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The Power Series Solution of Fingering Phenomenon Arising in Fluid Flow through Homogeneous Porous Media

Kinjal Patel, M. N. Mehta and Twinkle R. Patel

Department of Applied Mathematics & Humanities
S.V. National Institute of Technology
Surat-395007, India
kinjal.svnit@gmail.com;
mnmehta_2001@yahoo.com;
trpatel@ashd.svnit.ac.in

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Abstract

The present paper deals with the approximate solution of the fingering phenomenon occurring when water is pushed into oil in homogeneous porous media with capillary mean pressure. The phenomenon is formulated mathematically as a water-oil double phase flow problem. The solution of the nonlinear partial differential equation of fingering phenomenon has been discussed in terms of the power series using appropriate boundary conditions for any time $T > 0$. The solution is in ascending power series which represents saturation of injected fluid in fingering phenomenon & its graphical and numerical presentation is given in MATLAB coding.

Keywords: Instability phenomenon, double phase flow, capillary pressure, power series solution

MSC 2010 No.: 76T99, 76505

1. Introduction

In oil recovery process oil is produced by simple natural decompression without any pumping effort at the wells. This stage is referred to as primary recovery, and it ends when a pressure equilibrium between the oil field and the atmosphere occurs. Primary recovery usually leaves 70%–85% of oil in the reservoir. To recover part of the remaining oil, a fluid (usually water) is injected into some wells (injection wells) while oil is produced through other wells (production wells). This process serves to maintain high reservoir pressure and flow rates. It also displaces some of the oil and pushes it toward the production wells. This stage of oil recovery is called secondary recovery process and during this process, immiscible displacement of one fluid by another in porous medium several complex physical phenomena occur simultaneously of the pore scale, shown in Figure 1 (Chen and Hunan, 2006).

This paper presents the important phenomenon of fingering (instability) in a double phase flow through homogeneous porous medium. It is a very well-known physical fact that when a fluid flowing through a porous medium is displaced by another fluid of lesser viscosity then, instead of regular displacement of whole front, protuberance takes place which shoot through the porous medium at a relatively very high speed. This phenomenon is called fingering/instability as shown in Figure 2.

Many researchers have discussed this phenomenon from a variety view points. The injected fluid, intended to push the native fluid forward, tends to penetrate the more viscous native fluid through spontaneously formed multi-branched fingers (Marle, 1981). Saffman and Taylor (1958) derived a classical result for the shape of finger in the absence of capillary. Scheidegger (1960) considered the average cross-sectional area occupied by the fingers while the size and shape of the individual fingers were neglected. Verma (1969) has discussed the statistical behavior of the fingering phenomenon in a displacement process in heterogeneous porous medium with capillary pressure using perturbation solution.

Recently many researchers have discussed the shape, size and velocity prediction of fingers under different situation with different view (Lange et al., 1998; Brailovsky et al., 2006; Zhan and Yortsos, 2000 Wang and Feyen, 1998). Brailovsky et al. (2006) numerically formulated a non-linear evolution equation for water oil displacement front. They also discussed a way to control the unrestricted growth of fingers by the injection of water not uniformly but rather during suitably distributed intervals of time. Joshi and Mehta (2009) have discussed the solution by the Group Invariant method of the Instability phenomenon arising in fluid flow through porous media.

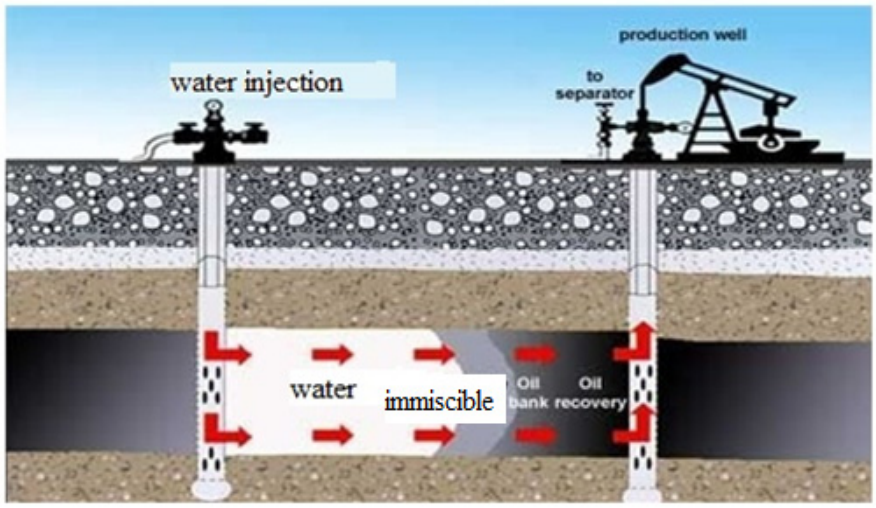

Figure 1: Secondary oil recovery process in oil reservoir

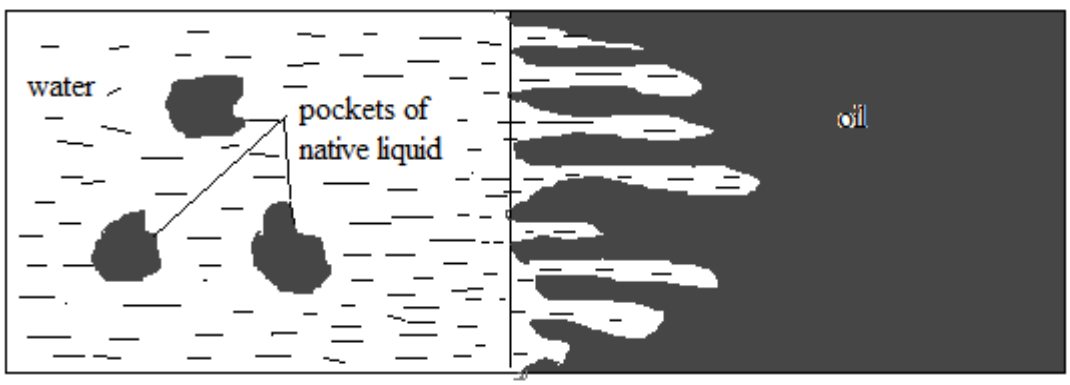

Figure 2: Actual formations of fingers in the porous media

2. Statement of the Problem

In secondary oil recovery process, when a fluid (water) is injected in oil formation to drive oil towards a production well, the phenomenon occurs. We considered that there is a uniform injection of less viscous fluid (injected fluid) into more viscous fluid (native fluid) in saturated homogeneous porous medium of length L such that the injected fluid (I) shoots through the native fluid (N) and give rise to protuberances as per figure (3). This furnishes a well developed fingers flow. Since the entire native fluid at the initial boundary $x = 0$ is displaced through a small distance due to injection, it is further assumed that nearly complete saturation exists at the initial boundary.

For the mathematical formulation, it has been assumed that Darcy's law is valid for the investigated flow system and assumed further that the macroscopic behavior of fingers is

governed by a statistical treatment. Hence only the average behavior of the two fluids involved is taken into consideration (Scheidegger, 1960). The saturation of the displacing fluid in a porous medium represents the average cross sectional area occupied by the fingers.

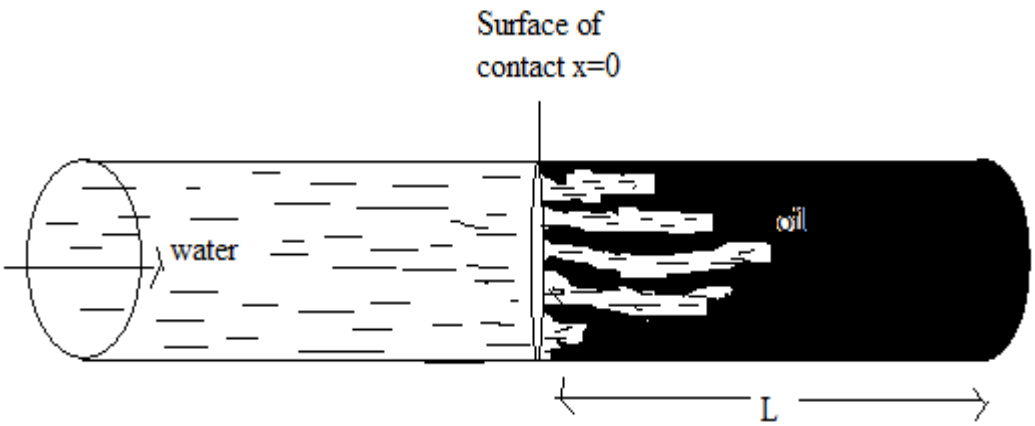

Figure 3: Formation of fingers in the cylindrical piece of porous media

3. Mathematical Formulation

During the injection process, let the injected fluid (I) and the native fluid (N) be two immiscible fluids governed by Darcy's law expressed as Bear (Bear, 1972)

$$V_i = -\frac{k_i}{\mu_i} K \frac{\partial p_i}{\partial x} \tag{1}$$

$$V_n = -\frac{k_n}{\mu_n} K \frac{\partial p_n}{\partial x}, \tag{2}$$

where K is the permeability of the homogenous porous medium, k_i and k_n the relative permeability's of the displacing fluid, which are function of saturations S_i and S_n, p_i and p_n are pressures of displacing injected fluid (water) and native fluid (oil), μ_i and μ_n are the constant kinematic viscosities of displacing fluids respectively.

The equation of continuity of the two fluid densities are written respectively as

$$P\frac{\partial S_i}{\partial t} + \frac{\partial V_i}{\partial x} = 0 \tag{3}$$

$$P\frac{\partial S_n}{\partial t}+\frac{\partial V_n}{\partial x}=0, \tag{4}$$

where P is the porosity of the homogeneous porous medium.

From the definition of phase saturation (Scheidegger, 1960) gives

$$S_i+S_n=1. \tag{5}$$

When the fluid is injected then the flow takes place in the interconnected capillary. Thus the capillary pressure p_c, defined as continuity of the flowing fluid across their common interface, is a function of the injected fluid saturation. It may be written (Scheidegger, 1960), as

$$p_c\left(S_i\right)=p_n-p_i\ , \tag{6}$$

$$p_c=-\beta S_i\ ;\ \beta \text{ is constant (Mehta, 1977)}\ . \tag{7}$$

For definiteness of the mathematical analysis, the relationship between phase saturation and relative permeability as, given by Scheidegger and Johnson (Scheidegger and Johnson, 1961) is used here.

$$\left.\begin{aligned} k_i &= S_i \\ k_n &= S_n = 1-S_i \end{aligned}\right\}. \tag{8}$$

The equation of motion for saturation is obtained by substituting the values of (v_i) and (v_n) from equations (1) and (2) to the equations (3) and (4) respectively,

$$P\frac{\partial S_i}{\partial t}=\frac{\partial}{\partial x}\left(\frac{k_i}{\mu_i}K\frac{\partial p_i}{\partial x}\right), \tag{9}$$

$$P\frac{\partial S_n}{\partial t}=\frac{\partial}{\partial x}\left(\frac{k_n}{\mu_n}K\frac{\partial p_n}{\partial x}\right). \tag{10}$$

Eliminating $\frac{\partial p_i}{\partial x}$ from equations (6) and (9) we have

$$\frac{\partial}{\partial x}\left[\frac{k_i}{\mu_i}K\left\{\left(\frac{\partial p_n}{\partial x}\right)-\left(\frac{\partial p_c}{\partial x}\right)\right\}\right]=P\frac{\partial S_i}{\partial t}. \tag{11}$$

By combining equation (10) and (11), we get

$$\frac{\partial}{\partial x}\left\{\left(\frac{k_i}{\mu_i}K+\frac{k_n}{\mu_n}K\right)\frac{\partial p_n}{\partial x}-\frac{k_i}{\mu_i}K\frac{\partial p_c}{\partial x}\right\}=0 \ . \tag{12}$$

Integrating equation (12) with respect to 'x', we get

$$\left\{\left(\frac{k_i}{\mu_i}K+\frac{k_n}{\mu_n}K\right)\frac{\partial p_n}{\partial x}-\frac{k_i}{\mu_i}K\frac{\partial p_c}{\partial x}\right\}=-V, \tag{13}$$

where V is a constant of integration. Simplify (13), we get

$$\frac{\partial p_n}{\partial x}=-\frac{V}{K\left(\frac{k_i}{\mu_i}+\frac{k_n}{\mu_n}\right)}+\frac{\left(\frac{\partial p_c}{\partial x}\right)}{1+\frac{k_n}{k_i}\frac{\mu_i}{\mu_n}} \ . \tag{14}$$

Now, from equations (11) and (14), the following is obtained

$$P\frac{\partial S_i}{\partial t}+\frac{\partial}{\partial x}\left[\frac{V}{1+\frac{k_n}{k_i}\frac{\mu_i}{\mu_n}}+\frac{\frac{k_n}{\mu_n}K\left(\frac{\partial p_c}{\partial x}\right)}{1+\frac{k_n}{k_i}\frac{\mu_i}{\mu_n}}\right]=0. \tag{15}$$

The value of the pressure of native fluid (p_n) can be written as (Oroveanu, 1963)

$$p_n=\frac{p_n+p_i+p_n-p_i}{2}$$

$$p_n=\overline{p}+\left(1/2\right)p_c\ ,\ \overline{p}=\left(p_n+p_i\right)/2 \ , \tag{16}$$

where $\overline{p}$ is the constant mean pressure.

On differentiating the above equation with respect to 'x' the following equation is obtained

$$\frac{\partial p_n}{\partial x}=\frac{1}{2}\frac{\partial p_c}{\partial x} \ . \tag{17}$$

On substituting the value of $\frac{\partial p_n}{\partial x}$ from (17) to (13) we can obtained

$$V = \left[\frac{K}{2} \left(\frac{k_i}{\mu_i} - \frac{k_n}{\mu_n} \right) \frac{\partial p_c}{\partial x} \right]. \tag{18}$$

On substituting the value of V in (15), we can obtained

$$P \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x} \left[\frac{\frac{K}{2} \left(\frac{k_i}{\mu_i} - \frac{k_n}{\mu_n} \right) \frac{\partial p_c}{\partial x}}{1 + \frac{k_n}{k_i} \frac{\mu_i}{\mu_n}} + \frac{\frac{k_n}{\mu_n} K \left(\frac{\partial p_c}{\partial x} \right)}{1 + \frac{k_n}{k_i} \frac{\mu_i}{\mu_n}} \right] = 0 \ . \tag{19}$$

Simplify this equation we get,

$$P \frac{\partial S_i}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left[K \left(\frac{k_i}{\mu_i} \frac{dp_c}{dS_i} \frac{\partial S_i}{\partial x} \right) \right] = 0. \tag{20}$$

On substituting the value of (p_c) and (k_i) from equation (6) & (8) in the above equation, the following equation is obtained

$$P \frac{\partial S_i}{\partial t} = \frac{\beta K}{2\mu_i} \frac{\partial}{\partial x} \left[\left(S_i \frac{\partial S_i}{\partial x} \right) \right]. \tag{21}$$

The appropriate sets of condition to solve nonlinear equation (21) are the saturation of the injected fluid at the common interface

$$S_i \left(0, t \right) = S_{i0} \qquad \text{at } x = 0 \text{ for } t > 0 \tag{22}$$

The saturation at distance $x = L$ in cylindrical porous matrix will be

$$S_i \left(L, t \right) = S_{i1} \qquad \text{at } x = L \text{ for } t > 0 \ . \tag{23}$$

The initial saturation of the injected fluid is

$$S_i \left(x, o \right) = S_{ic} \text{ at } t = 0 \text{ for } x > 0, \ 0 \le S_{ic} < S_{i0}. \tag{24}$$

The small radiation in the saturation of the injected fluid at the common interface will be

$$\frac{\partial S_i(0,t)}{\partial x} = \varepsilon \text{ at } x = 0 \text{ and } t > 0. \tag{25}$$

This equation (21) is a non-linear partial differential equation which describes the fingering phenomenon in fluid flow through a homogeneous porous medium.

4. Power Series Solution of the Problem

We choose new dimensionless variable

$$X = \frac{x}{L} \quad \text{and} \quad T = \frac{\beta K}{2PL^2 \mu_i} t.$$

Substituting this value in equation (21), together with boundary condition (22), (24) and (25), we have

$$\frac{\partial S_i}{\partial T} = \frac{\partial}{\partial X}\left[S_i \frac{\partial S_i}{\partial X} \right] \tag{26}$$

$$S_i(0,T) = S_{i0} \text{ at } X = 0 \text{ for } T > 0, \tag{27}$$

$$S_i(X,o) = S_{ic} \text{ at } T = 0 \text{ for } X > 0,\ 0 \le S_{ic} < S_{i0}, \tag{28}$$

$$\frac{\partial S_i(0,T)}{\partial X} = \varepsilon \text{ at } X = 0 \text{ and } T > 0. \tag{29}$$

The above condition is sufficient to solve equation (26).

We use similar transformation,

$$S_i(X,T) = f(\eta), \text{ where } \eta = \frac{X}{2\sqrt{T}} \text{ (Mehta, 1977) .} \tag{30}$$

The governing equation (26) reduce to the ordinary differential equation

$$f(\eta)\, f''(\eta) = -f'(\eta)\left[f'(\eta) - 2\eta\right] . \tag{31}$$

Together with boundary conditions

$$f(0) = S_{i0},\, X = 0,\, T > 0 \quad , \tag{32}$$

$$f(\infty) = S_{ic},\, T = 0,\, X > 0 \quad . \tag{33}$$

The resulting two point boundary value problems involve some difficulties, because condition (33) is not easy to satisfy. Fortunately (Mehta, 1977) equation (31) and the initial conditions (32) are such that each of the family of curves satisfying (32) tends to a horizontal asymptote as. $\eta \to \infty$ at initial stage. Hence the boundary value problem can be transformed into an initial value problem by replacing condition (33) by the variation condition (34).

$$f'(0) = \omega \neq 0 \; for\ any\ T > 0 \qquad \text{(very small)} . \tag{34}$$

To find successive coefficients of the Maclaurin's series at $\eta = 0$. We take nth derivative of equation (31) solve for $f^{(n+2)}(\eta)$ and evaluate at $\eta = 0$, giving

$$f^{(n+2)}(0) = \frac{1}{f(0)}\Big[\, n\, f^{n}(0)\left\{2 - f''(0)\right\} - f^{(n+1)}(0)(1+n)\, f'(0) + \\ + \sum_{k=2}^{n}\binom{n}{k}\left\{f^{k+1}(0)\, f^{(n-k+1)}(0) - f^{(k)}(0)\, f^{(n-k+2)}(0)\right\}\Big] \;,\; n \geq k \;,\; n = 1, 2, \ldots . \tag{35}$$

For the solution, it is necessary to determine the derivatives $f^{(n)}(0)$ for all n = 1, 2, 3, … . The derivative $f'(0)$ can be determined by means of formula (34) and $f''(0)$ from equation (31). Further, all other higher derivatives can be determined from formula (35) by putting $n \geq 1$, 2, 3, ... Thus, the desired value of $f(\eta)$ can be computed by the Maclaurin's series.

$$f(\eta) = \sum_{k=0}^{\infty} f^{(k)}(0)\, \frac{\eta^{k}}{k!} \tag{36}$$

$$f(\eta) = f(0) + \eta f'(0) + \frac{\eta^{2}}{2!} f''(0) + \frac{\eta^{3}}{3!} f'''(0) + \frac{\eta^{4}}{4!} f^{iv}(0) + \cdots \quad . \tag{37}$$

$$f(\eta) = S_{i0} + \eta\omega - \frac{\eta^2\omega^2}{2!(S_{i0})} + \frac{\eta^3}{3!(S_{i0})^2}\left[3\omega^3 + 2\omega(S_{i0})\right] - \frac{\eta^4}{4!(S_{i0})^3}\left[9\omega^4 + 8\,\omega^2(S_{i0})\right] \pm \cdots \tag{38}$$

From equation (30), we get

$$S_i(X,T) = S_{i0} + \frac{X\omega}{2\sqrt{T}} - \frac{X^2\omega^2}{8\left(\sqrt{T}\right)^2(S_{i0})} + \frac{X^3}{48\left(\sqrt{T}\right)^3(S_{i0})^2}\left[3\omega^3 + 2\,\omega(S_{i0})\right] - \\ - \frac{X^4}{384\left(\sqrt{T}\right)^4(S_{i0})^3}\left[9\omega^4 + 8\omega^2(S_{i0})\right] \pm \cdots . \tag{39}$$

Equation (36) represents the saturation of the injected fluid during the fingering phenomenon.

5. Convergence Study

Equation (37) represents Maclaurin's series in form of η while equation (39) represents the same series in the original values of X and T; so convergence of Maclaurin's series is sufficient to discuss convergence of equation (39). From equation (37) $u_{k+1} = f^{k+1}(0)\frac{\eta^{k+1}}{(k+1)!}$ and $u_k = f^k(0)\frac{\eta^k}{k!}$. Hence, by Ratio Test,

$$\lim_{k\to\infty}\frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty}\left|\frac{f^{k+1}(0)}{f^k(0)}\right|\left|\frac{\eta}{k}\right| = 0 < 1.$$

Since $f^{k+1}(0)$ and $f^k(0)$ are constant which is not zero and $\eta = \frac{X}{2\sqrt{T}} \neq 0$. Hence the equation (37) is absolutely convergent and is therefore a convergent series.

The uniqueness can also be discussed but the physical problem does not requires that at this stage, since to our main interest is getting classical results as well as numerical & graphical presentation.

6. Numerical & Graphical Presentations

Numerical and graphical presentations of equation (39) have been obtained by using MATLAB coding. Figure 4 shows the graph of $S_i(X,T)$ vs. X for time $T = 0.1, 0.2, 0.3, 0.4, 0.5$, and Table 1 represent the numerical data. Figure 5 is the graph of $S_i(X,T)$ vs. T for distance $X = 0.1, 0.2, 0.3, 0.4, 0.5$, Table 2 represents the numerical data. Both figures display the graphical representations of the phenomenon showing the behavior of the injected fluid.

Table 1

Distance X	$S_i(X,T)$ T=0.1	$S_i(X,T)$ T=0.2	$S_i(X,T)$ T=0.3	$S_i(X,T)$ T=0.4	$S_i(X,T)$ T=0.5
0.0	0.2	0.2	0.2	0.2	0.2
0.1	0.2016	0.2011	0.2009	0.2008	0.2007
0.2	0.2032	0.2023	0.2018	0.2016	0.2014
0.3	0.2048	0.2034	0.2028	0.2024	0.2022
0.4	0.2064	0.2046	0.2038	0.2033	0.2030
0.5	0.2080	0.2059	0.2049	0.2042	0.2038
0.6	0.2096	0.2072	0.2060	0.2053	0.2047
0.7	0.2111	0.2086	0.2072	0.2063	0.2057
0.8	0.2124	0.2100	0.2085	0.2075	0.2068
0.9	0.2136	0.2116	0.2099	0.2088	0.2080
1	0.2145	0.2131	0.2115	0.2103	0.2094

Table 2

Time T	$S_i(X,T)$ X =0.1	$S_i(X,T)$ X =0.2	$S_i(X,T)$ X =0.3	$S_i(X,T)$ X =0.4	$S_i(X,T)$ X =0.5
0.1	0.2016	0.2032	0.2048	0.2064	0.2080
0.2	0.2011	0.2023	0.2034	0.2046	0.2059
0.3	0.2009	0.2018	0.2028	0.2038	0.2049
0.4	0.2008	0.2016	0.2024	0.2033	0.2042
0.5	0.2007	0.2014	0.2022	0.2030	0.2038
0.6	0.2006	0.2013	0.2020	0.2027	0.2035
0.7	0.2006	0.2012	0.2018	0.2025	0.2033
0.8	0.2006	0.2011	0.2017	0.2024	0.2031
0.9	0.2005	0.2011	0.2016	0.2022	0.2029
1	0.2005	0.2010	0.2016	0.2021	0.2027

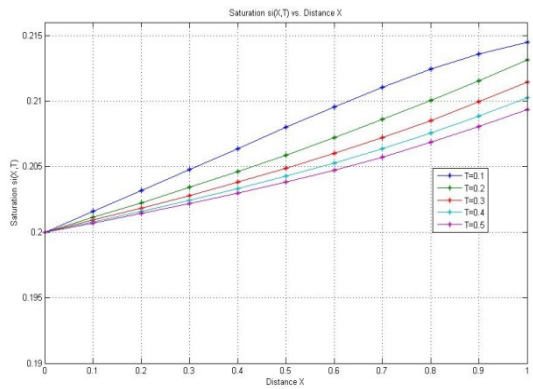

Figure 4: Saturation of injected fluid at different distance when
T = 0.1, 0.2, 0.3, 0.4, 0.5 and **ω** = 0.01, $\mathbf{S_{i0}}$ = 0.2 fixed

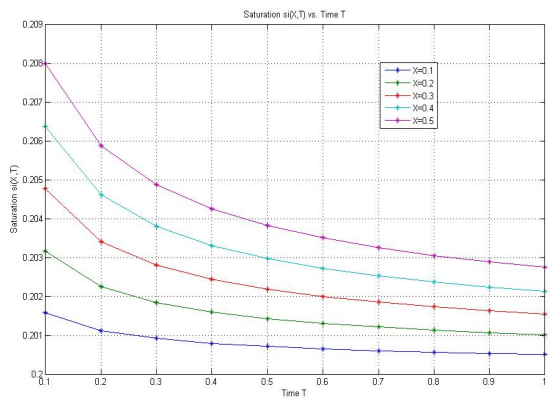

Figure 5: Saturation of injected fluid at different time when
X = 0.1, 0.2, 0.3, 0.4, 0.5 and **ω** = 0.01, $\mathbf{S_{i0}}$ = 0.2 fixed.

7. Conclusion

Here we have obtained an approximate solution of the fingering phenomenon in the form of a power series. The problem has great importance in petroleum technology and the behaviour of fingers is determined by statistical treatment. The equation (26) is yields a nonlinear differential equation and its solution are obtained by using Maclaurin's series under certain specific boundary conditions.

For numerical value it is necessary to find the co-efficient of Maclaurin's series by using the original conditions given for $\eta = 0$. Thus from (32) & (34) we have $f(0) = S_{i0}$ and $f'(0) = \omega \neq 0$. The equation (39) shows saturation of injected fluid at any distance X from common interface for any $T > 0$. These solutions (38) and (39) satisfy both boundary conditions

i.e., $f(0) = S_i(0,T), = S_{i0}$ *and* $f(\infty) = S_i(X,0) = S_{ic}$ When $S_{ic} \to \infty$ (initial maximum saturation).

The power series solution demonstrates that the saturation of the injected fluid $S_i(X,T)$ is increasing smoothly as the distance is increasing. It starts from the initial value $S_i(0,T) = S_{i0}$ and will subsequently stabilize after a long time. It also shows that the saturation of the injected fluid is decreasing when any injected fluid comes into contact with the native fluid. Hence saturation will decrease for any time T for a fixed X. It is consistent with the fingering phenomenon. The numerical and graphical presentation was obtained by using MATLAB coding.

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