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Distributional Properties of Record Values of the Ratio of Independent Exponential and Gamma Random Variables

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Abstract

Both exponential and gamma distributions play pivotal roles in the study of records because of their wide applicability in the modeling and analysis of life time data in various fields of applied sciences. In this paper, a distribution of record values of the ratio of independent exponential and gamma random variables is presented. The expressions for the cumulative distribution functions, moments, hazard function and Shannon entropy have been derived. The maximum likelihood, method of moments and minimum variance linear unbiased estimators of the parameters, using record values and the expressions to calculate the best linear unbiased predictor of record values, are obtained.

Keywords: Moments, exponential distribution, gamma distribution, Lomax distribution, ratio, record values

MSC 2010: 33C90, 60E05

1. Introduction

Suppose that $(X_n)_{n \geq 1}$ is a sequence of independent and identically distributed $(i.i.d.)$ random variables ($rv's$) with cumulative distribution function (cdf) F . Let

$Y_n = \max(\min)\left\{X_j \mid 1 \le j \le n\right\}$ for $n \ge 1$. We say X_j is an upper (lower) record value of $\{X_n \mid n \ge 1\}$, if $Y_j > (<)\, Y_{j-1}$, $j > 1$. By definition X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\left\{U(n), n \ge 1\right\}$, where $U(n) = \min\left\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n > 1\right\}$ and $U(1) = 1$. For detailed treatment of the record values, see Ahsanullah (2004). From here onward, for simplicity, the *nth* upper record value $X_{U(n)}$ will be denoted by $X(n)$. The development of the general theory of statistical analysis of record values began with the seminal work of Chandler (1952). Further contributions continued with the work of many authors and researchers, among them Ahsanullah (1980, 2004, and 2006), Arnold et al. (1998), Rao and Shanbhag (1998), Nevzorov (2000), Gulati and Padgett (2003), Balakrishnan et al. (2009), Ahsanullah et al. (2010), and Ahsanullah and Hamedani (2010) are notable. It appears from the literature that, despite many researches on record values such as estimation of parameters, prediction of record values, characterizations, reconstruction of past record values (which may also be viewed as the missing records) based on observed records, etc., not much attention was paid to the studies of record values from the ratio of independently distributed random variables. The distribution of the ratio of independent random variables arises naturally in many applied problems of biology, economics, engineering, genetics, hydrology, medicine, number theory, order statistics, physics, psychology, etc. Some of the notable examples are the ratios of inventory in economics, ratios of inheritance in genetics, ratios of target to control precipitation in meteorology, ratios of mass to energy in physics, among others. The distributions of the ratio $|X/Y|$, where X and Y are independent random variables and belong to the same family, have been studied by many researchers, see, for example, Press (1969), Malik and Trudel (1986), Pham-Gia (2000), Nadarajah (2005a, 2006, 2010), Nadarajah and Ali. (2005), Nadarajah and Gupta (2005, 2006), Nadarajah and Dey (2006), Ali et al. (2007), and Khoolenjani and Khorshidian (2009), among others. Recently, some researcher have started looking at the distributions of the ratio $|X/Y|$, when X and Y are independent random variables and belong to different families, see, for example, Nadarajah (2005b), Nadarajah and Kotz (2005a, 2005b, 2006, 2007), Shakil and Kibria (2006), Shakil et al. (2006, 2008), among others. In addition, some researchers have begun to study the record values of the ratio of independent random variables when they belong to the same family, see, for example, Shakil and Ahsanullah (2011) for the record values of the ratio of independent Rayleigh random variables, and Ahsanullah and Shakil (2011) for the record values of the ratio of independent exponential random variables. In this paper, the distribution of record values of the ratio X_1/X_2, when X_1 and X_2 are independent exponential and gamma random variables, is investigated. The organization is as follows. Section 2 contains the distributional properties of record values of the ratio of independent exponential and gamma random variables. In Section 3, some recurrence relations for calculating the higher moments are given. Section 4 contains the estimation of parameters using record values and prediction of record values. Concluding remarks are given in Section 5.

2. Distributional Properties of Record Values of the Ratio of Independent Exponential and Gamma Random Variables

In what follows, the distributional properties of record values of the ratio of independent exponential and gamma random variables are provided. For the sake of completeness, the ratio of Independent exponential and gamma random variables is first given below.

2.1. Ratio of Independent Exponential and Gamma Random Variables

Exponential Distribution: A *rv* X_1 is said to have an exponential distribution if its *cdf* F is given by

$$F(x_1) = 1 - \exp\left(-\tfrac{1}{\lambda} x_1\right),\ x_1 > 0,\ \lambda > 0, \tag{1}$$

where λ is known as the scale parameter of the exponential distribution.

Gamma Distribution: A *rv* X_2 is said to have a gamma distribution if its *cdf* F is given by

$$F(x_2) = \frac{\gamma\left(\sigma, \frac{1}{\beta} x_2\right)}{\Gamma(\sigma)},\ x_2 > 0,\ \beta > 0,\ \sigma > 0, \tag{2}$$

where σ and β are respectively called the shape and scale parameters of the gamma distribution, and $\gamma(a, z) = \int_0^z t^{a-1} e^{-t}\, dt$, $a > 0$, denotes the incomplete gamma function. When the scale parameter $\beta = 1$ in (2), the *rv* X_2 is said to have a standard gamma distribution. For detailed treatment on exponential and gamma distributions, the interested readers are referred to Johnson et al. (1994).

Suppose that $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\sigma)$ are independent exponential and gamma random variables distributed according to (1) and (2) respectively. Suppose $X = X_1 / X_2$. Then, using the Equation (3.381.4), Page 317, Gradshteyn and Ryzhik (1980), the *cdf* of the *rv* X is obtained as

$$F_X(x) = \int_0^\infty F_{X_1}(x\, x_2)\, f_{X_2}(x_2)\, dx_2 = 1 - \left(\tfrac{\lambda}{\lambda + x}\right)^\sigma, \tag{3}$$

where $x > 0$, $\lambda\ (> 0)$ is the scale parameter and $\sigma\ (> 0)$ is the shape parameter of the distribution of the ratio $X = X_1 / X_2$. Obviously, the *rv* X with the *cdf* (2.3) has the Lomax (or Pareto II) distribution [see Lomax (1954)]. Applications of the Lomax distribution mainly lie in the fields of business, economics and reliability modeling, see, for example, Giles et al. (2011), and references therein. Many researchers have also studied the applications of Lomax distribution to order statistics and record values, among them David (1981), Arnold and

Balakrishnan (1989), Ahsanullah (1991, 2004), Balakrishnan and Ahsanullah (1994) are notable. Also see Lee and Lim (2009). Note that the Lomax distribution considered by Balakrishnan and Ahsanullah (1994) is a particular case of Equation (2.3) for $\lambda = 1$. Many properties of the record values of the distribution of the ratio of independently distributed *rv's* $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\sigma)$ can be derived in a similar way to the record values of Lomax distribution. However, for the sake of completeness and without loss of generality, the distribution of the record values of the ratio X_1 / X_2 of the independently distributed *rv's* $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\sigma)$ is independently investigated in this paper. Using the Equation (3.194.3), Page 285, Gradshteyn and Ryzhik (1980), the *kth* moment of the *rv* $X = X_1 / X_2$ is obtained as

$$E[X^k] = \sigma \lambda^k B\left(k+1, \sigma - k\right), \tag{4}$$

where $\sigma > 0$, $\lambda > 0$, and $k < \sigma$, and $B(.)$ denotes the beta function. Clearly, the *kth* moment of the ratio $X = X_1 / X_2$ exists only when $k < \sigma$, where $k > 0$ is an integer. Also it is evident from Equation (4) that, since the $Var\left[X\right]$ exists only when $\sigma > 2$, the distribution of the *rv* $X = X_1 / X_2$ has very thick (that is, heavy) tails.

2.2. Record Values of the Ratio of Exponential and Gamma Random Variables

2.2.1. Many properties of the upper record value sequence can be expressed in terms of the cumulative hazard function, $R(x) = -\ln \overline{F}(x)$, where $0 < \overline{F}(x) < 1$ and $\overline{F}(x) = 1 - F(x)$. If we define $F_n(x)$ as the *cdf* of $X(n)$ for $n \geq 1$, then we have

$$F_n(x) = \int_{-\infty}^{x} \frac{(R(u))^{n-1}}{\Gamma(n)} dF(u),$$

where $-\infty < x < \infty$, see Ahsanullah (2004). Hence using the above equation, the *cdf* $F_n(x)$ of the *nth* record value $X(n)$ of the ratio $X = X_1 / X_2$, where $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\sigma)$, is given by

$$F_n(x) = \frac{\gamma\left(n, -\ln\left(\frac{\lambda}{\lambda + x}\right)^{\sigma}\right)}{\Gamma(n)}, \quad n = 1, 2, 3, \ldots, \tag{5}$$

where x, σ, $\lambda > 0$. Note that the distribution of the first record value is the distribution of the parent ratio of exponential and gamma random variables. The possible shapes of the *pdf* $f_n(x)$ corresponding to the *cdf* (5) of the *nth* record value $X(n)$, when $n = 2$ and 5, are provided for some select values of the parameters λ and σ in Figures 1 (a) and (b) respectively. The effects of the parameters can easily be seen from these graphs. For example, it is clear from the plotted Figures 1 (a) and (b), for selected values of the parameters, the distributions of the random variable $X(n)$ are unimodal and positively (that is, right) skewed with longer and heavier right tails.

(a) $n = 2 : \sigma = 0.5,\ \lambda = 2$, dashdot; $\sigma = 1,\ \lambda = 1$, solid; $\sigma = 2,\ \lambda = 0.5$, longdash.

(b) $n = 5 : \sigma = 0.5,\ \lambda = 5$, dashdot; $\sigma = 1,\ \lambda = 1$, solid; $\sigma = 2,\ \lambda = 0.5$, longdash.

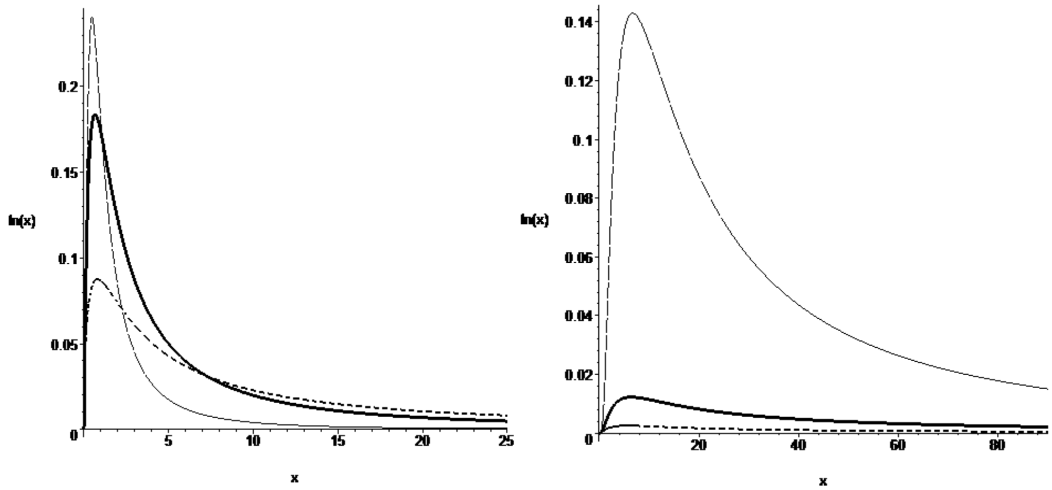

Figure 1: PDF Plots for (a) $n = 2$ (above left); and (b) $n = 5$ (above right).

2.2.2. Moments: The *kth* moment of the *nth* record value $X(n)$ with the *cdf* (5) is given by

$$E[X^k(n)] = \int_0^\infty x^k \frac{\left[-\ln\left(\frac{\lambda}{\lambda+x}\right)^\sigma\right]^{n-1}}{\Gamma(n)} \frac{\sigma\lambda^\sigma}{[\lambda+x]^{\sigma+1}}\, dx. \tag{6}$$

Substituting $-\ln\left(\frac{\lambda}{\lambda+x}\right)^\sigma = u$ in equation (6), the *kth* moment is obtained as

$$E[X^k(n)] = \sigma^n \lambda^k \sum_{j=0}^{\infty} (-1)^j \frac{k(k-1)\ldots(k-j+1)}{(j!)(\sigma-k+j)^n}, \tag{7}$$

where σ, $\lambda > 0$, $1 \le k < \sigma$, and $k \ge 1$ is an integer. Taking $n = 1$ in (7), and applying the series representation of the beta function (see Eq. 3.382.1, page 950, Gradshteyn and Ryzhik (1980)), the *kth* moment (4) of the first record value can easily be obtained.

2.2.3. Hazard Function: The hazard function of $X(n)$ with *cdf* (2.5) is given by

$$h_n(x) = \frac{f_n(x)}{1 - F_n(x)} = \frac{\sigma\lambda^\sigma\left[-\ln\left(\frac{\lambda}{\lambda+x}\right)^\sigma\right]^{n-1}}{\left[\Gamma(n) - \gamma\left(n, -\ln\left(\frac{\lambda}{\lambda+x}\right)^\sigma\right)\right][\lambda+x]^{\sigma+1}}, \tag{8}$$

where x, σ, $\lambda > 0$, and $n = 1, 2, 3, \ldots$. The possible shapes of the hazard function $h_n(x)$ in Eq. (8), when $n = 2$ and 5, are provided for some selected values of the parameter λ and σ in Figures 2 (a) and (b) respectively. The effects of the parameters can easily be seen from these graphs. For example, it is clear from the plotted Figures 2 (a) and (b), for selected values of the parameters, the hazard functions have upside down bathtub shapes, and are unimodal and positively (that is, right) skewed with longer and heavier right tails.

(a) $n = 2 : \sigma = 1, \lambda = 1$, dot; $\sigma = 1, \lambda = 2$, dash; $\sigma = 1, \lambda = 3$, longdash; $\sigma = 1, \lambda = 5$, solid.

(b) $n = 5 : \lambda = 1, \sigma = 1$, dot; $\lambda = 1, \sigma = 2$, dash; $\lambda = 1, \sigma = 3$, longdash; $\lambda = 1, \sigma = 5$, solid.

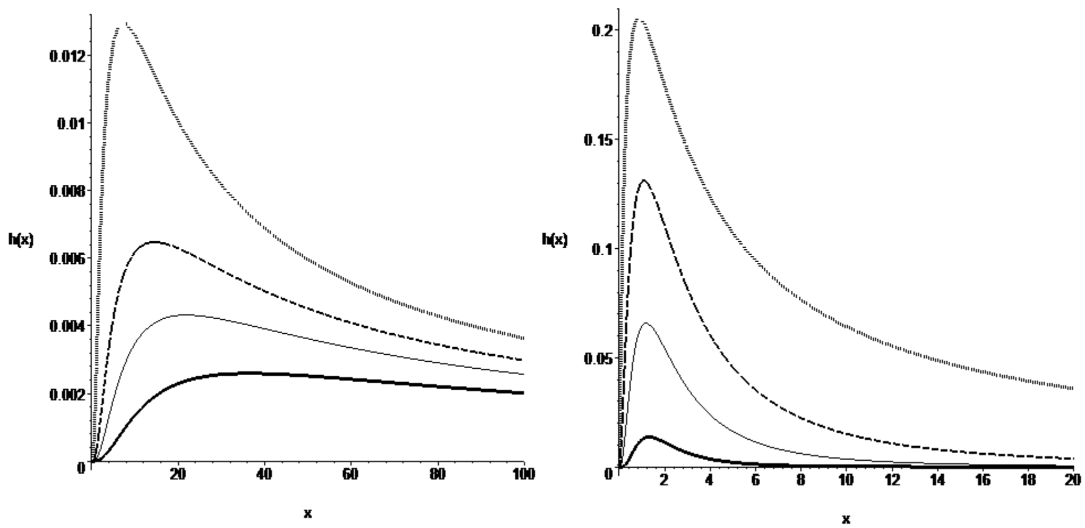

Figure 2: Hazard Function Plots for (a) $n = 2$, $\sigma = 1$ and $\lambda = 1, 2, 3, 5$ (above left); and (b) $n = 5$, $\lambda = 1$ and $\sigma = 1, 2, 3, 5$ (above right).

2.2.4. Entropy: Further, as proposed by Shannon (1948), entropy of an absolutely continuous random variable X having the probability density function $\phi_X(x)$ is defined as

$$H[X] = E\left[-\ln\{\phi_X(x)\}\right] = -\int_S \phi_X(x) \ln\{\phi_X(x)\} dx, \tag{9}$$

where $S = \{x : \phi_X(x) > 0\}$. Entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution (population). A large value of entropy implies the greater uncertainty in the data. Using the *pdf* corresponding to the *cdf* (5) in Eq. (9), and applying the Eqs. 3.194.3/p. 950, 4.293.14/p. 558, and 4.352.1/p. 576 of Gradshteyn and Ryzhik (1980), we obtain the Shannon entropy $H_{(n)}$ of the *nth* record value $X(n)$ with *cdf* (5) as

$$H_{(n)} = \ln\left\{\frac{\lambda\,\Gamma(n)}{\sigma}\right\} + n\left(1 + \frac{1}{\sigma}\right) - (n-1)\,\psi(n), \tag{10}$$

where $n \geq 1$ is an integer , and $\psi(z)$ denotes digamma function, see Abramowitz and Stegun (1970).

Remark: From Equation (10), it follows that

(i) the sequence $\{H_{(n)}\}$ is monotonic increasing in n, $\forall\ \sigma,\ \lambda > 0$, which follows from the definition of digamma function and the inequality $\frac{1}{2z} < \ln(z) - \psi(z) < \frac{1}{z}$, $z > 0$, see Alzer (1997);

(ii) the sequence $\{H_{(n)}\}$ is a monotonic increasing function of λ, $\forall\ \sigma,\ \lambda > 0$, and $n \geq 1$ (an integer), which follows from (10) by differentiating it with respect to λ;

(iii) the sequence $\{H_{(n)}\}$ is a monotonic decreasing convex function of σ, $\forall\ \sigma,\ \lambda > 0$, and $n \geq 1$ (an integer), which follows from (10) by differentiating it twice with respect to σ.

2.3. Representation of Records

Consider the *cdf* of $X = \dfrac{X_1}{X_2}$, where $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\sigma)$, given by the Equation (3), from which we have

$$F^{-1}(x) = \lambda[(1-x)^{-\frac{1}{\sigma}} - 1], \qquad ,$$

where x, σ, $\lambda > 0$. Suppose that

$$g_F(x) = -\ln(1 - F(x)) = \ln\left(\frac{\lambda + x}{\lambda}\right)^{\sigma}. \qquad .$$

Then,

$$\sum_{i=1}^{n} g_F(x_i) = \sum_{i=1}^{n} \ln\left(\frac{\lambda + x_i}{\lambda}\right)^{\sigma}$$

$$= \ln \prod_{i=1}^{n} \left(\frac{\lambda + x_i}{\lambda}\right)^{\sigma}.$$

Thus,

$$1 - e^{-\sum_{i=1}^{n} g_F(x_i)} = 1 - e^{-\ln \prod_{i=1}^{n}\left(\frac{\lambda + x_i}{\lambda}\right)^{\sigma}} = 1 - \prod_{i=1}^{n}\left(\frac{\lambda}{\lambda + x_i}\right)^{\sigma}.$$

Further, since the inverse of the function $g_F(x) = -\ln(1 - F(x))$ is given by

$$g_F^{-1}(x) = F^{-1}(1 - e^{-x}),$$

it follows that

$$g_F^{-1}\left(\sum_{i=1}^{n} g_F(x_i)\right) = F^{-1}\left[1 - \prod_{i=1}^{n}\left(\frac{\lambda}{\lambda + x_i}\right)^{\sigma}\right]$$

$$= \lambda\left\{\left[\prod_{i=1}^{n}\left(\frac{\lambda}{\lambda + x_i}\right)^{\sigma}\right]^{-\frac{1}{\sigma}} - 1\right\}.$$

Hence, using the Representation Theorem 8.4.1, page 256, Ahsanullah (2004), we have

$$X_{U(n)} \stackrel{d}{=} \lambda\left\{\prod_{i=1}^{n}\left(\frac{\lambda}{\lambda + X_i}\right)^{-1} - 1\right\}, \tag{11}$$

where X_i, $i = 1, 2, \ldots, n$ are $i.i.d.$ with the cdf (3). If $n = 1$, then from Equation (11), we have

$$X_{U(1)} \stackrel{d}{=} X\ .$$

Thus the moment of the first record value $X_{U(1)}$ is given by Equation (4) of the moment $E\left(X^k\right)$ of the parent distribution of the ratio $X = X_1 / X_2$. It is obvious from Equation (4) that, for $k \geq \sigma$, $E\left(X^k\right)$ fails to exist. Now, for some $\delta > 0$, we consider

$$E\left[X\right]^{k+\delta} = \int_0^{\infty} x^{k+\delta}\, \frac{\sigma\lambda^{\sigma}}{[\lambda + x]^{\sigma+1}}\, dx$$

$$= \frac{\sigma}{\lambda}\int_0^{\infty} \frac{x^{k+\delta}}{\left[1 + \frac{x}{\lambda}\right]^{\sigma+1}}\, dx,$$

from which, by letting $\dfrac{x}{\lambda} = u$, and applying Equation (3.194.3), Page 285, Gradshteyn and Ryzhik (1980), we have

$$E\left[X\right]^{k+\delta} = \sigma\lambda^{k+\delta} B\left(k + \delta + 1, \sigma - k - \delta\right),$$

which is finite only when $k + \delta < \sigma$ for some $\delta > 0$. Thus, by Theorem 1.3.1, page 15, Ahsanullah (2004), it follows that the *kth* moment $E\,[X^k(n)]$ of the *nth* upper record value $X_{U(n)}$ of the ratio $X = X_1 / X_2$, where $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\sigma)$, exists for all $n \geq 1$, provided $k + \delta < \sigma$ for some $\delta > 0$.

3. Recurrence Relations for Higher Moments

In this section, we derive some recurrence relations by which higher moments can easily be calculated. The expressions for the variance and covariance are also given.

3.1. Variance and Covariance

Equation (11) can be simplified to

$$X_{U(n)} \overset{d}{=} \lambda\left(\tfrac{X_1+\lambda}{\lambda}\tfrac{X_2+\lambda}{\lambda}\ldots\tfrac{X_n+\lambda}{\lambda}-1\right),$$

where the *pdf* of X_i is $\sigma\lambda^{\sigma}(\lambda+x)^{-(\sigma+1)}$. Now

$$E(X_{U(n)}+\lambda)=\lambda\left(E\left(\tfrac{X+\lambda}{\lambda}\right)\right)^n,$$

where $E\left(\frac{X+\lambda}{\lambda}\right)=\int_0^\infty \sigma\lambda^{\sigma-1}(\lambda+x)^{-\sigma}\,dx=\frac{\sigma}{\sigma-1},\ \sigma>1$. Thus,

$$E(X_{U(n)}+\lambda)=\lambda\left(\tfrac{\sigma}{\sigma-1}\right)^n,$$

from which we obtain

$$E(X_{U(n)})=\lambda\left(\left(\tfrac{\sigma}{\sigma-1}\right)^n-1\right)=\lambda\,\delta_n, \tag{12}$$

where

$$\delta_n=\left(\tfrac{\sigma}{\sigma-1}\right)^n-1,\ \sigma>1.$$

We can calculate higher moments of $X_{U(n)}$ as follows. From Equation (11), we have

$$E(X_{U(n)}+\lambda)^k=\lambda^k\left(E\left(\tfrac{X+\lambda}{\lambda}\right)^k\right)^n.$$

Taking $k=2$ in the above equation, we have, for $\sigma>2$,

$$E(X_{U(n)}+\lambda)^2=\lambda^2\left(\tfrac{\sigma}{\sigma-2}\right)^n,$$

since

$$E\left(\tfrac{X+\lambda}{\lambda}\right)^2=\int_0^\infty\left(\tfrac{x+\lambda}{\lambda}\right)^2\sigma\lambda^{\sigma}(\lambda+x)^{-\sigma-1}\,dx$$

$$=\int_0^\infty\sigma\lambda^{\sigma-2}(\lambda+x)^{-\sigma+1}\,dx$$

$$= \sigma \lambda^{\sigma - 2} \lambda^{2 - \sigma} \tfrac{1}{\sigma - 2}$$

$$= \tfrac{\sigma}{\sigma - 2}, \ \sigma > 2 .$$

The variance of $X_{U(n)}$ is obtained as

$$Var(X_{U(n)}) = Var(X_{U(n)} + \lambda)$$

$$= \lambda^2 [(\tfrac{\sigma}{\sigma - 2})^n - (\tfrac{\sigma}{\sigma - 1})^{2n}], \ \sigma > 2 . \tag{13}$$

Again, from Eq. (11), we have

$$\tfrac{X_{U(n)} + \lambda}{\lambda} \stackrel{d}{=} \tfrac{X_{U(n-k)} + \lambda}{\lambda} . (\tfrac{X_1 + \lambda}{\lambda}).(\tfrac{X_2 + \lambda}{\lambda}) ... (\tfrac{X_k + \lambda}{\lambda}),$$

where $X_1, X_2, ..., X_k$ are $i.i.d.$ with the cdf as $(\frac{\lambda}{\lambda + x})^{\sigma}$. Also $X_1, X_2, ..., X_k$ are independent of $X_{U(n-k)}$ and $X_{U(n)}$. Thus we have

$$E(\tfrac{X_{U(n)} + \lambda}{\lambda} \tfrac{X_{U(n-k)} + \lambda}{\lambda}) = E(\tfrac{X_{U(n-k)} + \lambda}{\lambda})^2 \prod_{i=1}^k E((\tfrac{X_i + \lambda}{\lambda})) = (\tfrac{\sigma}{\sigma - 2})^{n-k} (\tfrac{\sigma}{\sigma - 1})^k,$$

from which we obtain

$$E(\tfrac{X_{U(n)} + \lambda}{\lambda} \tfrac{X_{U(n-k)} + \lambda}{\lambda}) - E(\tfrac{X_{U(n)} + \lambda}{\lambda})(E \tfrac{X_{U(n-k)} + \lambda}{\lambda})$$

$$= (\tfrac{\sigma}{\sigma - 2})^{n-k} (\tfrac{\sigma}{\sigma - 1})^k - (\tfrac{\sigma}{\sigma - 1})^n (\tfrac{\sigma}{\sigma - 1})^{n-k}$$

$$= (\tfrac{\sigma}{\sigma - 1})^k [(\tfrac{\sigma}{\sigma - 2})^{n-k} - (\tfrac{\sigma}{\sigma - 1})^{2(n-k)}].$$

Thus,

$$Cov(X_{U(n-k)}, X_{U(n)}) = \lambda^2 Cov(\tfrac{X_{U(n)} + \lambda}{\lambda}, \tfrac{X_{U(n-k)} + \lambda}{\lambda})$$

$$= \lambda^2 (\tfrac{\sigma}{\sigma - 1})^k [(\tfrac{\sigma}{\sigma - 2})^{n-k} - (\tfrac{\sigma}{\sigma - 1})^{2(n-k)}]$$

$$= (\tfrac{\sigma}{\sigma - 1})^k Var(X_{U(n-k)}). \tag{14}$$

3.2. Recurrence Relations for Moments

From Equation (3), we have

$$(\lambda + x)\, f_X(x) = \sigma\left[1 - F_X(x)\right],\ x, \sigma, \lambda > 0,$$

where $f_X(x)$ denotes the *pdf* corresponding to the *cdf* (3) of the *rv* $X = X_1 / X_2$. Using the above equation, the following recurrence relations for the single and product moments of the record values of the *rv* $X = X_1 / X_2$ are obtained from Ahsanullah (2004), Section 4.2, with $\beta = 1/\sigma$ and $\sigma = \lambda$, or following Balakrishnan and Ahsanullah (1994). For the sake of brevity, the proofs are omitted here.

Result 1. For $n \geq 2$, $r = 0,1,2, \ldots,$ and $r+1 < \sigma$,

$$E(X_{U(n)}^{r+1}) = \lambda\left(\tfrac{r+1}{\sigma-r-1}\right) E(X_{U(n)}^{r}) + \left(\tfrac{\sigma}{\sigma-r-1}\right) E(X_{U(n+1)}^{r+1}).$$

Result 2. For $1 \leq m \leq n-2$, $r, s = 0,1,2, \ldots,$ $r+1 < \sigma$, and $s+1 < \sigma$,

$$E(X_{U(m)}^{r}\ X_{U(n)}^{s+1}) = \lambda\left(\tfrac{s+1}{\sigma-s-1}\right) E(X_{U(m)}^{r}\ X_{U(n)}^{s}) + \left(\tfrac{\sigma}{\sigma-s-1}\right) E(X_{U(m)}^{r}\ X_{U(n-1)}^{s+1}).$$

Result 3. For $1 \leq m \leq n-2$, and $\sigma > 1$,

$$Cov(X_{U(m)}, X_{U(n)}) = \left(\tfrac{\sigma}{\sigma-1}\right) Cov(X_{U(m)}, X_{U(n-1)}).$$

Result 4. For $1 \leq m \leq n-1$, and $\sigma > 1$,

$$Cov(X_{U(m)}, X_{U(n)}) = \left(\tfrac{\sigma}{\sigma-1}\right)^{n-m} Var(X_{U(m)}).$$

Result 5. For $2 \leq m \leq n-2$, and $r, s = 0,1,2, \ldots,$

$$E(X_{U(m)}^{r+1} X_{U(n)}^{s}) = \left(\tfrac{\sigma}{r+1}\right)\left[E(X_{U(m)}^{r+1}\ X_{U(n-1)}^{s}) - E(X_{U(m-1)}^{r+1}\ X_{U(n-1)}^{s})\right] - E(X_{U(m)}^{r} X_{U(n)}^{s}).$$

4. Estimation and Prediction

In this section, we derive the minimum variance linear unbiased estimators (MVLUE's) of the parameters of the record values from the distribution of the *rv* $X = X_1 / X_2$. The estimation of parameters using the methods of moments (MOM) and maximum likelihood estimation (MLE) are also given. The expressions to calculate the Best Linear Unbiased Predictor (BLUP) of record values are obtained. For details on MVLUE and BLUP, one can visit Lloyd (1952), Sarhan and Greenberg (1962), David (1981), and Ahsanullah (2004), among others.

4.1. Estimating λ for Known σ Based on n (Upper) Records

Knowing σ, the MVLUE of λ using upper record values can be obtained following Lloyd's method (1952) for deriving MVLUE as shown below. Let $X_{U(1)}$, $X_{U(2)}$, ... , $X_{U(n)}$ be n upper record values of the *rv* $X = X_1 / X_2$ with the *cdf* F given by (3). Let

$$X^{/} = (X_{U(1)} , X_{U(2)} ,..., X_{U(n)}) .$$

Suppose

$$\alpha' = (\delta_1 - 1, \delta_2 - 1, ..., \delta_n - 1) = H^{/} - L^{/} ,$$

$$\delta_j = (\tfrac{1}{\sigma - 1})^j , \ \sigma > 1 ,$$

$$H^{/} = (\delta_1, \delta_2, ..., \delta_n),$$

$$L' = (1,1,...,1) ,$$

and

$$V = (V_{i,j}) ,$$

where $V_{i,j}$ is the covariance of $X_{U(i)}$ and $X_{U(j)}$, given by

$$V_{i,j} = a_i b_j , \ 1 \le i < j \le n , \ V_{j,i} = V_{i,j} ,$$

with

$$a_i = (\tfrac{\sigma - 1}{\sigma})^i [(\tfrac{\sigma}{\sigma - 2})^i - (\tfrac{\sigma}{\sigma - 1})^{2i}], \text{ and } b_j = (\tfrac{\sigma}{\sigma - 1})^j .$$

Then $E(X) = \lambda\alpha$ and $Var(X) = \lambda^2 V$.

It follows that

$$V_{i,i} = Var(X_{U(i)}) = a_i b_i = [(\tfrac{\sigma}{\sigma - 2})^i - (\tfrac{\sigma}{\sigma - 1})^{2i}],$$

and

$$V_{i,j} = a_i b_j = (\tfrac{\sigma}{\sigma - 1})^{j-i} Var(X_{U(i)}) = (\tfrac{\sigma}{\sigma - 1})^{j - i} a_i \, b_i .$$

Note that V is a pattern matrix and its inverse is well known, see Graybill (1983). Let $V^{-1} = (V^{i,j})$. Then, the inverse V^{-1} can be expressed as

$$V^{i+1,i} = V^{i,i+1}$$

$$= -\frac{1}{a_{i+1}b_i - a_i b_{i+1}}, \; i = 1, 2, \ldots, n-1,$$

$$V^{i,i} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, \; i = 1, 2, \ldots, n-1,$$

$$V^{n,n} = \frac{b_{n-1}}{b_n(a_n b_{n-1} - a_{n-1}b_n)},$$

and

$$V^{i,j} = 0 \text{ for } |i - j| > 1,$$

where $a_0 = b_{n+1} = 0$ and $b_0 = a_{n+1} = 1$.

On simplification of the above, it can be shown that

$$V^{i,i} = (2\sigma^2 - 4\sigma + 1)c^i, \; i = 1, 2, \ldots, n-1,$$

$$V^{i+1,i} = V^{i,i+1} = -(\sigma^2 - 3\sigma + 2)c^i, \; i = 1, 2, \ldots, n-1,$$

$$V^{n,n} = (\sigma^2 - 2\sigma + 1)c^n,$$

$$Vi, j = 0 \text{ for } |i - j| > 1,$$

where

$$c = \frac{\sigma - 2}{\sigma}, \; \sigma > 2.$$

Now, let λ^* be the MVLUE of λ. Then, using Lloyd's method (1952) for deriving MVLUE, we have

$$\lambda^* = \frac{\alpha' V^{-1} X}{\alpha' V^{-1}\alpha} = \frac{H' V^{-1} X - L' V^{-1} X}{\alpha' V^{-1}\alpha},$$

from which we have

$$E(\lambda^*) = E(\tfrac{\alpha' V^{-1} X}{\alpha' V^{-1} \alpha}) = \lambda ,$$

and

$$Var(\lambda^*) = \tfrac{\lambda^2}{\alpha' V^{-1} \alpha} .$$

Suppose

$$q_1 = p_1 - \tfrac{(\sigma-2)^2}{\sigma T_1 - \sigma + 2} ,$$

$$q_j = -\left(\sigma - 1\right) p_j , \; j = 2, \ldots, n ,$$

$$p_1 = \tfrac{\sigma(\sigma-1)T_1 - (\sigma-2)}{\sigma(\sigma T_1 - \sigma + 2)} ,$$

$$p_j = \tfrac{-\left(\frac{\sigma-2}{\sigma}\right)^j}{\sigma T_1 - \sigma + 2} , \; j = 2, \ldots, n-1 ,$$

$$p_n = \tfrac{(\sigma-1) T_1 - \frac{(\sigma-1)^2 (\sigma-2)}{\sigma}}{\sigma T_1 - \sigma + 2} ,$$

$$T_1 = \sum_{j=1}^{n} \left(\tfrac{\sigma-2}{\sigma}\right)^j , \; \sigma > 2 ,$$

and

$$T_2 = \left(\sigma - 2\right)\left(\sigma T_1 - \sigma + 2\right), \; \sigma > 2 .$$

Then, we have

$$\lambda^* = \sum_{j=1}^{n} q_j \, X_{U(j)} ,$$

and

$$Var(\lambda^*) = \tfrac{\lambda^2 \left[T_1 + (\sigma-2)^2 \right]}{T_2} .$$

Further, based on the first n upper record values, the best linear invariant estimator (BLIE) $\tilde{\lambda}$ of λ is given by

$$\tilde{\lambda} = \tfrac{\lambda^* T_2}{T_1 \, (\sigma-1)^2} ,$$

see Ahsanullah (2004). Hence, the mean squared error (MSE) of $\tilde{\lambda}$ is given by

$$MSE\left(\tilde{\lambda}\right) = Var\left(\lambda^*\right)\frac{T_2}{T_1\,(\sigma-1)^2},$$

that is,

$$MSE\left(\tilde{\lambda}\right) = \frac{\lambda^2\left[T_1 + (\sigma-2)^2\right]}{T_1\,(\sigma-1)^2}.$$

In the following Table 1, the variances and the mean squared errors of MVLUE and BLIE, respectively, are calculated for some selected values of σ and n.

Table 1: The variances and the mean squared errors of MVLUE and BLIE $(\lambda = 1)$

n	σ	$Var(\lambda^*)$	$MSE(\tilde{\lambda})$
2			
	3	4.3333	0.8125
	5	1.8444	0.6484
	8	1.3819	0.5802
	10	1.2781	0.5610
	20	1.1170	0.5276
5			
	3	3.0333	0.7521
	5	0.8840	0.4691
	8	0.5186	0.3415
	10	0.4412	0.3062
	20	0.3268	0.2463
10			
	3	3.0001	0.7500
	5	0.7850	0.4398
	8	0.3887	0.2799
	10	0.3049	0.2336
	20	0.1834	0.1559

It is evident from the Table 1 that the mean squared errors of BLIE, that is, $MSE\left(\tilde{\lambda}\right)$, are considerably smaller than those of the MVLUE, that is, $Var(\lambda^*)$. As σ increases, both $MSE\left(\tilde{\lambda}\right)$ and $Var(\lambda^*)$ decrease.

4.2. Method of Moments (MOM) Estimation of the Parameters λ and σ

Suppose that the sample *kth* raw moment of the *nth* record value $X(n)$ with the *cdf* (5) be denoted by $\hat{\mu}_k$. Now, since from Equation (12), we have

$$E(X_{U(n)}) = \mu_1 = \lambda((\tfrac{\sigma}{\sigma-1})^n - 1) = \lambda\delta_n,$$

where

$$\delta_n = \left(\tfrac{\sigma}{\sigma-1}\right)^n - 1,\ \sigma > 1,$$

therefore we can take $\hat{\lambda}$ as the MOM estimator of λ given by

$$\hat{\lambda} = \tfrac{\hat{\mu}_1}{\delta_n}.$$

Using Equations (12) and (13), it can be seen that

$$\mu_2 + 2\lambda\mu_1 + \lambda^2 = \lambda^2\left(\frac{\sigma}{\sigma-2}\right)^n,$$

from which we obtain

$$\sigma = \frac{2}{1 - \left[\dfrac{(\hat{\lambda})^2}{\mu_2 + 2\hat{\lambda}\hat{\mu}_1 + (\hat{\lambda})^2}\right]^{\frac{1}{n}}},$$

which is the required MOM estimator of σ.

4.3. Maximum Likelihood Estimation (MLE) of the Parameters λ and σ

In this section, we find the MLE's of λ and σ based on n upper record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ of the *rv* $X = X_1 / X_2$ with the *cdf* F given by (3).

Using Equation (3), the log-likelihood function is given by

$$\begin{aligned}\ln L(\sigma,\lambda \mid x_1,x_2,\ldots,x_n) &= \ln f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n)\\ &= \ln\left[f(x_n)\prod_{i=1}^{n-1}\frac{f(x_i)}{1-F(x_i)}\right]\\ &= n\ln\sigma+\sigma\ \ln\lambda-\sigma\ \ln(\lambda+x_n)-\sum_{i=1}^{n}\ln(\lambda+x_i). \end{aligned} \tag{15}$$

Setting the partial derivatives of Equation (15) with respect to σ and λ equal to zero, that is, $\frac{\partial \ln L}{\partial\sigma}=0$ and $\frac{\partial \ln L}{\partial\lambda}=0$, and solving these iteratively gives the MLE's $\hat{\sigma}_{MLE}$ and $\hat{\lambda}_{MLE}$ of σ and λ respectively. Further, assuming that λ is known, then, using Eq. (15), the MLE $\hat{\sigma}_{MLE}$ of σ is obtained by solving the following equation

$$\frac{\partial \ln L}{\partial\sigma}=0,$$

which gives

$$\hat{\sigma}_{MLE}=\frac{n}{\ln(\lambda+x_n)-\ln\lambda}.$$

Also see Abd Ellah (2006) for Bayesian and non-Bayesian estimates using record statistics from Lomax distribution.

4.4. Best Linear Unbiased Predictor (BLUP)

In this section, we derive the expressions to calculate the BLUP of record values. Suppose

$$\alpha'=(\delta_1-1,\ \delta_2-1,\ldots,\ \delta_n-1),$$

$$\delta_j=\left(\tfrac{1}{\sigma-1}\right)^j,$$

$$X'=(X_{U(1)},X_{U(2)},\ldots,X_{U(n)}),$$

$$W'=(w_1,w_2,\ldots,w_n),$$

$$w_k=\left(\tfrac{\sigma}{\sigma-1}\right)^k v_{k,k},$$

and

$$v_{k,k} = Var(X_{U(k)}) = (\tfrac{\sigma}{\sigma-2})^k - (\tfrac{\sigma}{\sigma-1})^{2k}.$$

Then the BLUP of $X_{L(s)}, s > n,$ is given by

$$X^*_{U(s)} = \lambda^*((\tfrac{\sigma}{\sigma-1})^s - 1) + W V^{-1}(X - \lambda^* \alpha),$$

where λ^* is the MVLUE of λ given by

$$\lambda^* = \frac{\alpha' V^{-1} X}{\alpha' V^{-1} \alpha}.$$

It can be shown that

$$E(X^*_{U(s)}) = \lambda[(\tfrac{1}{\sigma-1})^s - 1] = E(X_{L(s)}).$$

Also, since $X^*_{U(s)}$ is an unbiased estimate of $X_{L(s)}$, we have

$$Var(X^*_{U(s)}) = Var[(\lambda^*(\tfrac{\sigma}{\sigma-1})^s - 1) + W' V^{-1}(X - \lambda^* \alpha)].$$

The above expressions can easily be used to compute the Best Linear Unbiased Predictor (BLUP) of record values for some selected values of σ and n.

5. Concluding Remarks

In this paper, we have discussed the distribution of record values when the parent distribution is the ratio of two independently distributed random $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\sigma)$. The expressions for the cumulative distribution functions, moments, hazard function and Shannon entropy have been derived. The maximum likelihood, method of moments and minimum variance linear unbiased estimators of the parameters, using record values and the expressions to calculate the best linear unbiased predictor of record values, are obtained. The findings of this paper can be useful for the practitioners in various fields of studies and further enhancement of research in record value theory and its applications.

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