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The First Integral Method to Nonlinear Partial Differential Equations

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Abstract

In this paper, we show the applicability of the first integral method for obtaining exact solutions of some nonlinear partial differential equations. By using this method, we found some exact solutions of the Landau-Ginburg-Higgs equation and generalized form of the nonlinear Schrödinger equation and approximate long water wave equations. The first integral method is a direct algebraic method for obtaining exact solutions of nonlinear partial differential equations. This method can be applied to nonintegrable equations as well as to integrable ones. This method is based on the theory of commutative algebra.

Keywords: First integral method; Landau-Ginburg-Higgs equation; Generalized form of the nonlinear Schrödinger equation; Approximate long water wave equations

MSC 2010: 35Q53; 35Q80; 35Q55; 35G25

1. Introduction

Many phenomena in physics and engineering are described by nonlinear partial differential equations (NPDEs). When we want to understand the physical mechanism of phenomena in nature, described by nonlinear PDEs, exact solution for the nonlinear PDEs have to be explored. Thus the methods for deriving exact solutions for the governing equations are important and have to be developed. To study exact solutions of nonlinear PDEs has become one of the most important topics in mathematical physics. For instances, the nonlinear wave phenomena observed in fluid dynamics, plasma, and optical fiber are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of their solutions. Nonlinear differential equations have a wide array of applications in many fields. They could describe the motion of isolated waves, localized in a small part of space. Their applications could extend to magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas. Looking for exact solitary wave solutions to nonlinear evolution equations has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields, such as solitons and propagation with a finite speed, and thus they may give more insight into the physical aspects of the problems. In order to obtain the periodic wave and soliton solutions of nonlinear evolution equations, a number of methods have been proposed, such as tanh-sech function method, extended tanh function method , hyperbolic function method, sine-cosine method, Jacobi elliptic function expansion method, F-expansion method, transformed rational function method and the first integral method.

The first integral method is a powerful solution method for the computation of exact traveling wave solutions. This method is one of the most direct and effective algebraic methods for finding exact solutions of nonlinear partial differential equations. Different from other traditional methods, the first integral method has many advantages, which is the avoidance of a great deal of complicated and tedious calculations resulting in more exact and explicit traveling solitary solutions with high accuracy.

In the pioneer work, Feng (2002) introduced the first integral method for a reliable treatment of the nonlinear PDEs. The first integral method is widely used by many such as in [Taghizadeh et al. (2011), Taghizadeh and Mirzazadeh (2011), Taghizadeh et al. (2012), Moosaei et al. (2011) and by the reference therein]. Taghizadeh et al. (2011) proposed the first integral method to solve the modified KdV–KP equation and the Burgers–KP equation. The method was utilized to construct exact solutions of the nonlinear Schrödinger equation. Taghizadeh and Mirzazadeh (2011) used the first integral method to obtain the exact solutions of some complex nonlinear partial differential equations and Konopelchenko-Dubrovsky equation. Moosaei et al. (2011) solved the perturbed nonlinear Schrödinger’s equation with Kerr law nonlinearity by using the first integral method. Recently, it was successfully used for constructing the exact solutions of the Eckhaus equation [Taghizadeh et al. (2012)].

The paper is arranged as follows. In section 2, we describe briefly the first integral method. In section 3, we apply this method to the Landau-Ginburg-Higgs equation and generalized form of the nonlinear Schrödinger equation and approximate long water wave equation.

2. The First Integral Method

Step 1. Consider a general nonlinear PDE in the form

$$E(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0. \tag{1}$$

To find the travelling wave solutions to Equation (1), we introduce the wave variable

$$\xi = x - ct, \tag{2}$$

so that

$$u(x,t) = u(\xi). \tag{3}$$

Based on this we use the following changes

$$\begin{aligned}
&\frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.),\\
&\frac{\partial}{\partial t}(.) = -c\frac{\partial}{\partial \xi}(.),\\
&\frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.),\\
&\frac{\partial^2}{\partial t \partial x}(.) = -c\frac{\partial^2}{\partial \xi^2}(.),
\end{aligned} \tag{4}$$

and so on for the other derivatives.

Using (4) changes the PDE (1) to an ODE

$$H(u, \frac{\partial u}{\partial \xi}, \frac{\partial^2 u}{\partial \xi^2}, ...) = 0, \tag{5}$$

where $u = u(\xi)$ is an unknown function, H is a polynomial in the variable u and its derivatives.

Step 2. Suppose the solution of ODE (5) can be written as follows:

$$u(x,t) = f(\xi), \tag{6}$$

and furthermore, we introduce a new independent variable

$$X(\xi) = f(\xi), \quad Y(\xi) = \frac{\partial f(\xi)}{\partial \xi}. \tag{7}$$

Step 3. Under the conditions of Step 2**,** Equation (5) can be converted to a system of nonlinear ODEs as follows

$$\begin{aligned}
&X'(\xi) = Y(\xi),\\
&Y'(\xi) = F(X(\xi), Y(\xi)).
\end{aligned} \tag{8}$$

If we can find the integrals to Equation (8), then the general solutions to Equation (8) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that can tell us how to find its first integrals, nor a logical way for telling us what these first integrals are. We will apply the so-called Division Theorem to obtain one first integral to Equation (8) which reduces Equation (5) to a first order integrable ODE. An exact solution to Equation (1) is then obtained by solving this equation.

Division Theorem. Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$, and $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that

$$Q(w,z) = P(w,z)G(w,z).$$

The Divisor Theorem follows immediately from the Hilbert–Nullstellensatz Theorem.

Hilbert–Nullstellensatz Theorem. Let K be a field and L be an algebraic closure of K. Then:

(i) Every ideal γ of $K[X_1, X_2, ..., X_n]$ not containing 1 admits at least one zero in L^n.

(ii) Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be two elements of L^n. For the set of polynomials of $K[X_1, X_2, ..., X_n]$ zero at x to be identical with the set of polynomials of $K[X_1, X_2, ..., X_n]$ zero at y, it is necessary and sufficient that there exists a $K-$automorphism S of L such that $y_i = S(x_i)$ for $1 \le i \le n$.

(iii) For an ideal α of $K[X_1, X_2, ..., X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that α is the set of polynomials of $K[X_1, X_2, ..., X_n]$ zero at x.

(iv) For a polynomial Q of $K[X_1, X_2, ..., X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $K[X_1, X_2, ..., X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.

3. Application of the First Integral Method to Npdes

3.1. The first integral method for obtaining exact solutions of NPDEs

3.1. A. Landau-Ginburg-Higgs equations

Consider the Landau-Ginburg-Higgs equations [Khuri (2008)]:

$$u_{tt} - u_{xx} - m^2 u + n^2 u^3 = 0, \qquad (9)$$

where m and n are real constants. By make the transformation $u(x,t) = f(\xi) = f(k(x-\lambda t))$, where λ is the wave speed and $\xi = k(x-\lambda t)$, Equation (9) becomes

$$k^2(\lambda^2-1)\frac{\partial^2 f(\xi)}{\partial \xi^2} - m^2 f(\xi) + n^2 (f(\xi))^3 = 0. \tag{10}$$

If we let $X = f(\xi), Y = \dfrac{df(\xi)}{d\xi}$, the Equation (10) is equivalent to the two dimensional autonomous system

$$\begin{cases} X' = Y, \\ Y' = \dfrac{m^2}{k^2(\lambda^2-1)} X(\xi) - \dfrac{n^2}{k^2(\lambda^2-1)} X^3(\xi). \end{cases} \tag{11}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$, are the nontrivial solutions of (11) also

$$q(X,Y) = \sum_{i=0}^{m} a_i(X) Y^i = 0,$$

is an irreducible polynomial in the complex domain $C[X, Y]$, such that

$$q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi)) Y^i(\xi) = 0, \tag{12}$$

where $a_i(X)(i = 0,1,\ldots,m)$, are polynomials of X and $a_m(X) \neq 0$. Equation (12) is called the first integral to (11). Suppose that $m = 1$ in (12). Note that $\dfrac{dq}{d\xi}$ is a polynomial in X and Y, and $q[X(\xi), Y(\xi)] = 0$ implies $\left.\dfrac{dq}{d\xi}\right|_{(11)} = 0$. According to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in $C[X,Y]$ such that

$$\begin{aligned} \left.\frac{dq}{d\xi}\right|_{(11)} &= \left.\left(\frac{dq}{dX}.\frac{dX}{d\xi} + \frac{dq}{dY}.\frac{dY}{d\xi}\right)\right|_{(11)} \\ &= \left(\sum_{i=0}^{1} a_i'(X) Y^i\right)(Y) + \left(\sum_{i=0}^{1} i a_i(X) Y^{i-1}\right) \times \left(\frac{m^2}{k^2(\lambda^2-1)} X - \frac{n^2}{k^2(\lambda^2-1)} X^3\right) \end{aligned}$$

$$=(g(X)+h(X)Y)\sum_{i=0}^{1}a_i(X)Y^i, \tag{13}$$

where prime denotes differentiation with respect to the variable X. By comparing with the coefficients of $Y^i\,(i=2,1,0)$ of both sides of (13), we have

$$a_1'(X)=h(X)a_1(X), \tag{14}$$

$$a_0'(X)=g(X)a_1(X)+h(X)a_0(X), \tag{15}$$

$$a_1(X)[\frac{m^2}{k^2(\lambda^2-1)}X-\frac{n^2}{k^2(\lambda^2-1)}X^3]=g(X)a_0(X). \tag{16}$$

Since $a_i(X)(i=0,1)$ are polynomials, then from (14) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=1$ only. Suppose that $g(X)=B_0+A_1X$, then we find $a_0(X)$.

$$a_0(X)=\frac{1}{2}A_1X^2+B_0X+A_0, \tag{17}$$

where A_0 is the arbitrary integration constant. Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (16) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0=0,\quad A_0=\mp\frac{m^2}{kn\sqrt{2(1-\lambda^2)}},\quad A_1=\pm\frac{2n}{k\sqrt{2(1-\lambda^2)}}, \tag{18}$$

where λ is an arbitrary constant.

Using the conditions (18) in (12), we obtain

$$Y(\xi)\mp\frac{1}{k\sqrt{2(1-\lambda^2)}}(\frac{m^2}{n}-nX^2(\xi))=0. \tag{19}$$

Combining (19) with (11), we obtain the exact solution to equation (10) and then the exact solution to Landau-Ginburg-Higgs equation can be written as

$$u(x,t)=\pm\frac{m}{n}\tanh\left(\frac{m}{k\sqrt{2(1-\lambda^2)}}(k(x-\lambda t)+\xi_0)\right), \tag{20}$$

for $\lambda^2<1$.

$$u(x,t)=\pm\frac{im}{n}\tan\left(\frac{m}{k\sqrt{2(\lambda^2-1)}}(k(x-\lambda t)+\xi_0)\right), \tag{21}$$

for $\lambda^2>1$.

3.1. B. The Non-integrable Equation

The nonintegrable equation

$$\hat{M}(D_t,D_x)v(t,x)=3v^2+v^3,\quad \hat{M}(D_t,D_x)=D_t^2-D_x^2-2, \tag{22}$$

is given by Baikov and Khusnutdinova (1996). We are interested in the exact solution to Equation (22).

Substituting $v(t,x)=f(\xi)=f(x-ct)$, into Equation (22), we obtain

$$(c^2-1)f''(\xi)-2f(\xi)-3f^2(\xi)-f^3(\xi)=0. \tag{23}$$

If we let $X=f(\xi), Y=\dfrac{df(\xi)}{d\xi}$, the Equation (23) is equivalent to the two dimensional autonomous system

$$\begin{cases} X'=Y, \\ Y'=\dfrac{2}{c^2-1}X(\xi)+\dfrac{3}{c^2-1}X^2(\xi)+\dfrac{1}{c^2-1}X^3(\xi). \end{cases} \tag{24}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$, are the nontrivial solutions of (24) also

$$q(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i=0,$$

is an irreducible polynomial in the complex domain $C[X,Y]$, such that

$$q(X(\xi),Y(\xi))=\sum_{i=0}^{m}a_i(X(\xi))Y^i(\xi)=0, \tag{25}$$

where $a_i(X)(i=0,1,...,m)$, are polynomials of X and $a_m(X)\neq 0$. Equation (25) is called the first integral to (10). Due to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$, such that

$$\frac{dq}{d\xi}=\frac{dq}{dX}.\frac{dX}{d\xi}+\frac{dq}{dY}.\frac{dY}{d\xi}=(g(X)+h(X)Y)\sum_{i=0}^{m}a_i(X)Y^i. \tag{26}$$

Assuming that $m=1$, by comparing with the coefficients of $Y^i(i=2,1,0)$ of both sides of (26), we have

$$a_1'(X)=h(X)a_1(X), \tag{27}$$

$$a_0'(X)=g(X)a_1(X)+h(X)a_0(X), \tag{28}$$

$$a_1(X)[\frac{2}{c^2-1}X+\frac{3}{c^2-1}X^2+\frac{1}{c^2-1}X^3]=g(X)a_0(X). \tag{29}$$

Since $a_i(X)(i=0,1)$ are polynomials, then from (27) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=1$, only. Suppose that $g(X)=A_1X+B_0$, then we find $a_0(X)$.

$$a_0(X)=A_0+B_0X+\frac{1}{2}A_1X^2, \tag{30}$$

where A_0 is arbitrary integration constant. Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (29) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0=0,\quad B_0=\mp\frac{2}{\sqrt{2(c^2-1)}},\quad A_1=\pm\frac{2}{\sqrt{2(c^2-1)}}, \tag{31}$$

where c is arbitrary constant.

Using the conditions (31) in (25), we obtain

$$Y(\xi)\mp\frac{1}{\sqrt{2(c^2-1)}}(X^2(\xi)+2X(\xi))=0. \tag{32}$$

Combining (32) with (24), we obtain the exact solution to Equation (23) and then the exact solution to the nonintegrable equation (22) can be written as

$$v(t,x)=\pm\frac{2e^{\sqrt{\frac{2}{c^2-1}}(x-ct+\xi_0)}}{1-e^{\sqrt{\frac{2}{c^2-1}}(x-ct+\xi_0)}},$$

where $c^2>1$.

Comparing our results with Baikov's results [Baikov and Khusnutdinova (1996)], it can be seen that the results are same.

3.2. The first integral method for obtaining exact solutions of complex NPDEs

In this section we study the GNLS equation [Moghaddam et al. (2009)]:

$$iu_t+au_{xx}+bu|u|^2+icu_{xxx}+id(u|u|^2)_x=ke^{i(\chi(\xi)-\omega t)}, \tag{33}$$

where $\xi=\alpha(x-vt)$ a real is function and a,b,c,d,ω,α,v are non-zero constants and $u=u(x,t)$ is a complex-valued function of two real variables x,t.

We use the wave transformation

$$u(x,t)=e^{i(\chi(\xi)-\omega t)}f(\xi),\quad \chi(\xi)=\beta\xi+x_0,\quad \xi=\alpha(x-vt), \tag{34}$$

where $\alpha,\omega,v,\beta,x_0$ are constants.

By replacing Equation (34) into Equation (33) and separating the real and imaginary parts of the result, we obtain the two following ordinary differential equations:

$$c\alpha^3f'''(\xi)+(2a\beta\alpha^2-\alpha v-3c\alpha^3\beta^2)f'(\xi)+3d\alpha f^2(\xi)f'(\xi)=0, \tag{35}$$

$$\begin{aligned}(a\alpha^2-3c\alpha^3\beta)f''(\xi)+(\alpha\beta v+\omega-a\alpha^2\beta^2+c\alpha^3\beta^3)f(\xi)\\ +(b-d\alpha\beta)f^3(\xi)-k=0.\end{aligned} \tag{36}$$

Integrating Equation (35) once, with respect to ξ, yields:

$$c\alpha^2f''(\xi)+(2a\beta\alpha-v-3c\alpha^2\beta^2)f(\xi)+df^3(\xi)-R=0, \tag{37}$$

where R is an integration constant.

Since the same function $f(\xi)$ satisfies two Equations (35) and (37), we obtain the following constraint condition:

$$\frac{c\alpha^2}{a\alpha^2-3c\alpha^3\beta}=\frac{2a\beta\alpha-v-3c\alpha^2\beta^2}{\alpha\beta v+\omega-a\alpha^2\beta^2+c\alpha^3\beta^3}=\frac{d}{b-d\alpha\beta}=\frac{R}{k}.$$

If we let $X=f(\xi), Y=\frac{df(\xi)}{d\xi}$, the Equation (37) is equivalent to the two dimensional autonomous system

$$\begin{cases} X'=Y, \\ Y'=\left(\dfrac{3c\alpha^2\beta^2+v-2a\beta\alpha}{c\alpha^2}\right)X(\xi)-\dfrac{d}{c\alpha^2}X^3(\xi)+\dfrac{R}{c\alpha^2}. \end{cases} \tag{38}$$

Equation (12) is called the first integral to (38). According to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$ in $C[X,Y]$ such that

$$\begin{aligned} \left.\frac{dq}{d\xi}\right|_{(38)} &= \left.\left(\frac{dq}{dX}.\frac{dX}{d\xi}+\frac{dq}{dY}.\frac{dY}{d\xi}\right)\right|_{(38)} \\ &= \left(\sum_{i=0}^{m} a_i'(X)Y^i\right)(Y)+\left(\sum_{i=0}^{m} i a_i(X)Y^{i-1}\right) \\ &\quad\times\left(\left(\frac{3c\alpha^2\beta^2+v-2a\beta\alpha}{c\alpha^2}\right)X(\xi)-\frac{d}{c\alpha^2}X^3(\xi)+\frac{R}{c\alpha^2}\right) \\ &= (g(X)+h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i, \end{aligned} \tag{39}$$

where prime denotes differentiation with respect to the variable X. Assuming that $m=2$, by comparing with the coefficients of $Y^i\,(i=3,2,1,0)$ of both sides of (39), we have

$$\dot{a}_2(X)=h(X)a_2(X), \tag{40}$$

$$\dot{a}_1(X)=g(X)a_2(X)+h(X)a_1(X), \tag{41}$$

$$\dot{a}_0(X) = -2a_2(X)\left[\left(\frac{3c\alpha^2\beta^2 + v - 2a\beta\alpha}{c\alpha^2}\right)X - \frac{d}{c\alpha^2}X^3 + \frac{R}{c\alpha^2}\right] + g(X)a_1(X) + h(X)a_0(X), \tag{42}$$

$$a_1(X)\left[\left(\frac{3c\alpha^2\beta^2 + v - 2a\beta\alpha}{c\alpha^2}\right)X - \frac{d}{c\alpha^2}X^3 + \frac{R}{c\alpha^2}\right] = g(X)a_0(X). \tag{43}$$

Since $a_i(X)(i = 0,1,2)$ are polynomials, then from (40) we deduce that $a_2(X)$, is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees $g(X), a_1(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$, only. Suppose that $g(X) = A_1X + B_0$, then we find $a_0(X)$ and $a_1(X)$ as

$$a_1(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{44}$$

$$a_0(X) = p + \left(B_0A_0 - \frac{2R}{c\alpha^2}\right)X + \left(\frac{B_0^2}{2} + \frac{A_0A_1}{2} - \frac{3c\alpha^2\beta^2 + v - 2a\beta\alpha}{c\alpha^2}\right)X^2 + \frac{1}{2}A_1B_0X^3 + \left(\frac{A_1^2}{8} + \frac{d}{2c\alpha^2}\right)X^4, \tag{45}$$

where p is arbitrary integration constant. Substituting $a_0(X), a_1(X), a_2(X)$ and $g(X)$, in the last equation in (43) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it with aid Maple, we obtain

$$B_0 = 0, \quad A_0 = \pm\frac{\sqrt{-2cd}\,(2a\beta\alpha - 3c\alpha^2\beta^2 - v)}{c\alpha d}, \quad A_1 = \pm\frac{2\sqrt{-2cd}}{c\alpha}, \quad R = 0,$$
$$p = -\frac{4a^2\beta^2\alpha^2 - 12a\beta^3\alpha^3c - 4a\beta\alpha v + 9c^2\alpha^4\beta^4 + 6c\alpha^2\beta^2v + v^2}{c\alpha^2 d}, \tag{46}$$

where α, β and v are arbitrary constants.

Using the conditions (46) into (12), we get

$$Y(\xi) = \pm\frac{\sqrt{-2cd}\,(2a\beta\alpha - 3c\alpha^2\beta^2 - v + dX^2(\xi))}{2cd\alpha}. \tag{47}$$

Combining (47) with (38), we obtain the exact solution to (37) and then the exact solutions to the GNLS equations can be written as

$$u(x,t)=\pm\sqrt{\frac{v-2a\beta\alpha+3c\alpha^2\beta^2}{d}}e^{i(\beta\alpha(x-vt)-\omega t+x_0)}\tanh\left[\sqrt{\frac{2a\beta\alpha-3c\alpha^2\beta^2-v}{2c\alpha^2}}(\alpha(x-vt)+\xi_0)\right].$$

3.3. The first integral method for obtaining exact solutions of systems of NPDEs

Consider the following systems of partial differential equations:

$$\begin{aligned}&\Psi_1(u,v,u_t,v_t,u_x,v_x,u_{tt},v_{tt},u_{xx},v_{xx},...)=0,\\&\Psi_2(u,v,u_t,v_t,u_x,v_x,u_{tt},v_{tt},u_{xx},v_{xx},...)=0.\end{aligned} \tag{48}$$

We use the transformations

$$u(x,t)=f(\xi),\quad v(x,t)=g(\xi),\quad \xi=x-ct. \tag{49}$$

Using Equation (4) to transfer the *systems of NPDEs* (48) to the *systems of ODEs*

$$\begin{aligned}&\Gamma_1(f,g,f',g',...)=0,\\&\Gamma_2(f,g,f',g',...)=0.\end{aligned} \tag{50}$$

Using some mathematical operations, *the systems of ODEs* (50) is converted into a second-order ODE as

$$\Omega(f,f',f'',...)=0. \tag{51}$$

If we let $X(\xi)=f(\xi),\quad Y(\xi)=f'(\xi)$, the Equation (51) is equivalent to the two dimensional autonomous system

$$\begin{cases}X'=Y,\\Y'=\Delta(X,Y).\end{cases} \tag{52}$$

New, we will apply Division Theorem to obtain one first integral to Equation (52) which reduces Equation (51) to a first order integrable ODE. An exact solution to *systems of NPDEs* (48) is then obtained by solving this equation.

3.3. A. Now, we will consider the approximate long water wave equations [Wang et al. (2008)]:

$$\begin{cases} u_t - uu_x - v_x + \alpha u_{xx} = 0, \\ v_t - (uv)_x - \alpha v_{xx} = 0. \end{cases} \tag{53}$$

Making the transformation $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$, $\xi = kx + lt$, we change the ALWW system (53) to the following system of ODEs

$$\begin{cases} lu' - kuu' - kv' + \alpha k^2 u'' = 0, \\ lv' - k(uv)' - \alpha k^2 v'' = 0. \end{cases} \tag{54}$$

By integrating the first equation we have

$$lu - \frac{k}{2}u^2 - kv + \alpha k^2 u' = R_1, \tag{55}$$

where R_1 is integration constant. Rewrite this equation as follows

$$v(\xi) = \frac{l}{k}u - \frac{u^2}{2} + \alpha k u' - \frac{R_1}{k}. \tag{56}$$

Inserting Equation (56) into the second system (54) and integrating the resulting equation, we obtain

$$(\frac{l^2}{k} + R_1)u - \frac{3l}{2}u^2 + \frac{k}{2}u^3 - \alpha^2 k^3 u'' = R_2, \tag{57}$$

where R_2 is integration constant.

If we let $X = f(\xi), Y = \frac{df(\xi)}{d\xi}$, the Equation (57) is equivalent to the two dimensional autonomous system

$$\begin{cases} X' = Y, \\ Y' = \left(\frac{1}{2\alpha^2 k^2}\right) X^3(\xi) - \frac{3l}{2\alpha^2 k^3} X^2(\xi) + \left(\frac{l^2}{\alpha^2 k^4} + \frac{R_1}{\alpha^2 k^3}\right) X(\xi) - \frac{R_2}{\alpha^2 k^3}. \end{cases} \tag{58}$$

Equation (12) is called the first integral to (58). According to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in $C[X,Y]$ such that

$$\begin{aligned}\left.\frac{dq}{d\xi}\right|_{(58)} &= \left.\left(\frac{dq}{dX}.\frac{dX}{d\xi}+\frac{dq}{dY}.\frac{dY}{d\xi}\right)\right|_{(58)} \\ &= \left(\sum_{i=0}^{1} a_i'(X)Y^i\right)(Y)+\left(\sum_{i=0}^{1} i a_i(X)Y^{i-1}\right) \\ &\qquad \times\left(\left(\frac{1}{2\alpha^2k^2}\right)X^3-\frac{3l}{2\alpha^2k^3}X^2+\left(\frac{l^2}{\alpha^2k^4}+\frac{R_1}{\alpha^2k^3}\right)X-\frac{R_2}{\alpha^2k^3}\right) \\ &= (g(X)+h(X)Y)\sum_{i=0}^{1} a_i(X)Y^i,\end{aligned} \tag{59}$$

where prime denotes differentiation with respect to the variable X. By comparing with the coefficients of $Y^i\,(i=2,1,0)$ of both sides of (59), we have

$$a_1'(X)=h(X)a_1(X), \tag{60}$$

$$a_0'(X)=g(X)a_1(X)+h(X)a_0(X), \tag{61}$$

$$a_1(X)\left[\left(\frac{1}{2\alpha^2k^2}\right)X^3-\frac{3l}{2\alpha^2k^3}X^2+\left(\frac{l^2}{\alpha^2k^4}+\frac{R_1}{\alpha^2k^3}\right)X-\frac{R_2}{\alpha^2k^3}\right]=g(X)a_0(X). \tag{62}$$

Since $a_i(X)(i=0,1)$ are polynomials, then from (60) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=1$ only. Suppose that $g(X)=B_0+A_1X$, then we find $a_0(X)$.

$$a_0(X)=\frac{1}{2}A_1X^2+B_0X+A_0, \tag{63}$$

where A_0 is arbitrary integration constant. Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (62) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0=-\frac{l}{k^2\alpha},\quad A_1=\frac{1}{ka},\quad R_1=k^2\alpha A_0,\quad R_2=l\alpha kA_0, \tag{64}$$

where k,l and A_0 are arbitrary constant.

Using the conditions (64), we obtain

$$Y(\xi)=-A_0+\frac{l}{\alpha k^2}X(\xi)-\frac{1}{2\alpha k}X^2(\xi). \tag{65}$$

Combining (65) with (58), we obtain the exact solution to equation (57) and then the exact solution to the ALWW system (53) can be written as

$$\begin{cases} u(x,t) = \dfrac{l}{k} - \dfrac{\sqrt{2\alpha k^3 A_0 - l^2}}{k} \tan\left[\dfrac{\sqrt{2\alpha k^3 A_0 - l^2}}{2k^2\alpha}(kx + lt + \xi_0) \right], \\ v(x,t) = \left(\dfrac{l^2}{2k^2} - \dfrac{2\alpha k^3 A_0 - l^2}{2k^2} - \alpha k A_0 \right) - \dfrac{2\alpha k^3 A_0 - l^2}{k^2} \tan^2\left[\dfrac{\sqrt{2\alpha k^3 A_0 - l^2}}{2k^2\alpha}(kx + lt + \xi_0) \right], \end{cases}$$

for $2\alpha k^3 A_0 > l^2$.

$$\begin{cases} u(x,t) = \dfrac{l}{k} - \dfrac{\sqrt{l^2 - 2\alpha k^3 A_0}}{k} \tanh\left[\dfrac{\sqrt{l^2 - 2\alpha k^3 A_0}}{2k^2\alpha}(kx + lt + \xi_0) \right], \\ v(x,t) = \left(\dfrac{l^2}{2k^2} - \dfrac{2\alpha k^3 A_0 - l^2}{2k^2} - \alpha k A_0 \right) - \dfrac{l^2 - 2\alpha k^3 A_0}{k^2} \tanh^2\left[\dfrac{\sqrt{l^2 - 2\alpha k^3 A_0}}{2k^2\alpha}(kx + lt + \xi_0) \right], \end{cases}$$

for $2\alpha k^3 A_0 < l^2$.

4. Conclusion

In this paper, the first integral method has been used to construct exact traveling wave solutions of nonlinear partial differential equations, the Landau-Ginburg-Higgs equation and generalized form of the nonlinear Schrödinger equation and approximate long water wave equations. The performance of this method is found to be reliable and effective and it gives more solutions. The method has the advantages of being direct and concise. The method proposed in this paper can also be extended to solve some nonlinear evolution equations in mathematical physics.

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