two Numerical Algorithms for Solving a Partial Integro-Differential Equation with a Weakly Singular Kernel

Jeong-Mi Yoon, Shishen Xie and Volodymyr Hrynkiw

Department of Computer and Mathematical Sciences
University of Houston-Downtown
Houston, TX, 77002

yoonyj@uhd.edu; xies@uhd.edu; hrynkiwV@uhd.edu

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Abstract

Two numerical algorithms based on variational iteration and decomposition methods are developed to solve a linear partial integro-differential equation with a weakly singular kernel arising from viscoelasticity. In addition, analytic solution is re-derived by using the variational iteration method and decomposition method.

Keywords: Variational Iteration Method, Decomposition Method, Partial Integro-differential Equations, Singular Kernel, Numerical Methods

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1. Introduction

Mathematical modeling of real world phenomena often leads to functional equations, like integral and integro-differential equations, stochastic equations, etc. While solution techniques for many types of these problems are well known, there is a large class of problems that lack standard solution methods, namely, partial integro-differential equations. In general, partial integro-differential equations are difficult to solve analytically and, as a result, one has to resort to numerical approximation of the solution. Main challenges in solving these kinds of problems, both numerically and analytically, are due to different factors, such as large range of variables, nonlinearity and non-local phenomena, multi-dimensionality, physical constraints, etc. Problems involving partial integro-differential equations arise in fluid dynamics, viscoelasticity, engineering, mathematical biology, financial mathematics, and other areas. In the study of fluids

involving viscoelastic forces [Olmstead et al. (1986); Lodge et al. (1985)] or in the modeling of heat flow in materials with memory [Gurtin and Pipkin (1968); Miller (1978)] or in the modeling of linear viscoelastic mechanics [Christensen (1971); Renardy (1989)], the following partial integro-differential equation can be found:

$$u_t = \mu u_{xx} + \int_0^t (t-s)^{-1/2} u_{xx}(x,s) ds, \quad (1)$$

where the unknown real function $u(x,t)$ is sought for $0 \leq t \leq T, 0 \leq x \leq 1$, with the initial condition

$$u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \quad (2)$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq T. \quad (3)$$

In (1), μu_{xx} term represents Newtonian contribution to the viscosity. In this paper we study a special case of the problem (1)-(3) when $\mu = 0$, that is, non-Newtonian fluids. Namely, we study the problem

$$u_t = \int_0^t (t-s)^{-1/2} u_{xx}(x,s) ds, \quad (4)$$

where the initial and boundary conditions are given by (2)-(3). The memory integrals in (1) and (4) can be thought of as representing viscoelastic forces. It is interesting to note that equation (4) can be considered as an equation intermediate between the classical parabolic heat and wave equations [Sanz-Serna (1988); Lopez-Marcos (1990)]. Also, it should be noted that the analysis of equation (4) is an important step in the study of equation (1). Numerical studies of these types of problems have been considered by many authors [Dehghan (2006); Sanz-Serna (1988); Tang (1993); Lopez-Marcos (1990); and references therein], and were concerned with solving the problems by finite differences.

In this paper we used the variational iteration method and Adomian decomposition method to solve equation (4) numerically. As a nice by-product of these methods we re-derived the existing formula (9) for analytical solution of equation (4). Throughout this paper we choose

$$u(x,0) = u_0(x) = \sin(\pi x), \quad 0 \leq x \leq 1, \quad (5)$$

[Dehghan (2006); Tang (1993)]. To our knowledge this is the first paper that deals with application of VIM and ADM to a partial integro-differential equation with a singular kernel.

The paper is organized as follows. In section 2 we discuss the variational iteration and Adomian decomposition methods. As a nice by-product of these methods, we re-derive the analytical solution to the given problem in section 3. Conclusions and discussion of the results are in section 4.

2. Two Numerical Algorithms

2.1. The Variational Iteration Method

The variational iteration method (VIM), [He (1999, 1997, 2007)], was proposed by J.H. He to solve nonlinear differential equations using an iterative scheme. In this section we develop a VIM algorithm to solve partial integro-differential equation (4). To illustrate the main idea of VIM, consider the following, in general nonlinear equation

$$Lu(t) + Ru(t) = g(t),$$

where L is a linear operator, R is a nonlinear operator, and g is a given function. One constructs a correction functional as follows

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^t \lambda [Lu_n(s) + R\tilde{u}_n(s) - g(s)] ds,$$

where λ is a Lagrange multiplier, and \tilde{u}_n is considered a restricted variation, that is, $\delta\tilde{u}_n = 0$, [He (1999, 1997, 2007)]. This gives the desired iterative scheme.

Applying the VIM algorithm to (4), we obtain the following iteration scheme

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\tau) \left[\frac{\partial}{\partial \tau} u_n(x, \tau) - \int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} u_n(x, s) ds \right] d\tau, \quad (6)$$

where λ is a general Lagrangian multiplier, which can be identified optimally via the variation theory. Taking variation with respect to u_n and noticing that $\delta R\tilde{u}_n = 0$ (where R is a nonlinear operator that contains partial derivatives with respect to x), we obtain

$$\begin{aligned} \delta u_{n+1}(x, t) &= \delta u_n(x, t) + \delta \int_0^t \lambda(\tau) \left[\frac{\partial}{\partial \tau} u_n(x, \tau) - \int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} \tilde{u}_n(x, s) ds \right] d\tau \\ &= \delta u_n(x, t) + \lambda(\tau) \delta u_n(x, \tau) \Big|_{\tau=t} - \int_0^t \lambda'(\tau) \delta u_n(x, \tau) d\tau = 0. \end{aligned}$$

This yields the stationary conditions

$$\lambda'(\tau) = 0, \text{ and } 1 + \lambda(\tau)|_{\tau=t} = 0.$$

Therefore, the Lagrangian multiplier $\lambda = -1$. Substituting the identified multiplier into (6), we obtain the following iteration formula

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left[\frac{\partial}{\partial \tau} u_n(x, \tau) - \int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} u_n(x, s) ds \right] d\tau, \quad (7)$$

for $n = 0, 1, 2, 3, \dots$ For $n = 0$, we choose

$$u_0(x, t) = u(x, 0) = \sin(\pi x). \quad (8)$$

Thus, when $n = 1$,

$$\begin{aligned} u_1(x, t) &= u_0(x, t) - \int_0^t \left[\frac{\partial}{\partial \tau} u_0(x, \tau) - \int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} u_0(x, s) ds \right] d\tau \\ &= \sin(\pi x) - \frac{4}{3} \pi^2 \sin(\pi x) t^{3/2}. \end{aligned}$$

Similarly, for $n = 2, 3, \dots$, we get

$$\begin{aligned} u_2(x, t) &= u_1(x, t) - \int_0^t \left[\frac{\partial}{\partial \tau} u_1(x, \tau) - \int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} u_1(x, s) ds \right] d\tau \\ &= \sin(\pi x) - \frac{4}{3} \pi^2 \sin(\pi x) t^{3/2} + \frac{1}{6} \pi^5 \sin(\pi x) t^3, \\ u_3(x, t) &= \sin(\pi x) - \frac{4}{3} \pi^2 \sin(\pi x) t^{3/2} + \frac{1}{6} \pi^5 \sin(\pi x) t^3 - \frac{32}{945} \pi^7 \sin(\pi x) t^{9/2}, \text{ etc.} \end{aligned}$$

From the iteration process (7), it can be observed that the numerical solution of VIM shows a reasonably rapid convergence of iterates after around twenty iterations.

It is known [Sanz-Serna (1988); Tang (1993)] that

$$u(x, t) = \sum_{n=0}^{\infty} (-1)^n \Gamma\left(\frac{3}{2}n + 1\right)^{-1} (\pi^{5/2} t^{3/2})^n \sin(\pi x) \quad (9)$$

is the analytic solution satisfying the partial integro-differential equation (4) with the given boundary and initial conditions, (3) and (5), where Γ denotes the Gamma function. Table 1 shows the errors ($|u_{analytic} - u_{VIM}|$) between the analytic solutions and the 20th iteration u_{20} of VIM at $T = 1.0$. For a comparison, the errors ($|u_{analytic} - u_{CN}|$) between the analytic solution

and Crank-Nicolson method with $\Delta x = 0.1$ and $\Delta t = 0.005$ (that is, 10 steps in x direction and 200 steps in t direction) are also listed in the table.

It can be seen from the table that the solutions obtained by variational iteration method are much closer to the analytic solutions than Crank-Nicolson method. Furthermore, it is important to point out that our VIM algorithm is designed in such a way that the iterations can be computed rapidly in Maple with only a few seconds to complete the 20 iterations. In comparison, the computing time in Maple for the Crank-Nicolson method is nearly one minute, which is much slower than the VIM algorithm.

Table 1. Error comparison between VIM and Crank-Nicolson solution for $T=1.0$

x	Analytic Solution	$ u_{\text{analytic}} - u_{\text{VIM}} $	$ u_{\text{analytic}} - u_{\text{CN}} $
0.1	0.00796108296	2.1×10^{-9}	3.2×10^{-5}
0.2	0.01514289252	1.2×10^{-7}	6.1×10^{-5}
0.3	0.02084234944	1.7×10^{-7}	8.4×10^{-5}
0.4	0.02450164262	6.8×10^{-8}	9.9×10^{-5}
0.5	0.02576258933	4.0×10^{-8}	1.0×10^{-4}
0.6	0.02450164262	1.5×10^{-7}	9.9×10^{-5}
0.7	0.02084234944	8.5×10^{-8}	8.4×10^{-5}
0.8	0.01514289252	9.7×10^{-8}	6.1×10^{-5}
0.9	0.00796108296	1.7×10^{-8}	3.2×10^{-5}

2.3. Adomian Decomposition Method

Similarly to the VIM, the Adomian decomposition method (ADM), [Adomian (1984, 1990)], was proposed by G. Adomian to solve differential equations using a recursive formula. In this section we apply the ADM algorithm to solve partial integro-differential equation (4). By the decomposition algorithm, we assume a series solution

$$u(x, t) = \sum_{n=0}^{\infty} v_n(x, t) = v_0(x, t) + v_1(x, t) + v_2(x, t) + \dots$$

for problem (3)-(5). Integrating both sides of (4) from 0 to t , we obtain the following recursive relation

$$v_{n+1}(x, t) = \int_0^t \left[\int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} v_n(x, s) ds \right] d\tau$$

For $n=1, 2, 3, \dots$, with $v_0(x, t)$ chosen to be $v_0(x, t) = u_0(x, 0) = \sin(\pi x)$.

Then, for $n = 1$,

$$v_1(x, t) = \int_0^t \left[\int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} v_0(x, s) ds \right] d\tau = -\frac{4}{3} \sin(\pi x) t^{3/2}.$$

Similarly, for $n=2, 3, \dots$,

$$v_2(x, t) = \int_0^t \left[\int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} v_1(x, s) ds \right] d\tau = \frac{1}{6} \pi^5 \sin(\pi x) t^3.$$

$$v_3(x, t) = \int_0^t \left[\int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} v_2(x, s) ds \right] d\tau = -\frac{32}{945} \pi^7 \sin(\pi x) t^{3/2}, \text{ etc.}$$

We can observe that each term of the ADM solution is the same as corresponding term of the series solution obtained by VIM. Therefore, the numerical results of ADM are the same as those of VIM shown in Table 1.

3. Analytic Solution by VIM

In this section the VIM scheme (7) is used to re-derive the analytic solution obtained in [8, 9] for the problem (3) – (5). The results are summarized in the following theorem.

Theorem 3.1. The series solution obtained using the variational iteration scheme (7) for the partial integro-differential equation (4) with conditions (3) and (5) is

$$u(x, t) = \sum_{n=0}^{\infty} (-1)^n \Gamma\left(\frac{3}{2}n + 1\right)^{-1} (\pi^{5/2} t^{3/2})^n \sin(\pi x),$$

which is the analytic solution (9) of the problem.

Proof:

We prove it using mathematical induction. First, it is obvious for $n = 0$ that

$$u_0(x, t) = \sum_{n=0}^0 (-1)^n \Gamma\left(\frac{3}{2}n + 1\right)^{-1} (\pi^{5/2} t^{3/2})^n \sin(\pi x) = \sin(\pi x),$$

which is the same as the definition for $u_0(x, 0)$ in (8). Assume by induction that for some k

$$u_k(x, t) = \sum_{n=0}^k (-1)^n \Gamma\left(\frac{3}{2}n + 1\right)^{-1} (\pi^{5/2} t^{3/2})^n \sin(\pi x), \quad (10)$$

For $n = k + 1$, it can be obtained by variational iteration scheme (7) that

$$u_{k+1}(x, t) = u_k(x, t) - \int_0^t \left[\frac{\partial}{\partial \tau} u_k(x, \tau) - \int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} u_k(x, s) ds \right] d\tau$$

$$\begin{aligned}
 &= u_k(x, t) - u_k(x, t) + u_k(x, 0) + \int_0^t \int_0^\tau (\tau - s)^{-1/2} \frac{\partial^2}{\partial x^2} u_k(x, s) ds d\tau \\
 &= \sin(\pi x) + \int_0^t \int_0^\tau (\tau - s)^{-1/2} \sum_{n=0}^k (-1)^n \frac{\pi^{\frac{5}{2}n} s^{\frac{3}{2}n}}{\Gamma(\frac{3}{2}n+1)} [-\pi^2 \sin(\pi x)] ds d\tau. \tag{11}
 \end{aligned}$$

The last equality is derived from the assumption (10). The integration in (11) can be further simplified with a substitution $s = \tau y$.

$$\begin{aligned}
 &\int_0^t \int_0^\tau (\tau - s)^{-1/2} \sum_{n=0}^k (-1)^n \frac{\pi^{\frac{5}{2}n} s^{\frac{3}{2}n}}{\Gamma(\frac{3}{2}n+1)} [-\pi^2 \sin(\pi x)] ds d\tau \\
 &= \sum_{n=0}^k (-1)^{n+1} \frac{\pi^{\frac{5}{2}n+2} \sin(\pi x)}{\Gamma(\frac{3}{2}n+1)} \int_0^t \tau^{\frac{3}{2}n+\frac{1}{2}} d\tau \int_0^1 y^{\frac{3}{2}n} (1-y)^{-\frac{1}{2}} dy \\
 &= \sum_{n=0}^k (-1)^{n+1} \frac{\pi^{\frac{5}{2}n+2} \sin(\pi x)}{\Gamma(\frac{3}{2}n+1)} \frac{t^{\frac{3}{2}n+\frac{3}{2}}}{\frac{3}{2}n+\frac{3}{2}} B\left(\frac{3}{2}n+1, \frac{1}{2}\right), \tag{12}
 \end{aligned}$$

where $B(p, q)$ is the Beta function defined by $B(p, q) = \int_0^1 y^{p-1} (1-y)^{q-1} dy$, and $p, q > 0$.

Using the relation $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ between Beta and Gamma functions, and properties of Gamma function

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \text{ and } z\Gamma(z) = \Gamma(z+1),$$

we can rewrite (12) as

$$\begin{aligned}
 &\sum_{n=0}^k (-1)^{n+1} \frac{\pi^{\frac{5}{2}n+2} \sin(\pi x)}{\Gamma(\frac{3}{2}n+1)} \frac{t^{\frac{3}{2}n+\frac{3}{2}}}{\frac{3}{2}n+\frac{3}{2}} \frac{\Gamma(\frac{3}{2}n+1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}n+\frac{3}{2})} \\
 &= \sum_{n=0}^k (-1)^{n+1} \frac{\pi^{\frac{5}{2}n+2} \sin(\pi x) t^{\frac{3}{2}n+\frac{3}{2}} \pi^{\frac{1}{2}}}{\left(\frac{3}{2}n+\frac{3}{2}\right)\Gamma(\frac{3}{2}n+\frac{3}{2})} \\
 &= \sum_{n=0}^k (-1)^{n+1} \frac{\pi^{\frac{5}{2}(n+1)} \sin(\pi x) t^{\frac{3}{2}(n+1)}}{\Gamma\left(\frac{3}{2}(n+1)+1\right)}
 \end{aligned}$$

$$= \sum_{m=1}^{k+1} (-1)^m \frac{\pi^{5m} \sin(\pi x) t^{\frac{3}{2}m}}{\Gamma\left(\frac{3}{2}m + 1\right)} \quad (13)$$

The last summation is obtained by a change of the index: $m = n+1$. Combining expressions (11) and (13) yields

$$\begin{aligned} u_{k+1}(x, t) &= \sin(\pi x) + \sum_{m=1}^{k+1} (-1)^m \frac{\pi^{5m} \sin(\pi x) t^{\frac{3}{2}m}}{\Gamma\left(\frac{3}{2}m + 1\right)} \\ &= \sum_{m=0}^{k+1} (-1)^m \Gamma\left(\frac{3}{2}m + 1\right)^{-1} (\pi^{5/2} t^{3/2})^m \sin(\pi x). \end{aligned}$$

The proof explains the excellent numerical results listed in Table 1 in Section 2.1. Also it is obvious that the proof of the theorem can be done by ADM since it generates identical terms in the series solution as VIM.

4. Conclusions and Discussions

Even though VIM and ADM are already well known, we have shown that those algorithms can be used successfully to solve a partial integro-differential equation with a weakly singular kernel. From the numerical analysis, we observed that the numerical solution obtained from VIM and ADM shows a rapid convergence of iterates after a reasonable number of iterations. Furthermore, VIM and ADM are used to derive the analytic solution via the mathematical induction. In our work Maple was used to calculate the exact integrations of the series solutions obtained from VIM and ADM numerically. The advantage of these two methods is the ability to solve integro-differential equations rapidly without discretizing variables for numerical integration.

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