Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 6, Issue 1 (June 2011) pp. 223 – 231
(Previously, Vol. 6, Issue 11, pp. 1964 – 1972)

Applications and Applied Mathematics:
An International Journal
(AAM)

Exact Travelling Wave Solutions of the Coupled Klein-Gordon Equation by the Infinite Series Method

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Received: September 22, 2010; Accepted: February 10, 2011

Abstract

In this paper, we employ the infinite series method for travelling wave solutions of the coupled Klein-Gordon equations. Based on the idea of the infinite series method, a simple and efficient method is proposed for obtaining exact solutions of nonlinear evolution equations. The solutions obtained include solitons and periodic solutions.

Keywords: Infinite series method; Cosine function method; Coupled Klein-Gordon equation

MSC 2010 No.: 47F05; 35Q53; 35Q55; 35G25.

1. Introduction

Over the last few decades, directly searching for exact solutions of nonlinear partial differential equations has become a more attractive topic in the physical and nonlinear sciences. The investigation of the travelling wave solutions of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics and fluid dynamics.

In order to better understand these nonlinear phenomena, many mathematicians and physical scientists make the effort to seek more exact solutions. Several powerful methods have been proposed to obtain exact solutions of nonlinear evolution equations, such as the tanh method [Malfliet (1992), Malfliet and Hereman (1996)], the extended tanh method [El-Wakil et al. (2007), Fan (2000), and Wazwaz (2005)], the hyperbolic function method [Xia and Zhang (2001)], the sine-cosine method [Wazwaz (2004], Yusufoglu and Bekir (2006), Jacobi elliptic function expansion method [Inc and Ergut (2005)], F-expansion method [Zhang (2006)], and the direct algebraic method [Hereman et al. (1986), Hereman and Takaoka (1990)] .

Ablowitz and Segur (1981) implemented the inverse scattering transform method to handle the nonlinear equations of physical significance where soliton solutions and rational solutions were developed. Recently, the tanh method established in [Malfliet (1992), Malfliet and Hereman (1996)], was effectively used in [Hereman and Takaoka (1990), Goktas and Hereman (1997), Hereman and Nuseir (1997)] among many others. The tanh method was subjected by some modifications using the Riccati equation.

Xu and Li and Liu (2003) introduced a unified method for finding travelling wave solutions of nonlinear evolution equations. This method is based on the arbitrariness of the manifold in Painleve' analysis. The original idea behind all the direct methods goes back to Hirota (1980), who systematically solved large classes of nonlinear evolution equations using a bilinear transformation.

The technique we use in this paper is due to Hereman et al. (1986). By this method, solutions are developed as series in real exponential functions which physically corresponds to mixing of elementary solutions of the linear part due to nonlinearity.

The method of Hereman et al. (1986) falls into the category of direct solution methods for nonlinear partial differential equations. This method is currently restricted to traveling wave solutions. In addition, depending on the number of nonlinear terms in the partial differential equation with arbitrary numerical coefficients, it is sometimes necessary to specialize to particular values of the velocity in order to find closed form solutions. On the other hand, the Hereman et al. series method does give a systematic means of developing recursion relations. Hereman et al. (1986) direct series method can be used to solve both dissipative and non dissipative equations. They regard solutions of the linear equation to be of the form

$$\exp[-k(c)(x-ct)],$$

where $k(c)$ is a function of the velocity c. The velocity is assumed constant but in general is related to the wave amplitude. It is from the solutions of the linear part that the solution of the full nonlinear partial differential equation is synthesized. With wave number k, the dispersion relation $w=k(c)$ gives the angular frequency.

Here, we apply the infinite series method for solving the coupled Klein-Gordon equation.

2. Infinite Series Method

Consider the nonlinear partial differential equation:

$$F(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{1}$$

where $u = u(x,t)$ is the solution of the Eq. (1). We use transformations

$$u(x,t) = f(\xi), \qquad \xi = x + \lambda t, \tag{2}$$

where λ is constant, to obtain

$$\frac{\partial}{\partial t}(.) = \lambda \frac{\partial}{\partial \xi}(.), \qquad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \qquad \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), \ldots \;. \tag{3}$$

We use (3) to change the nonlinear partial differential equation (1) to the nonlinear ordinary differential equation

$$G\left(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, \ldots\right) = 0. \tag{4}$$

Next, we apply the approach of Hereman et al. (1986) we solve the linear terms and then suppose the solution in the form

$$f(\xi) = \sum_{n=1}^{\infty} a_n g^n(\xi), \tag{5}$$

where $g(\xi)$ is a solution of linear terms and the expansion coefficients $a_n\,(n = 1, 2, \ldots)$ are to be determined. To deal with the nonlinear terms, we need to apply the extension of Cauchy's product rule for multiple series.

Lemma 1 (Extension of Cauchy's product rule):

If

$$F^{(i)} = \sum_{n_1=1}^{I} a_{n_1}^{(i)}, \qquad i = 1, \ldots, I, \tag{6}$$

represents I infinite convergent series, then

$$\prod_{i=1}^{I} F^{(i)} = \sum_{n=1}^{\infty} \sum_{r=I-1}^{n-1} \cdots \sum_{m=2}^{k-1} \sum_{l=1}^{m-1} a_l^{(1)} a_{m-l}^{(2)} \ldots a_{n-r}^{(I)}. \tag{7}$$

Proof:

See Hereman et al. (1986).

Substituting (5) into (4) yields recursion relation which gives the values of the coefficients.

3. Coupled Klein-Gordon Equation

The nonlinear coupled Klein-Gordon equation Alagesan et al. (2004) is very important equation in the area of Theoretical Physics. The nonlinear coupled Klein-Gordon equation was first studied by Alagesan et al. (2004), and then Shang (2010) and Yusufoglu and Bekir (2008) gave further result by using the ideas of the tanh method and the general integral method. They also obtained the solutions and periodic solutions. In paper Sassaman and Biswas (2009), the quasilinear coupled Klein-Gordon, which have several forms of power law nonlinearity, are well studied by using soliton perturbation theory.

Let us consider the coupled Klein-Gordon equation Alagesan et al. (2004)

$$\begin{aligned} &u_{xx} - u_{tt} - u + 2u^3 + 2uv = 0, \\ &v_x - v_t - 4uu_t = 0, \end{aligned} \tag{8}$$

by using the infinite series method.

We use the wave transformations

$$u(x,t) = u(\xi), \qquad v(x,t) = v(\xi), \qquad \xi = x - ct. \tag{9}$$

Substituting (9) into equation (8), we have the ordinary differential equations (ODEs) for $u(\xi)$ and $v(\xi)$

$$\begin{aligned} &(1-c^2)u''(\xi) - u(\xi) + 2u^3(\xi) + 2u(\xi)v(\xi) = 0, \\ &(1+c)v'(\xi) + 4cu(\xi)u'(\xi) = 0. \end{aligned} \tag{10}$$

By integrating the second equation with respect to ξ, and neglecting the constant of integration we obtain

$$v(\xi) = -\frac{2c}{1+c}u^2(\xi). \tag{11}$$

Substituting (11) into the first equation of equation (10) and integrating the resulting equation, we find

$$(1-c^2)u''(\xi)-u(\xi)+\frac{2(1-c)}{1+c}u^3(\xi)=0. \tag{12}$$

3.1. Using Infinite Series Method

The linear equation from (12) has the solution in the form

$$g(\xi)=\exp(\frac{\xi}{\sqrt{1-c^2}}).$$

Thus, we look for the solution of (12) in the form

$$u(\xi)=\sum_{n=1}^{\infty}a_n\exp(\frac{n\xi}{\sqrt{1-c^2}}). \tag{13}$$

Substituting (13) into (12) and by using Lemma 1, we obtain the recursion relation follows

a_1 is arbitrary,

$$a_2=0,$$

$$(n^2-1)a_n+\frac{2(1-c)}{(1+c)}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}a_l a_{m-l}a_{n-m}=0, \qquad n\geq 3. \tag{14}$$

Then, by (14), we have

$$a_{2d}=0,$$

$$a_{2d+1}=(-1)^d 2^d(\frac{1-c}{1+c})^d\frac{a_1^{2d+1}}{2^{3d}}, \qquad d=1,2,3,\ldots. \tag{15}$$

Substituting (15) into (13) gives

$$u(\xi)=\sum_{d=0}^{\infty}(-1)^d 2^d(\frac{1-c}{1+c})^d\frac{a_1^{2d+1}}{2^{3d}}\exp(\frac{(2d+1)\xi}{\sqrt{1-c^2}})=\frac{a_1\exp(\frac{\xi}{\sqrt{1-c^2}})}{1+(\frac{1-c}{1+c})\frac{a_1^2}{4}\exp(\frac{2\xi}{\sqrt{1-c^2}})}.$$

By using (11), we get

$$v(\xi)=-\frac{2c}{1+c}(\frac{a_1\exp(\frac{\xi}{\sqrt{1-c^2}})}{1+(\frac{1-c}{1+c})\frac{a_1^2}{4}\exp(\frac{2\xi}{\sqrt{1-c^2}})})^2.$$

In $(x,t)-$variables, we have the exact soliton solution of the coupled Klein-Gordon equation in the following form

$$u(x,t)=\frac{a_1\exp(\frac{x-ct}{\sqrt{1-c^2}})}{1+(\frac{1-c}{1+c})\frac{a_1^2}{4}\exp(\frac{2(x-ct)}{\sqrt{1-c^2}})}, \tag{16}$$

$$v(x,t)=-\frac{2c}{1+c}(\frac{a_1\exp(\frac{x-ct}{\sqrt{1-c^2}})}{1+(\frac{1-c}{1+c})\frac{a_1^2}{4}\exp(\frac{2(x-ct)}{\sqrt{1-c^2}})})^2. \tag{17}$$

In (16) and (17) if we choose $a_1=\pm 2\sqrt{\frac{1+c}{1-c}}$, then

$$u(x,t)=\pm\sqrt{\frac{1+c}{1-c}}\frac{2\exp(\frac{x-ct}{\sqrt{1-c^2}})}{1+\exp(\frac{2(x-ct)}{\sqrt{1-c^2}})}=\pm\sqrt{\frac{1+c}{1-c}}\sec h[\frac{x-ct}{\sqrt{1-c^2}}],$$

since

$$\sec h\theta=\frac{1}{\cosh\theta}=\frac{2}{e^{\theta}+e^{-\theta}}=\frac{2e^{\theta}}{1+e^{2\theta}}.$$

Thus, the exact solutions of the coupled Klein-Gordon equation can be expressed as:

$$\begin{aligned} u(x,t)&=\pm\sqrt{\frac{1+c}{1-c}}\sec h(\frac{x-ct}{\sqrt{1-c^2}}),\\ v(x,t)&=-\frac{2c}{1-c}\sec h^2(\frac{x-ct}{\sqrt{1-c^2}}), \end{aligned} \tag{18}$$

for $c^2<1$, and

$$\begin{aligned} u(x,t)&=\pm\sqrt{\frac{1+c}{1-c}}\sec(\frac{x-ct}{\sqrt{c^2-1}}),\\ v(x,t)&=-\frac{2c}{1-c}\sec^2(\frac{x-ct}{\sqrt{c^2-1}}), \end{aligned} \tag{19}$$

for $c^2>1$.

3.2. Using the Cosine-Function Method

In this section, the cosine function method [8 − 10] is applied to the coupled Klein-Gordon equation.

The solution of equation (12) can be expressed in the form:

$$u(\xi) = \lambda \cos^{\beta}(\mu\xi), \quad |\xi| \le \frac{\pi}{2\mu}, \tag{20}$$

where λ, β and μ are unknown parameters which will be determined. Then we have:

$$\begin{aligned} u'(\xi) &= -\lambda\beta\mu \cos^{\beta-1}(\mu\xi)\sin(\mu\xi), \\ u''(\xi) &= -\lambda\mu^2\beta^2 \cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi). \end{aligned} \tag{21}$$

Substituting (20) and (21) into equation (12) gives

$$\begin{aligned} &(1-c^2)[-\lambda\mu^2\beta^2 \cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi)] \\ &-\lambda\cos^{\beta}(\mu\xi) + \frac{2(1-c)}{(1+c)}\lambda^3 \cos^{3\beta}(\mu\xi) = 0. \end{aligned} \tag{22}$$

By equating the exponents and the coefficients of each pair of the cosine function we obtain the following system of algebraic equations:

$$\begin{aligned} &(c^2-1)\lambda\mu^2\beta^2 - \lambda = 0, \\ &(1-c^2)\lambda\mu^2\beta(\beta-1) + \frac{2(1-c)}{(1+c)}\lambda^3 = 0, \\ &3\beta = \beta - 2. \end{aligned} \tag{23}$$

Solving the system (23), we have

$$\lambda = \pm\sqrt{\frac{1+c}{1-c}}, \qquad \beta = -1, \qquad \mu = \pm\frac{1}{\sqrt{c^2-1}}. \tag{24}$$

Combining (24) with (20), we obtain the exact solution to equation (12) as follows:

$$u(\xi) = \pm\sqrt{\frac{1+c}{1-c}}\sec\left(\frac{\xi}{\sqrt{c^2-1}}\right), \qquad c^2 > 1. \tag{25}$$

Then, the exact solution to the coupled Klein-Gordon equation can be written as

$$u(x,t) = \pm\sqrt{\frac{1+c}{1-c}}\sec\left(\frac{x-ct}{\sqrt{c^2-1}}\right),$$
$$v(x,t) = -\frac{2c}{1-c}\sec^2\left(\frac{x-ct}{\sqrt{c^2-1}}\right), \tag{26}$$

for $c^2 > 1$.

Remark 1:

In (16) and (17) if we choose $a_1 = \pm 2\sqrt{\frac{1+c}{1-c}}$, then the solutions are obtained of the coupled Klein-Gordon equation by using the infinite series and cosine function methods are equal.

4. Conclusion

The infinite series method has been successfully applied here for solving the coupled Klein-Gordon equation. This is an efficient method for obtaining exact solutions of some nonlinear partial differential equations. It may also be applied to nonintegrable equations as well as integrable ones. The method does not apply to nonlinear partial differential equations whose linear part cannot be separated from the nonlinear part. The solution is new and the method can be extended to solve problems of nonlinear systems arising in the theory of solitons and other areas.

Acknowledgement

The authors are very grateful to the referees for their detailed comments and kind help.

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