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Algorithms to Solve Singularly Perturbed Volterra Integral Equations

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Abstract

In this paper, we apply the Differential Transform Method (DTM) and Variational Iterative Method (VIM) to develop algorithms for solving singularly perturbed volterra integral equations (SPVIEs). The study outlines the significant features of the two methods. A comparison between the two methods for the solution of SPVIs is given for three examples. The results show that both methods are very efficient, convenient and applicable to a large class of problems.

Keywords: Differential Transform Method, Variational Iterative Method, Singularly Perturbed Volterra Integral Equations

MSC (2010) No.: 35A 15, 34B 10, 45D 05, 45G 05

1. Introduction

In recent years, much attention has been paid to finding solutions for singularly perturbed volterra integral equations (SPVIEs). The aim of this paper is to continue this trend and consider new analytical techniques, the Differential Transform Method (DTM) and the Variational Iterative Method (VIM) for solving SPVIEs of the form

$$\varepsilon y(x) = g(x) + \int_0^x K(x,t,y(t))dt, \qquad 0 < x < \eta, \tag{1.1}$$

where $\varepsilon > 0$ is a small positive parameter called the 'perturbation parameter' that gives rise to the singularly perturbed nature of the problem Alnaser (2000), Lange and Smith (1988), Angel and Olmstead (1987). The kernel K and the function $g(x)$ are given smooth functions. Under appropriate condition on g and K, for every $\varepsilon > 0$, (1.1) has a unique continuous solution on $[0,\eta]$ see Brunner (1986) and Alnaser (2000). It should be mentioned that in order to use the DTM, the solution of (1.1) must be analytic.

The singularly perturbed nature of (1.1) arises when the properties of the solution with $\varepsilon > 0$ are incompatible with those when $\varepsilon = 0$. For $\varepsilon > 0$, (1.1) is an integral equation of the second kind. When $\varepsilon = 0$, (1.1) reduces to an integral equation of the first kind whose solution may be incompatible with the case $\varepsilon > 0$.

Problems of this nature imply incompatibility in the behavior of y near $x = 0$. This suggests the existence of boundary layer near the origin where the solution undergoes a rapid transition Brunner (1986).

Angel and Olmstead (1987), Lange and Smith (1988) developed a formal methodology to obtain asymptotic solution for (1.1). Alnaser (2000) applied a multi-step method to solve singular perturbation problem in Volterra integral equation. Finally, Alnaser and Momany (2008) used Homotopy perturbation method to solve the presented problem.

In Section (2), we apply DTM to solve our problem. In Section (3), we use VIM to give approximate solution for the proposed problem. Test examples with known exact solutions are presented at the end of each section to discuss the accuracy and efficiency of the methods. Finally, our conclusion will be given in Section (4).

2. Solving SPVIEs Using DTM

Consider the general form of SPVIE which is given in (1.1). Now, applying differential transform to (1.1) we get

$$\varepsilon Y(k) = G(k) + \frac{H(k-1)}{k}, \qquad k \geq 1, \tag{2.1}$$

where $H(k)$ is the differential transform of the kernel $K(x,t,y(t))$. Thus, the recurrence formula is

$$Y(k) = \frac{1}{\varepsilon}\left\{G(k) + \frac{H(k-1)}{k}\right\}, \qquad k \geq 1. \tag{2.2}$$

Substituting $x = 0$ in (1.1), we get

$$y(0) = \frac{g(0)}{\varepsilon}.$$

Therefore, the transformed initial condition at $x = 0$ is

$$Y(0) = \frac{g(0)}{\varepsilon}.$$

Starting with $Y(0)$ and the recurrence formula in (2.2), $Y(1)$ can be determined. Now, using $Y(0)$, $Y(1)$, then $Y(2)$ is easily identified. Continuing in this manner, the first N-differential transforms of $y(x)$ can be identified. Finally, the inverse transform of $Y(k)$ is

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k. \tag{2.3}$$

Details about DTM and its properties can be found in Alquran and Al-Khaled (2010), Kanth and Aruna (2009) and Erturk (2007).

2.1. Numerical Examples

In this section we discuss three different examples. The result will be compared with the exact solution for various values of ε.

Example 1.

Consider the following linear problem

$$\varepsilon y(x) = \int_0^x \left(1 + t - y(t)\right)dt, \tag{2.4}$$

which has the exact solution

$$y(x) = x + 1 - e^{\frac{-x}{\varepsilon}} - \varepsilon\left(1 - e^{\frac{-x}{\varepsilon}}\right). \tag{2.5}$$

Equation (2.4) can be written in the form

$$\varepsilon y(x) = x + \frac{x^2}{2} - \int_0^x y(t)dt. \tag{2.6}$$

Applying DT to (2.6), we get

$$Y(k) = \frac{1}{\varepsilon}\left\{\delta(k-1) + \frac{1}{2}\delta(k-2) - \frac{Y(k-1)}{k}\right\}, \quad k \geq 1. \tag{2.7}$$

Since $y(0) = 0$, then the transformed initial condition is $Y(0) = 0$. Now, we coded (2.7) in Mathematica and obtained

$$Y(1) = \frac{1}{\varepsilon}, \quad Y(2) = \frac{\varepsilon - 1}{2\varepsilon^2}, \quad Y(3) = -\frac{\varepsilon - 1}{6\varepsilon^3},$$

$$Y(4) = \frac{\varepsilon - 1}{24\varepsilon^4}, \quad Y(5) = -\frac{\varepsilon - 1}{120\varepsilon^5}, \quad Y(6) = \frac{\varepsilon - 1}{720\varepsilon^6}, \ldots\ldots$$

Thus, the approximate solution around $x_0 = 0$ can be expressed as:

$$y_{appr}(x) = \frac{1}{\varepsilon}x + \frac{\varepsilon - 1}{2\varepsilon^2}x^2 - \frac{\varepsilon - 1}{6\varepsilon^3}x^3 + \frac{\varepsilon - 1}{24\varepsilon^4}x^4 - \frac{\varepsilon - 1}{120\varepsilon^5}x^5 + \frac{\varepsilon - 1}{720\varepsilon^6}x^6 + \ldots.$$

Figure 1 represents the absolute errors between the exact solution and the approximate solution for ($0 < \varepsilon < 1$) and considering 25 terms of the DT series.

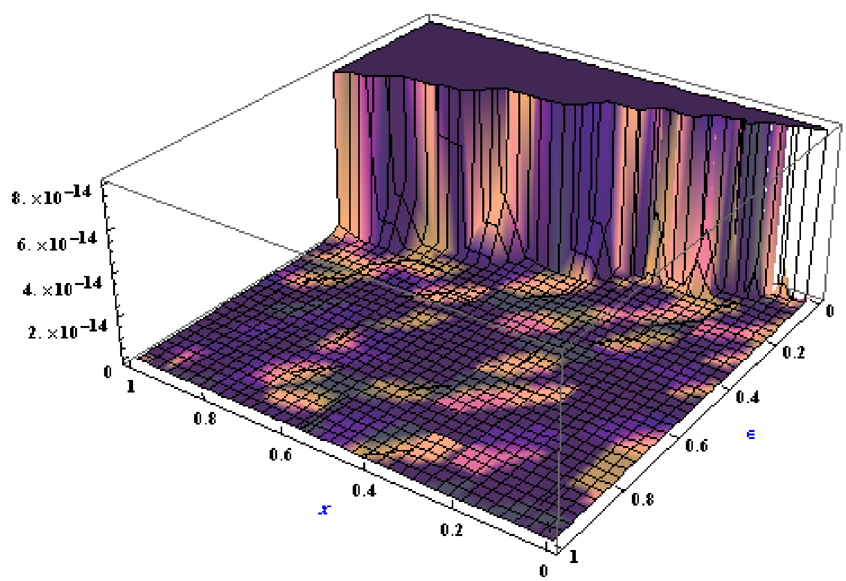

Figure 1. Absolute errors of equation (2.4) using DTM, for $N = 25$

Example 2.

Consider the following linear problem

$$\varepsilon y(x) = \int_0^x (1+x-t)(1+t-y(t))dt, \tag{2.8}$$

the exact solution is given by

$$y(x) = 1+x+\frac{-e^{x\gamma_2}\left(\gamma_1 - 1 + \frac{1}{\varepsilon}\right) + e^{x\gamma_1}\left(\gamma_2 - 1 + \frac{1}{\varepsilon}\right)}{\gamma_1 - \gamma_2}, \tag{2.9}$$

where

$$\gamma_1 = \frac{\sqrt{1-4\varepsilon}-1}{2\varepsilon},$$

and

$$\gamma_2 = \frac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}.$$

Equation (2.9) can be written as

$$\varepsilon y(x) = x + x^2 + \frac{x^3}{6} - \int_0^x y(t)dt - x\int_0^x y(t)dt + \int_0^x ty(t)dt. \tag{2.10}$$

Note that for $f(x) = x\int_0^x y(t)dt$, then its DT is

$$F(k) = \sum_{i=1}^{k} \frac{\delta(k-i-1)Y(i-1)}{i},$$

also, for $g(x) = \int_0^x ty(t)$, then its DT is

$$G(k) = \frac{1}{k}\sum_{i=0}^{k-1}\delta(i-1)Y(k-i-1).$$

Accordingly, the differential transform for (2.10) is

$$\begin{aligned} Y(k) = \frac{1}{\varepsilon}\{&\delta(k-1)+\delta(k-2)+\frac{1}{6}\delta(k-3)-\frac{Y(k-1)}{k} \\ &-\sum_{i=1}^{k}\frac{\delta(k-i-1)Y(i-1)}{i}+\frac{1}{k}\sum_{i=0}^{k-1}\delta(i-1)Y(k-i-1)\},\ k\geq 1 \end{aligned} \tag{2.11}$$

and the transformed initial condition used is $Y(0)=0$. By coding (2.11) in Mathematica, we obtain

$$Y(1)=\frac{1}{\varepsilon},\quad Y(2)=\frac{-1+2\varepsilon}{2\varepsilon^2},\quad Y(3)=\frac{1-3\varepsilon+\varepsilon^2}{6\varepsilon^3},$$

$$Y(4)=-\frac{1-4\varepsilon+3\varepsilon^2}{24\varepsilon^4},\quad Y(5)=-\frac{-1+5\varepsilon-6\varepsilon^2+\varepsilon^3}{120\varepsilon^5},\ \ldots.$$

The approximate solution is

$$y_{appr}(x)=\frac{1}{\varepsilon}x+\frac{-1+2\varepsilon}{2\varepsilon^2}x^2+\frac{1-3\varepsilon+\varepsilon^2}{6\varepsilon^3}x^3-\frac{1-4\varepsilon+3\varepsilon^2}{24\varepsilon^4}x^4-$$

$$\frac{-1+5\varepsilon-6\varepsilon^2+\varepsilon^3}{120\varepsilon^5}x^5+\ldots \tag{2.12}$$

Figure 2 represents the absolute errors between the exact solution and the approximate solution for ($0.2<\varepsilon<1$) and considering 18 terms of the DT series.

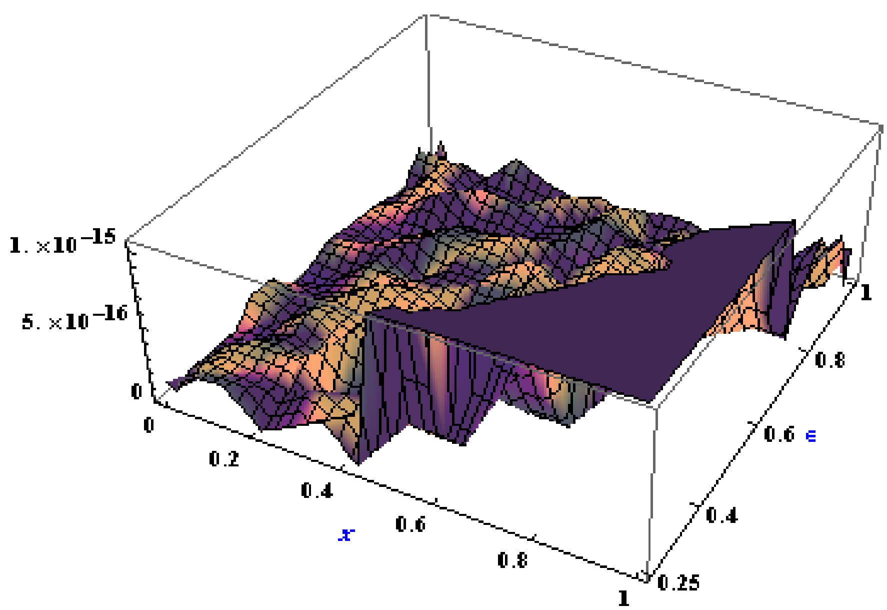

Figure 2. Absolute errors of Equation (2.8) using DTM, for $N=18$

Example 3.

Consider the following non-linear problem

$$\varepsilon y(x) = \int_0^x e^{x-t}\left(y^2(t) - 1\right)dt, \tag{2.13}$$

which has the exact solution

$$y(x) = \frac{2\left(1 - e^{x\gamma}\right)}{1 + \gamma + e^{x\gamma}(-1+\gamma)\varepsilon}, \tag{2.14}$$

where

$$\gamma = \frac{\sqrt{4+\varepsilon^2}}{\varepsilon}.$$

Equation (2.13) can be written in the form

$$\varepsilon y(x) = 1 - e^x + e^x \int_0^x e^{-t} y^2(t)dt. \tag{2.15}$$

Let $f(x) = e^x$, then

$$F(k) = \frac{1}{k!}.$$

Also, let $h(x) = e^{-t} y^2(t)$, then the differential transform of $h(x)$ is

$$H(k) = \sum_{i_2=0}^{k}\sum_{i_1=0}^{i_2} \frac{(-1)^{i_1}}{i_1!} Y(i_2 - i_1)Y(k - i_2).$$

Applying DT to (2.15), we get

$$Y(k) = \frac{1}{\varepsilon}\left\{\delta(k) - \frac{1}{k!} + \sum_{k_1=1}^{k} \frac{1}{k_1} G(k-k_1)H(k_1 - 1)\right\}, k \geq 1, \tag{2.16}$$

since the initial condition is $y(0) = 0$, then its transform $Y(0) = 0$. By using $Y(0) = 0$ and the recurrence formula in (2.16) we obtain

$$Y(1) = -\frac{1}{\varepsilon}, \quad Y(2) = -\frac{1}{2\varepsilon}, \quad Y(3) = -\frac{-2+\varepsilon^2}{6\varepsilon^3}, \quad Y(4) = -\frac{-8+\varepsilon^2}{24\varepsilon^3},$$

$$Y(5) = -\frac{16 - 22\varepsilon^2 + \varepsilon^4}{120\varepsilon^5}, \quad Y(6) = -\frac{136 - 52\varepsilon^2 + \varepsilon^4}{720\varepsilon^5}, \ldots.$$

Thus, the approximate solution around $x_0 = 0$ is

$$y_{appr}(x) = -\frac{1}{\varepsilon}x - \frac{1}{2\varepsilon}x^2 - \frac{-2+\varepsilon^2}{6\varepsilon^3}x^3 - \frac{-8+\varepsilon^2}{24\varepsilon^3}x^4 - \frac{16-22\varepsilon^2+\varepsilon^4}{120\varepsilon^5}x^5 + -\frac{136-52\varepsilon^2+\varepsilon^4}{720\varepsilon^5}x^6 + \dots \quad . \tag{2.17}$$

The following table shows the absolute error of (2.13) for different values of ε

Table 1: The absolute error of Equation (2.13) using DTM for $N = 32$

x	$\varepsilon = 1$	$\varepsilon = 0.9$	$\varepsilon = 0.8$	$\varepsilon = 0.5$
$x = 0.0$	0	0	0	0
$x = 0.2$	1.22×10^{-15}	1.49×10^{-2}	3.32×10^{-2}	1.23×10^{-1}
$x = 0.4$	1.25×10^{-10}	2.53×10^{-2}	5.83×10^{-2}	1.52×10^{-1}
$x = 0.6$	7.56×10^{-8}	2.91×10^{-2}	5.80×10^{-2}	1.19×10^{-1}
$x = 0.8$	7.05×10^{-8}	2.72×10^{-2}	5.11×10^{-2}	6.99×10^{-1}

3. Solving SPVIEs Using VIM

Consider the general form of SPVIE given in (1.1). To apply the VIM to this problem, we have to differentiate (1.1) to get

$$\varepsilon y'(x) = g'(x) + \frac{d}{dx}\left[\int_0^x K(x,t,y(t))dt\right], \qquad 0 < x < \eta. \tag{3.1}$$

To solve (3.1) we assume that the kernel function $K(x,t,y(t))$ is nonlinear with respect to $y(x)$. According to VIM we can construct the following correction functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\left(\varepsilon y_{n'}(t) - g'(t) - \frac{d}{dt}\left[\int_0^t K(t,s,y_n(s))ds\right]\right)dt, \tag{3.2}$$

where $\lambda(x,t)$ is the general lagrange multiplier, which can be identified using the variational theory, and n denoted the n^{th} iteration. Now, making this correction functional stationary, $\delta y_n(0) = 0$, we obtain:

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(x,t)\left(\varepsilon y_{n'}(t) - g'(t) - \frac{d}{dt}\left[\int_0^t K(t,s,y_n(s))ds\right]\right)dt,$$

$$= \delta y_n(x) + \varepsilon \int_0^x \lambda(x,t)\delta y_n'(t)dt = 0, \tag{3.3}$$

where $g'(t)$ and $\frac{d}{dt}\left[\int_0^t K(t,s,y_n(s))ds\right]$ are restricted variations i.e., $\delta g'(t) = 0$ and

$$\delta \frac{d}{dt}\left[\int_0^t K(t,s,y_n(s))ds\right] = 0 .$$

Integrating (3.3) by parts yields

$$\delta y_{n+1}(x) = \delta y_n(x) + \varepsilon\lambda(x,t)\delta y_n(t)|_{t=x} - \varepsilon\int_0^x \frac{\partial}{\partial t}\lambda(x,t)\delta y_n(t)dt$$
$$= \left(1+\varepsilon\lambda(x,t)\right)\delta y_n(t)|_{t=x} - \varepsilon\int_0^x \frac{\partial}{\partial t}\lambda(x,t)\delta y_n(t)dt = 0.$$

Therefore, the general lagrange multiplier satisfies

$$\frac{\partial}{\partial t}\lambda(x,t) = 0 ,$$

subject to

$$\lambda(x,t)|_{t=x} = -\frac{1}{\varepsilon} .$$

Solving the above equation, we get

$$\lambda(x,t) = -\frac{1}{\varepsilon} . \tag{3.4}$$

Thus, the correction functional becomes

$$y_{n+1}(x) = y_n(x) - \frac{1}{\varepsilon}\int_0^x \left(\varepsilon y_n'(t) - g'(t) - \frac{d}{dt}\left[\int_0^t K(t,s,y_n(s))ds\right]\right)dt, \tag{3.5}$$

and the considered initial guess is the initial condition for the problem, i.e.,

$$(y_0(x) = y(0) = \frac{g(0)}{\varepsilon}).$$

Details about the VIM can be found in He (1997), Alawneh and Al-Khaled (2010), Biazar and Ghazuini (2007), Tatari and Dehghan (2007) and Khaleghi, Ganji and Sadighi (2007).

3.1. Numerical Examples

In this section we apply the VIM to the same examples considered in section 2 and compare the results with the exact solution.

Example 4.

Consider the problem in Example 1, and differentiate the equation to get

$$\varepsilon y'(x) = 1 + x - y(x).$$

According to VIM, the correction functional is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\{\varepsilon y_{n'}(t) + y_n(t) - t - 1\}dt,$$

and the correction functional stationary is

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(x,t)\{\varepsilon y_{n'}(t) + y_n(t) - t - 1\}dt$$

$$= \delta y_n(x) + \varepsilon\lambda(x,t)\delta y_n(t)|_{t=x} -$$

$$\varepsilon\int_0^x \left\{\frac{\partial}{\partial t}\lambda(x,t)\delta y_n(t) - \lambda(x,t)\delta y_n(t)\right\}dt,$$

$$= \delta y_n(t)\big(1 + \varepsilon\lambda(x,t)\big)|_{t=x} -$$

$$\varepsilon\int_0^x \left\{\frac{\partial}{\partial t}\lambda(x,t) - \lambda(x,t)\right\}\delta y_n(t)dt = 0.$$

Therefore, the general lagrange multiplier satisfies

$$\varepsilon\frac{\partial}{\partial t}\lambda(x,t) - \lambda(x,t) = 0,$$

subject to

$$\lambda(x,t)|_{t=x} = -\frac{1}{\varepsilon}.$$

Solving the above equation yields

$$\lambda(x,t) = -\frac{e^{\frac{t-x}{\varepsilon}}}{\varepsilon}, \tag{3.6}$$

Thus, the correction functional becomes

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(\frac{e^{\frac{t-x}{\varepsilon}}}{\varepsilon} \right) \{ \varepsilon y_{n'}(t) + y_n(t) - t - 1 \} \, dt. \tag{3.7}$$

Starting with the initial condition $y(0) = 0$ as the initial guess $y_0(x) = 0$. Then, by (3.7)

$$y_1(x) = x + 1 - e^{\frac{-x}{\varepsilon}} - \varepsilon \left(1 - e^{\frac{-x}{\varepsilon}} \right),$$

which is the exact solution.

Example 5.

Consider the problem in Example 2, which can be written in the form

$$\varepsilon y(x) = x + x^2 + \frac{x^3}{6} - \int_0^x y(t)dt - x\int_0^x y(t)dt + \int_0^x t y(t)dt. \tag{3.8}$$

We differentiate (3.8) to get

$$\varepsilon y'(x) = 1 + 2x + \frac{x^2}{2} - y(x) - \int_0^x y(t)dt\ . \tag{3.9}$$

The correction functional is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t) \left\{ \varepsilon y_{n'}(t) + y_n(t) + \int_0^t y_n(s)ds - \frac{t^2}{2} - 2t - 1 \right\} dt,$$

and the correction functional stationary is

$$\delta y_{n+1}(x) = \delta y_n(x) + \int \{ \varepsilon \lambda(x,t) \delta y_n'(t) + \lambda(x,t) \delta y_n(t) \} dt = 0.$$

The obtained Lagrange multiplier is

$$\lambda(x,t) = -\frac{e^{\frac{t-x}{\varepsilon}}}{\varepsilon}.$$

Thus, the correction functional becomes

$$y_{n+1}(x)=y_n(x)+\int_0^x\left(-\frac{e^{\frac{t-x}{\varepsilon}}}{\varepsilon}\right)\left\{\varepsilon y_{n'}(t)+y_n(t)+\int_0^t y_n(s)ds-\frac{t^2}{2}-2t-1\right\}dt. \tag{3.10}$$

Starting with ($y_0(x)=0$). Then, by using (3.10) and with the help of Mathematica, we get

$$y_1(x)=\frac{1}{2}\left(x^2-2x(-2+\varepsilon)+2(-1+\varepsilon)^2-2e^{-\frac{x}{\varepsilon}}(-1+\varepsilon)^2\right),$$

recursively,

$$y_2(x)=\frac{1}{6}e^{-\frac{x}{\varepsilon}}\{-6(-1+\varepsilon)(-1+x(-1+\varepsilon)-\varepsilon+4\varepsilon^2$$
$$+e^{x/\varepsilon}[-x^3+x^2(-3+6\varepsilon)+x(6+18\varepsilon-18\varepsilon^2)+6(1-5\varepsilon^2+4\varepsilon^3)]\}.$$

Continuing in this manner $y_6(x)$ is determined as

$$y_6(x)=\frac{1}{5040}e^{-\frac{x}{\varepsilon}}(-42(x^5(-1+\varepsilon)^2+5x^4(1+4\varepsilon-13\varepsilon^2+8\varepsilon^3)+$$
$$20x^3(1+3\varepsilon+10\varepsilon^2-49\varepsilon^3+36\varepsilon^4)+.... \quad . \tag{3.11}$$

Figure 3 shows the absolute error between the exact solution and the approximate solution for ($0.2<\varepsilon<1$) and considering six iterations of the VIM sequence.

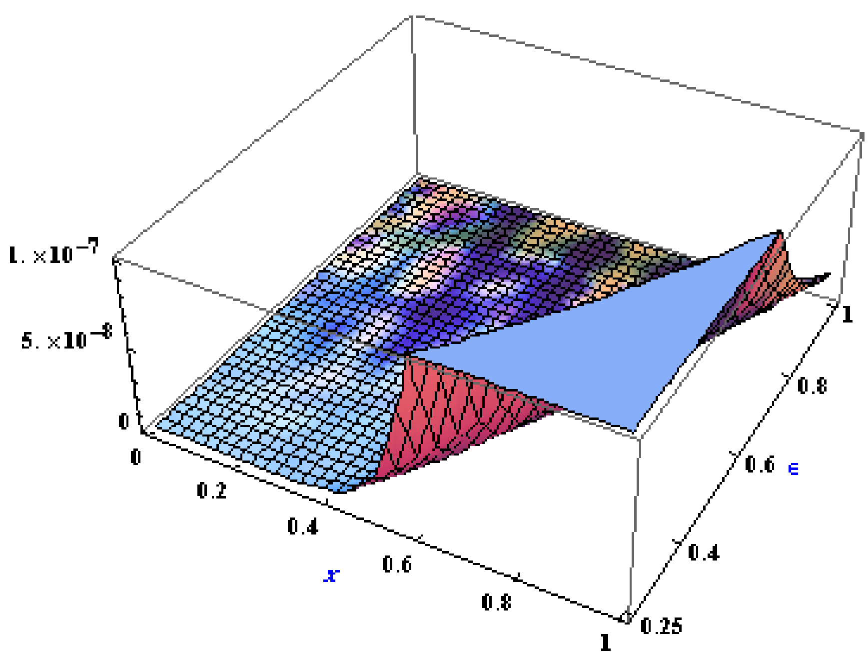

Figure 3. Absolute errors of equation (2.8) using VIM for N = 5

Example 6.

Consider the problem in Example 3, and differentiate to get

$$\varepsilon y'(x) = \frac{d}{dx}\left[\int_0^x e^{x-t}\left(y^2(t)-1\right)dt\right]. \tag{3.12}$$

The correction functional is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\left\{\varepsilon y_{n'}(t) - \frac{d}{dt}\left[\int_0^t e^{t-s}\left(y^2(s)-1\right)ds\right]\right\}dt.$$

Now, by using the obtained result in (3.4), then the general lagrange multiplier is

$$\lambda(x,t) = -\frac{1}{\varepsilon}.$$

Thus, the correction functional becomes

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(\frac{1}{\varepsilon}\right)\left\{\varepsilon y_{n'}(t) - \frac{d}{dt}\left[\int_0^t e^{t-s}\left(y^2(s)-1\right)ds\right]\right\}dt.$$

Starting with ($y_0(x) = 0$), then

$$y_1(x) = \frac{1-e^x}{\varepsilon}$$

and, then after

$$y_2(x) = \frac{\varepsilon^2 - e^x\left(2x+\varepsilon^2\right) + 2e^x \text{Sinh}[x]}{\varepsilon^3}.$$

Continuing in this manner, $y_4(x)$ is determined as

$$y_4(x) = \frac{1}{113400\varepsilon^{15}}(1800e^{8x} - 4200e^{7x}(-4+6x+3\varepsilon^2) +$$

$$113400(-1+4\varepsilon^2-6\varepsilon^4+6\varepsilon^6-5\varepsilon^8+2\varepsilon^{10}+\varepsilon^{14}) + \dots .$$

The following table shows the absolute error of (2.13) for different values of ε

Table 2. The absolute error of equation (2.13) using VIM for $N = 5$

x	$\varepsilon = 1$	$\varepsilon = 0.9$	$\varepsilon = 0.8$	$\varepsilon = 0.5$
$x = 0.0$	1.48×10^{-13}	2.5×10^{-11}	1.71×10^{-10}	4.28×10^{-4}
$x = 0.2$	5.15×10^{-11}	1.49×10^{-2}	3.32×10^{-2}	1.23×10^{-1}
$x = 0.4$	1.53×10^{-7}	2.53×10^{-2}	5.38×10^{-2}	1.46×10^{-1}
$x = 0.6$	1.79×10^{-5}	2.91×10^{-2}	5.82×10^{-2}	1.37×10^{-1}
$x = 0.8$	5.24×10^{-4}	2.86×10^{-2}	5.44×10^{-2}	3.44×10^{-1}

4. Conclusion

In this paper, a comparative study of VIM and DTM has been conducted. These methods were applied to solve linear and nonlinear SPVIEs. The three examples considered in this work support our belief that the results of these methods are in excellent agreement with exact solutions. The comparison revealed that, although the numerical results are similar, VIM is much easier, more convenient, and more efficient; it does not require intermediate complex calculations, such as finding Taylor series expansion ininvolved in the DTM.

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