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The Multisoliton Solutions of Some Nonlinear Partial Differential Equations

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Abstract

In this paper, we obtain multisoliton solutions of the Camassa-Holm equation and the Joseph-Egri (TRLW) equation by using the formal linearization method. The formal linearization method is an efficient instrument for constructing multisoliton solution of some nonlinear partial differential equations. This method can be applied to nonintegrable equations as well as to integrable ones.

Keywords: Formal linearization method; Camassa-Holm equation; Joseph-Egri (TRLW) equation

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1. Introduction

The direct linearization of certain famous integrable nonlinear equations was carried out in [Rosales (1978)]. Solutions of the KdV equation were connected with solutions of the Hopf equation by using formal series in [Baikov et al. (1989)]. Convergent exponential series were used in a variety of papers [Bobylev (1980), (1981), (1984); Vedenyapin (1981), (1988), Mischenko and Petrina (1988)] for constructing solutions of the Boltzmann equations. The possibility to adopt such series for some other equations was discussed in [Bobylev (1981)]. Fourier series were applied in the construction of solutions to the perturbed *KdV* equation in [Nikolenko (1980)]. In this paper we consider the class of equations and systems containing arbitrary linear differential operators with constant coefficients and arbitrary nonlinear analytic functions of dependent variables and their derivatives up to some finite order under the assumption that these equations possess a constant solution.

The formal linearization method consists of linearizing a nonlinear partial differential equation to the system of linear ordinary differential equations, describing some finite-dimensional subspace of the solution space of the linearized equation. It allows us to develop a very simple technique of finding the linearizing transformation and to apply the method to nonintegrable equations as well as to integrable ones; the solutions are in the form of exponential or Fourier series. We note that a similar approach with a different technique was independently developed for the wide class of evolution equations and the convergence of the constructed exponential series investigated [Vedenyapin]. The multisoliton solution of the Klein-Gordon's equation by using the formal linearization method was also done [Taghizadeh and Mirzazadeh (2008)]. The aim of this paper is to find multisoliton solutions of the Camassa-Holm equation and the Joseph-Egri (TRLW) equation.

2. Formal Linearization Method

Let us consider equations of the following form

$$\hat{L}(D_{x_1}, D_{x_2}, D_{x_3})u(x_1, x_2, x_3) = N[u], \tag{1}$$

where

$$\hat{L}(D_{x_1}, D_{x_2}, D_{x_3}) = \sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}\sum_{k_3=0}^{K_3} l_{k_1k_2k_3} D_{x_1}^{k_1} D_{x_2}^{k_2} D_{x_3}^{k_3} \tag{2}$$

is a linear differential operator with constant coefficients and

$$N[u] = N(u, u_1, u_2, \ldots, u_p),$$

$$u_p = \frac{\partial^{p_1+p_2+p_3} u}{\partial x_1^{\,p_1} \partial x_2^{\,p_2} \partial x_2^{\,p_2}},$$

$$p = (p_1, p_2, p_3),$$

is an arbitrary analytic function of u and of its derivatives up to some finite order p . We suppose that Eq. (1) possesses the constant solution. Without loss of generality we assume that

$$N[u]=0, \qquad \frac{\partial N[0]}{\partial u}=0, \qquad \frac{\partial N[0]}{\partial u_1}=0, \quad \ldots, \quad \frac{\partial N[0]}{\partial u_p}=0.$$

We consider Eq. (1) in connection with the equation linearized near a zero solution:

$$\hat{L}(D_{x_1},D_{x_2},D_{x_3})w(x_1,x_2,x_3)=0. \tag{3}$$

Let L be the vector space of solutions of Eq. (3) and $P^N \subset L$ be the *N*-dimensional

$$w_i = W_i \exp(\alpha_i \xi_i), \qquad \xi_i = x_3 - a_i x_1 - b_i x_2, \qquad i = 1,2,\ldots,N .$$

Here a_i, b_i and W_i are some constants. The constants $\alpha_i = \alpha_i(a_i, b_i)$ are assumed to satisfy the dispersion relation

$$\hat{L}(-\alpha_i a_i, -\alpha_i b_i, \alpha_i)=0.$$

The subspace $P^N = \{\sum_{i=1}^{N} C_i \omega_i \mid C_i = const\}$ is specified by the system of N linear ordinary differential equations

$$\frac{dw_i}{d\xi_i}=\alpha_i w_i, \qquad i = 1,2,\ldots,N .$$

We use the following notation:

$$w_{(N)}^{\delta} = w_1^{\delta_1} w_2^{\delta_2} \ldots . w_N^{\delta_N},$$

$$\delta = (\delta_1,\delta_2,\ldots,\delta_N), \qquad |\delta| = \sum_{i=1}^{N}\delta_i.$$

It is obvious that the monomials $\omega_{(N)}^{\delta}$ are the eigenfunctions of the operator (2):

$$\hat{L}(D_{x_1},D_{x_2},D_{x_3})w_{(N)}^{\delta} = \lambda_\delta w_{(N)}^{\delta}$$

with the eigenvalues

$$\lambda_\delta = \sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}\sum_{k_3=0}^{K_3} l_{k_1k_2k_3}(-\sum_{i=1}^{N}\alpha_i a_i \delta_i)^{k_1}(-\sum_{i=1}^{N}\alpha_i a_i \delta_i)^{k_2}(\sum_{i=1}^{N}\alpha_i \delta_i)^{k_3}.$$

Theorem:

If $\lambda_\delta \neq 0$ for every multiindex δ with positive integer components $\delta_i \in Z_+, i = 1,...,N$, satisfying the condition $|\delta| \neq 0,1,$ then equation (1) possesses solutions connected with solutions form P^N by the formal transformation

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2, ..., w_n), \tag{4}$$

where

$$\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(N)}^{\delta} \tag{5}$$

are homogeneous polynomials of degree n in the variables w_i. This transformation is unique (for the first term $\phi_1 \in P^N$ fixed).

Remark 1:

Here ε is the grading parameter, finally we can put $\varepsilon = 1$.

The proof of the theorem is constructive. Substituting (4) into (1) , expanding $N[u]$ into the power series in ε, and then collecting equal powers of ε, we obtain the determining equations for the functions ϕ_n and show that if $\lambda_\delta \neq 0,$ then these equations possess the solution (5) with the coefficients $(A_n)_\delta$ uniquely determined through the coefficients $(A_1)_\delta$ by the recursion relation. Thus, the theorem gives us the method for constructing particular solutions of equation (1).

3. Application

3.1. Camassa-Holm Equation

Let us consider the Camassa-Holm equation [Kalisch and Lenells (2005)]:

$$\hat{L}(D_t,D_x)u(t,x) = -3uu_x + 2u_x u_{xx} + uu_{xxx},$$
$$\hat{L}(D_t,D_x) = D_t - D_t D_x^2 + 2\alpha D_x. \tag{6}$$

For simplicity we look for a solution of (6) in the form

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1,w_2), \tag{7}$$

where

$$w_i = W_i \exp[\sqrt{\frac{a_i - 2\alpha}{a_i}}(x - a_i t)], \qquad i = 1,2$$

is the basis of the subspace $P^2 \subset L$ (let s_i and W_i be some real constants). Substituting (7) into (6) and collecting equal powers of ε we obtain the determining equations for the functions ϕ_n as follows

$$\hat{L}\phi_1 = 0,$$
$$\hat{L}\phi_n = -3\sum_{k=1}^{n-1}\phi_k D_x \phi_{n-k} + 2\sum_{k=1}^{n-1}\phi_k D_x^2 \phi_{n-k} + \sum_{k=1}^{n-1}\phi_k D_x^3 \phi_{n-k},$$
$$n \geq 2.$$

These equations possess the solution $\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(2)}^{\delta}$, $\delta = (\delta_1,\delta_2)$, which can be rewritten in this case in the following form

$$\phi_n = \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k} \qquad (\phi_1 \in P^2), \tag{8}$$

the coefficients A_k^n can be found through A_0^1 and A_1^1 (we can assume that either $A_0^1 = A_1^1 = 1$ or $A_0^1 = 0, A_1^1 = 1$) by the recursion relation:

If $n \geq 2$, $0 \leq k \leq n$ then

$$A_k^n = \frac{1}{\lambda_{(k,n-k)}}\left\{\begin{array}{l} -3\sum_{l=1}^{n-1}\sum_{m=0}^{n-l}(\sqrt{\frac{a_1-2\alpha}{a_1}}m+\sqrt{\frac{a_2-2\alpha}{a_2}}(n-l-m))A_{k-m}^{l}A_m^{n-l} \\ +2\sum_{l=1}^{n-1}\sum_{m=0}^{n-l}(\sqrt{\frac{a_1-2\alpha}{a_1}}m+\sqrt{\frac{a_2-2\alpha}{a_2}}(n-l-m))(\sqrt{\frac{a_1-2\alpha}{a_1}}m+\sqrt{\frac{a_2-2\alpha}{a_2}}(n-l-m))^2 A_{k-m}^{l}A_m^{n-l} \\ +\sum_{l=1}^{n-1}\sum_{m=0}^{n-l}(\sqrt{\frac{a_1-2\alpha}{a_1}}m+\sqrt{\frac{a_2-2\alpha}{a_2}}(n-l-m))^3 A_{k-m}^{l}A_m^{n-l} \end{array}\right\}.$$

If $k<0$ or $k>n$, then $A_k^n=0.$

$$\lambda_{(k,n-k)} = (a_1-2\alpha)\sqrt{\frac{a_1-2\alpha}{a_1}}k(k^2-1)+(a_2-2\alpha)\sqrt{\frac{a_2-2\alpha}{a_2}}(n-k)((n-k)^2-1)$$
$$+\frac{a_1-2\alpha}{a_1}(2a_1+a_2)\sqrt{\frac{a_2-2\alpha}{a_2}}k^2(n-k)+\frac{a_2-2\alpha}{a_2}(2a_2+a_1)\sqrt{\frac{a_1-2\alpha}{a_1}}k(n-k)^2.$$

If $a_1>2\alpha, a_2>2\alpha, a_1<0, a_2<0,$ then $\lambda_{(k,n-k)}\neq 0$ for every pair $(k,n-k)$ with $k,n\in Z_+$, $n\geq 2,\ \ 0\leq k\leq n$.

Remark 2:

If $A_0^1=0,$ then $\phi_1\in P^1$ and we get from (7) the expansion for a 1-soliton solution. For obtaining the N-soliton solutions, we must take $\phi_1\in P^N$.

3.2. Joseph-Egri (TRLW) Equation

We next consider the Joseph-Egri (TRLW) equation [Hereman et al. (1986)]:

$$\begin{aligned} &\hat{L}(D_t,D_x)u(t,x) = -\alpha u u_x, \\ &\hat{L}(D_t,D_x) = D_t + D_x + D_x D_t^2. \end{aligned} \tag{9}$$

In this case, the subspace P^2 is generated by the functions

$$w_i = W_i \exp[\frac{\sqrt{a_i-1}}{a_i}(x-a_i t)], \quad i=1,2.$$

Our procedure gives the solution

$$u = \sum_{n=1}^{\infty} \varepsilon^n \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k}, \tag{10}$$

where if $n \geq 2, \quad 0 \leq k \leq n,$ then

$$A_k^n = -\frac{\alpha}{\lambda_{(k,n-k)}} \sum_{l=1}^{n-1} \sum_{m=0}^{n-l} [\frac{\sqrt{a_1 - 1}}{a_1} m + \frac{\sqrt{a_2 - 1}}{a_2} (n - l - m)] A_{k-m}^l A_m^{n-l}.$$

If $k < 0$ or $k > n,$ then $A_k^n = 0.$

$$\begin{aligned}\lambda_{(k,n-k)} = {} & \frac{a_1 - 1}{a_1} \sqrt{a_1 - 1} k(k^2 - 1) + \frac{a_2 - 1}{a_2} \sqrt{a_2 - 1} (n - k)((n - k)^2 - 1) \\ & + \frac{1}{a_1 a_2} \{((a_2 - 1)\sqrt{a_1 - 1}(a_2 + 2a_1))(n - k) \\ & + ((a_1 - 1)\sqrt{a_2 - 1}(2a_2 + a_1))k\} k(n - k).\end{aligned}$$

Here, either $A_0^1 = A_1^1 = 1$ or $A_0^1 = 0, A_1^1 = 1.$

If $a_1 > 1$ and $a_2 > 1,$ then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $k, n \in Z_+,$ $n \geq 2,$ $0 \leq k \leq n$.

In (10), if $A_0^1 = 0,$ then we get

$$u = \varepsilon w_1 - \frac{\alpha}{6(a_1 - 1)} (\varepsilon w_1)^2 + \frac{\alpha^2}{48(a_1 - 1)^2} (\varepsilon w_1)^3 - \cdots = \frac{12(a_1 - 1)}{\alpha} \sum_{n=1}^{\infty} (-1)^{n+1} n w^n,$$

where

$$w = \frac{\varepsilon \alpha}{12(a_1 - 1)} w_1.$$

In $(t, x)-$variables we have

$$u(t, x) = \pm \frac{3(a_1 - 1)}{\alpha} \operatorname{sec} h^2 [\frac{\sqrt{a_1 - 1}}{a_1} (x - a_1 t + x_0)], \tag{11}$$

where x_0 is arbitrary constant.

Then, (10) is a 2-soliton solution of the Joseph-Egri equation and (11) is a 1-soliton solution of the Joseph-Egri equation.

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