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GLOBAL STABILITY RESULTS AND WELL POSEDNESS OF AN SI AGE-STRUCTURED EPIDEMIC MODEL WITH VERTICAL TRANSMISSION

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Abstract

An SI age-structured epidemic model for a vertically as well as horizontally transmitted disease is investigated when the fertility and mortality rates depend on age and the force of infection of proportionate mixing assumption type. We prove the well posedness of the model as well as the global stability for endemic equilibriums.

Keywords: Vertical transmission; Horizontal transmission; Age-structure; Epidemic; Global stability; Proportionate mixing; Well posedness.
MSC 2000: 45K05; 45M10; 35A05; 35B30; 35B35; 35B45; 35L40; 92D30; 92D25.

1 Introduction

Several recent papers and books have dealt with SI age-structured epidemic models, for example, Busenberg, et al. (1993), (1991), El-Doma (2006), (2004a), (2004b), (2004c),

(2000), (1987), Brauer (2002a), (2002b), Louie, et al. (1994), Anderson, et al. (1991), May, et al. (1988) and Hoppenstead (1975).

In this paper, we study an SI age-structured epidemic model, where age is the chronological age i.e., the elapse of time since birth. The disease causes so few fatalities that they can be neglected and is horizontally as well as vertically transmitted. Horizontal transmission of infection is the transfer of infection through some direct or indirect contact with infected individuals, for example, malaria is horizontally transmitted. A particular form of horizontal transmission known as proportionate mixing is assumed in this paper. Vertical transmission of infection is the passing of infection to offspring of infected parentage, for example, AIDS, chagas and hepatitis B are vertically (as well as horizontally) transmitted diseases. This form of transmission plays an important role in maintaining some diseases, for example, see Busenberg, et al. (1993). We note that some sexually transmitted diseases such as herpes and AIDS, are of SI type.

We establish the well posedness of the model and prove global stability results and show that, if $q \neq 1$, then the endemic equilibrium is globally stable. We also show that if $q = 1$, and other conditions hold, then the endemic equilibrium is also globally stable. The local stability of the endemic equilibrium as well as the global stability of the disease-free equilibrium are reported in El-Doma (2004a). We note that if $q = 1$, then either there exists a unique endemic equilibrium, which is globally stable, or problem (2.1) gives rise to a continuum of endemic equilibriums, in the case of non-fertile infectibles.

The global stability results that we obtain are under very general conditions, and, in fact, we do not require any condition other than the existence of a unique endemic equilibrium.

The organization of this paper is as follows: in section 2 we describe the model and obtain the model equations; in section 3 we reduce the model equations to several subsystems and prove the well posedness of the model equations; in section 4 we determine the steady states; in section 5 we prove global stability results; in section 6 we conclude our results.

2 The Model

We consider an age-structured population of variable size exposed to a communicable disease. The disease is vertically as well as horizontally transmitted and causes so few fatalities that they can be neglected. We assume the following.

1. $s(a,t)$ and $i(a,t)$, respectively, denote the age-density for susceptible and infective of age a at time t. Then
   $\int_{a_1}^{a_2} s(a,t)da$ = total number of susceptible at time t of ages between a_1 and a_2,
   $\int_{a_1}^{a_2} i(a,t)da$ = total number of infective at time t of ages between a_1 and a_2. We assume that the total population consists entirely of susceptible and infective.

2. Let $k(a,a')$ denote the probability that a susceptible individual of age a is infected by an infective of age a'. We further assume that, $k(a,a') = k_1(a).k_2(a')$, which is known as the "proportionate mixing assumption", see Dietz, et al. (1985). Therefore the horizontal transmission of the disease occurs at the following rate:

$$k_1(a)s(a,t)\int_0^\infty k_2(a')i(a',t)da',$$

where $k_1(a)$ and $k_2(a)$ are bounded, nonnegative, continuous functions of a. The term

$$k_1(a)\int_0^\infty k_2(a')i(a',t)da',$$

is called "force of infection" and we let

$$\lambda(t)=\int_0^\infty k_2(a)i(a,t)da.$$

3. The fertility rate $\beta(a)$ is a nonnegative, continuous function, with compact support $[0,A]$, $(A>0)$. The number of births of susceptible per unit time is given by $s(0,t)=\int_0^\infty \beta(a)[s(a,t)+(1-q)i(a,t)]da, q\in[0,1]$, where q is the probability of vertically transmitting the disease. Accordingly, all newborns from susceptible parents are susceptible but a portion q of newborns from infected parents are infective, i.e., they acquire the disease via birth (vertical transmission) and therefore, $i(0,t)=q\int_0^\infty \beta(a)i(a,t)da$.

4. The death rate, $\mu(a)$, is the same for susceptible and infective and $\mu(a)$ is a nonnegative, continuous function and $\exists\ a_0\in[0,\infty)$ such that $\mu(a)>\bar{\mu}>0,\ \forall\ a>a_0$ and $\mu(a_2)>\mu(a_1),\ \forall\ a_2>a_1>a_0$.

5. The initial age distributions $s(a,0)=s_0(a)$ and $i(a,0)=i_0(a)$ are continuous, nonnegative and integrable functions of $a\in[0,\infty)$.

These assumptions lead to the following system of nonlinear integro-partial differential equations with non-local boundary conditions, which describes the dynamics of the transmission of the disease.

$$\begin{cases}
\dfrac{\partial s(a,t)}{\partial a}+\dfrac{\partial s(a,t)}{\partial t}+\mu(a)s(a,t)=-k_1(a)s(a,t)\lambda(t),\ a>0,\ t>0,\\[2ex]
\dfrac{\partial i(a,t)}{\partial a}+\dfrac{\partial i(a,t)}{\partial t}+\mu(a)i(a,t)=k_1(a)s(a,t)\lambda(t),\ a>0,\ t>0,\\[2ex]
s(0,t)=\displaystyle\int_0^\infty \beta(a)[s(a,t)+(1-q)i(a,t)]da,\ t\geq 0,\\[2ex]
i(0,t)=q\displaystyle\int_0^\infty \beta(a)i(a,t)da,\ t\geq 0,\\[2ex]
\lambda(t)=\displaystyle\int_0^\infty k_2(a)i(a,t)da,\ t\geq 0,\\[2ex]
s(a,0)=s_0(a),\ i(a,0)=i_0(a),\ a\geq 0.
\end{cases}\tag{2.1}$$

We note that problem (2.1) is an SI age-structured epidemic model that has been studied in El-Doma (2004a), where the steady states are determined and the local asymptotic stability of the endemic equilibrium and the disease-free equilibrium as well as the

the global stability of the disease-free equilibrium are reported. The same model but with k_1 and k_2 constants is studied in El-Doma (1987). Also, in Brauer (2002a), (2002b), problem (2.1) is studied when k_1 and k_2 are constants and $q = 0$ (the case of no vertical transmission).

In what follows, we establish the well posedness of the model equations and prove the global stability of the endemic equilibrium when $q \neq 1$. We also show that if $q = 1$ and another condition holds, then the endemic equilibrium is globally stable.

3 Reduction of the Model and its well posedness

In this section, we develop some preliminary formal analysis of problem (2.1) and show that problem (2.1) is well posed. We define $p(a,t)$ by $p(a,t) = s(a,t) + i(a,t)$. Then from (2.1), by adding the equations, we find that $p(a,t)$ satisfies the following McKendrick-Von Forester equation:

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial a} + \dfrac{\partial p(a,t)}{\partial t} + \mu(a)p(a,t) = 0, \ a > 0, \ t > 0, \\ \\ p(0,t) = B(t) = \displaystyle\int_0^\infty \beta(a)p(a,t)da, \ t \geq 0, \\ \\ p(a,0) = p_0(a) = s_0(a) + i_0(a), \ a \geq 0. \end{cases} \tag{3.1}$$

Note that problem (3.1) has a unique solution that exists for all time, see Bellman, et al. (1963), Feller (1941) and Hoppensteadt (1975). The unique solution is given by

$$p(a,t) = \begin{cases} p_0(a-t)\pi(a)/\pi(a-t), & a > t, \\ B(t-a)\pi(a), & a < t, \end{cases} \tag{3.2}$$

where $\pi(a)$ is given by

$$\pi(a) = e^{-\int_0^a \mu(\tau)d\tau},$$

and $B(t)$ has the following asymptotic behavior as $t \to \infty$:

$$B(t) = [c + \theta(t)]e^{p^*t}, \tag{3.3}$$

where p^* is the unique real number which satisfies the following characteristic equation:

$$\int_0^\infty \beta(a)\pi(a)e^{p^*t}da = 1, \tag{3.4}$$

$\theta(t)$ is a function such that $\theta(t) \to 0$ as $t \to \infty$ and c is a constant.

Also, from (2.1), $s(a,t)$ and $i(a,t)$ satisfy the following systems of equations:

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial a} + \dfrac{\partial s(a,t)}{\partial t} + \mu(a)s(a,t) = -k_1(a)s(a,t)\lambda(t), \ a > 0, \ t > 0, \\ \\ s(0,t) = \displaystyle\int_0^\infty \beta(a)[s(a,t) + (1-q)i(a,t)]da, \ t \geq 0, \\ \\ s(a,0) = s_0(a), \ a \geq 0, \end{cases} \tag{3.5}$$

$$\begin{cases} \dfrac{\partial i(a,t)}{\partial a} + \dfrac{\partial i(a,t)}{\partial t} + \mu(a)i(a,t) = k_1(a)s(a,t)\lambda(t),\ a > 0,\ t > 0, \\ \\ i(0,t) = q\displaystyle\int_0^\infty \beta(a)i(a,t)da,\ t \geq 0, \\ \\ i(a,0) = i_0(a),\ a \geq 0. \end{cases} \tag{3.6}$$

Using (3.1)-(3.2), we obtain that $B(t)$ satisfies

$$B(t) = \int_0^\infty \beta(a)\pi(a)B(t-a)da + \int_t^\infty \beta(a)p_0(a-t)\frac{\pi(a)}{\pi(a-t)}da. \tag{3.7}$$

Using (3.7) and *Gronwall's* inequality, we obtain

$$B(t) \leq \|\beta(a)\|_\infty \|p_0(a)\|_{L^1[0,\infty)} e^{(\|\ \beta(a)\|_\infty - \mu_*)t}, \tag{3.8}$$

where μ_* is given by

$$\mu_* = \inf_{a\in[0,\infty)} \mu(a). \tag{3.9}$$

From (3.2) and (3.8), we obtain the following a priori estimate:

$$\int_0^\infty p(a,t)da \leq \|p_0(a)\|_{L^1[0,\infty)} e^{(\|\ \beta(a)\|_\infty - \mu_*)t}. \tag{3.10}$$

By integrating problem (3.6) along characteristic lines $t - a = const.$, and using $s(a,t) = p(a,t) - i(a,t)$, we find that $i(a,t)$ satisfies

$$i(a,t) = \begin{cases} i_0(a-t)e^{-\int_0^t [\mu(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)]d\tau} \\ \\ +\dfrac{p_0(a-t)\pi(a)}{\pi(a-t)}\left[1 - e^{-\int_0^t k_1(a-t+\tau)\lambda(\tau)d\tau}\right], \quad a > t, \\ \\ i(0,t-a)\pi(a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau} \\ \\ +B(t-a)\pi(a)\left[1 - e^{-\int_0^t k_1(\tau)\lambda(t-a+\tau)d\tau}\right], \quad a < t. \end{cases} \tag{3.11}$$

It is worth noting that if we can establish a solution for problem (3.11), then a solution for problem (3.5) via (3.2) is determined, and consequently a solution for problem (2.1) is determined. To establish the existence and uniqueness of solution to problem (2.1), we define the following set E to satisfy:

$$E = \left\{ i(a,t) : i(.,t) \in L^1([0,\infty)); C[0,t_0]), a \in [0,\infty), t \in [0,t_0], \|i(a,t)\| = \sup_{t\in[0,t_0]} \|i(a,t)\|_{L^1} \right\},$$

where $C[0,t_0]$ denotes the Banach space of continuous functions in $[0,t_0]$ and $L^1([0,\infty))$ denotes the space of equivalent classes of Lebesgue integrable functions. We note that E is a Banach space.

In order to facilitate our future calculations, we need the following lemma:

Lemma 3.1 *Suppose that* $x, y \geq 0$, *then* $|e^{-y} - e^{-x}| \leq |y - x|$.

Proof. Let $f(x) = e^{-x}$, then use the mean value theorem to establish the required result.

In the next theorem, we prove the existence and uniqueness of solution to problem (2.1) via a fixed-point theorem.

Theorem 3.2 *Problem (2.1) has a unique solution that exists for all time.*

Proof. Define the set Q by $Q = \{i(a,t) \in E, i(a,t) \geq 0, \|i(a,t)\| \leq M\}$, where M is a constant which satisfies the following:

$$M > \|p_0(a)\|_{L^1} e^{(\|\beta(a)\|_\infty - \mu_*)t_0}. \tag{3.12}$$

We note that Q is a closed set in E. Now, for fixed initial age-distributions $s_0(a), i_0(a)$, and $p_0(a)$, define the mapping $T : Q \subset E \to E$ by

$$Ti(a,t) = \begin{cases} i_0(a-t)e^{-\int_0^t [\mu(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)]d\tau} \\ \\ +\dfrac{p_0(a-t)\pi(a)}{\pi(a-t)}\left[1 - e^{-\int_0^t k_1(a-t+\tau)\lambda(\tau)d\tau}\right], \quad a > t, \\ \\ i(0,t-a)\pi(a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau} \\ \\ +B(t-a)\pi(a)\left[1 - e^{-\int_0^t k_1(\tau)\lambda(t-a+\tau)d\tau}\right], \quad a < t. \end{cases} \tag{3.13}$$

We note that since $s_0(a), i_0(a)$, and $p_0(a)$, and $B(t)$ are non-negative, continuous and integrable functions, we can use (3.8) and (3.10) to show that T maps Q into Q. Now, we look for a fixed point of this mapping to provide existence and uniqueness of solution for problem (2.1). To this end, we let $i(a,t)$ and $i_1(a,t)$ be elements of Q, then using (3.8)-(3.10), and Lemma (3.1), we obtain the following:

$$\|Ti(.,t) - Ti_1(.,)\|_{L^1} \leq K(M,t_0)\int_0^t \|i(.,\sigma) - i_1(.,\sigma)\|_{L^1} d\sigma, \tag{3.14}$$

where $K(M,t_0)$ is a constant which depends on M and t_0. Therefore,

$$\|Ti(.,t) - Ti_1(.,t)\| \leq t_0 K(M,t_0)\|i(.,\sigma) - i_1(.,\sigma)\|. \tag{3.15}$$

And thus, by induction, for each positive integer n, we obtain

$$\|T^n i(.,t) - T^n i_1(.,t)\| \leq \frac{[t_0 K(M,t_0)]^n}{n!}\|i(.,t) - i_1(.,t)\|. \tag{3.16}$$

Inequality (3.16) implies that there exists a positive integer N such that T^N is a strict contraction on Q. Thus T has a unique fixed point in Q. Since t_0 is arbitrary, it follows that problem (2.1) has a unique solution that exists for all time. This completes the proof of the theorem. ■

In the next theorem, we show that solutions of problem (2.1) depend continuously on the initial age-distributions, therefore, problem (2.1) is well posed.

Theorem 3.3 *Let $p(a,t)$ and $p_1(a,t)$ be two solutions of problem (2.1) corresponding to initial age-distributions $p_0(a), s_0(a), i_0(a)$ and $p_{01}(a), s_{01}(a), i_{01}(a)$, respectively. Also, suppose that $p(0,t) = B(t)$ and $p_1(0,t) = B_1(t)$, and let $i(a,t)$ and $i_1(a,t)$ be the corresponding solutions of problem (3.6). Then the following properties hold:*

$$|B(t) - B_1(t)| \leq \|\beta(a)\|_\infty \|p_0(a) - p_{01}(a)\|_{L^1} e^{(\|\beta(a)\|_\infty - \mu_*)t}, \tag{3.17}$$

$$\|p(.,t) - p_1(.,t)\|_{L^1} \leq \|p_0(a) - p_{01}(a)\|_{L^1} e^{(\|\beta(a)\|_\infty - \mu_*)t}, \tag{3.18}$$

$$\|i(.,t) - i_1(.,t)\|_{L^1} \leq \Big[\|i_0(a) - i_{01}(a)\|_{L^1} + 2\|p_0(a) - p_{01}(a)\|_{L^1} e^{\|\beta\|_\infty t_0} \Big] e^{tK(M,t_0)}. \tag{3.19}$$

Proof. Note that (3.17) and (3.18) follow directly from (3.8) and (3.10), respectively, by linearity.

To obtain (3.19), first we use (3.11) and (3.17)-(3.18), and then (3.14) to obtain the following:

$$\|i(.,t) - i_1(.,t)\|_{L^1} \leq \Big[\|i_0(a) - i_{01}(a)\|_{L^1} + 2\|p_0(a) - p_{01}(a)\|_{L^1} e^{\|\beta\|_\infty t_0} \Big] + K(M,t_0) \int_0^t \|i(.,\sigma) - i_1(.,\sigma)\|_{L^1} d\sigma.$$

Now, the foregoing inequality yields (3.19) by the aid of *Gronwall's* inequality. This completes the proof of the theorem. ■

We note that (3.17)-(3.19), show that solutions of problem (2.1) depend continuously on the initial age-distributions, and therefore, problem (2.1) is well posed.

4 The Steady States

In this section, we look at the steady state solution of problem (2.1), under the assumption that the total population has already reached its steady state distribution $p_\infty(a) = c\pi(a)$, i.e., we assume that (3.4) is satisfied with $p^* = 0$, see, for example, Busenberg, et al. (1988).

A steady state $s^*(a)$, $i^*(a)$ and λ^* must satisfy the following equations:

$$\begin{cases} \dfrac{ds^*(a)}{da} + \mu(a)s^*(a) = -k_1(a)s^*(a)\lambda^*, \ \ a > 0, \\ \\ s^*(0) = c - i^*(0), \end{cases} \tag{4.1}$$

$$\begin{cases} \dfrac{di^*(a)}{da} + \mu(a)i^*(a) = k_1(a)s^*(a)\lambda^*, \ \ a > 0, \\ \\ i^*(0) = q\displaystyle\int_0^\infty \beta(a)i^*(a)da, \end{cases} \tag{4.2}$$

$$\lambda^* = \int_0^\infty k_2(a) i^*(a) da. \tag{4.3}$$

Anticipating our future needs, we define a threshold parameter R_0, and is given by

$$R_0 = c \int_0^\infty \int_0^a k_1(\sigma) k_2(a) \pi(a) d\sigma da + \frac{cq \int_0^\infty \int_0^a \beta(a)\pi(a) k_1(\sigma) d\sigma da \left[\int_0^\infty k_2(a)\pi(a) da \right]}{[1-q]}. \tag{4.4}$$

Here, we note that the threshold parameter R_0, usually called the basic reproduction number, and is interpreted as the expected number of secondary cases produced, in a lifetime, by an infective, in a totally susceptible population.

In the following result, we determine the steady state solution of problem (2.1) when $q \neq 1$.

Theorem 4.1 *(see El-Doma (2004a) for a proof.) Suppose that $q \neq 1$, then:*

(1) *If $R_0 > 1$, then $\lambda^* = 0$ and $\lambda^* > 0$ are possible steady states. A steady state with $\lambda^* > 0$ is unique when it exists and it satisfies*

$$1 = c \int_0^\infty \int_0^a k_1(\sigma) k_2(a) \pi(a) e^{-\lambda^* \int_\sigma^a k_1(\tau) d\tau} d\sigma da$$

$$+ \frac{cq \int_0^\infty \int_0^a \beta(a)\pi(a) e^{-\lambda^* \int_\sigma^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \left[\int_0^\infty k_2(a)\pi(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]}{\left[1 - q \int_0^\infty \beta(a)\pi(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]}. \tag{4.5}$$

And in this case $s^*(a)$ and $i^*(a)$ are given by

$$s^*(a) = s^*(0)\pi(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau}, \tag{4.6}$$

$$i^*(a) = i^*(0)\pi(a) + \lambda^* s^*(0)\pi(a) \int_0^a k_1(\sigma) e^{-\lambda^* \int_0^\sigma k_1(\tau) d\tau} d\sigma, \tag{4.7}$$

where $i^*(0)$ satisfies

$$i^*(0) = \frac{cq\lambda^* \int_0^\infty \int_0^a \beta(a)\pi(a) k_1(\sigma) e^{-\lambda^* \int_\sigma^a k_1(\tau) d\tau} d\sigma da}{\left[1 - q \int_0^\infty \beta(a)\pi(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]}. \tag{4.8}$$

(2) *If $R_0 \leq 1$, then the disease-free equilibrium, $\lambda^* = 0$, is the only steady state, i.e., $s^*(a) = c\pi(a)$ and $i^*(a) = 0$.*

Here, we note that the disease will die out if $R_0 \leq 1$ and persists if $R_0 > 1$. The effect of vertical transmission via its parameter q is seen, since the right-hand side of (4.4) is an increasing function of q and therefore, a contributing factor for an endemic disease to occur.

In order to determine the solution of problem (2.1), when $q = 1$, we consider the following transformation, called the age profile of infective:

$$v(a,t) = \frac{i(a,t)}{p_\infty(a)},$$

and note that $s(a,t) = p_\infty(a) - i(a,t)$, since we are assuming that the total population has already reached its steady state distribution $p_\infty(a) = c\pi(a)$. Therefore, from (3.6), $v(a,t)$ satisfies the following:

$$\begin{cases} \dfrac{\partial v(a,t)}{\partial a} + \dfrac{\partial v(a,t)}{\partial t} = k_1(a)[1 - v(a,t)]\lambda(t), \quad a > 0, t > 0, \\ \\ v(0,t) = q\displaystyle\int_0^\infty \beta(a)\pi(a)v(a,t)da, \quad t \geq 0, \\ \\ v(a,0) = v_0(a) = i_0(a)/p_\infty(a), \quad a \geq 0, \\ \\ \lambda(t) = c\displaystyle\int_0^\infty k_2(a)\pi(a)v(a,t)da, \quad t \geq 0. \end{cases} \tag{4.9}$$

Then from (4.9), we see that $v(a,t) = 1$ is a solution of the above problem when q = 1, and therefore by uniqueness of solution (see section 3), it is the only solution for this case. Therefore, $i(a,t) = p_\infty(a) = c\pi(a)$ is the solution for problem (2.1) when $q = 1$.

We note that from (4.7) and $q = 1$, we obtain that

$$\lambda^* s^*(0) \int_0^\infty \int_0^a \beta(a)\pi(a)k_1(\sigma)e^{-\lambda^* \int_0^\sigma k_1(\tau)d\tau} d\sigma da = 0. \tag{4.10}$$

Thus if $\lambda^* = 0$ in (4.10), then using (4.7), we obtain that $i^*(0) = 0$ (we are assuming that $k_2(a)$ is not identically zero), and therefore $i^*(a) = 0$. Accordingly, we obtain the following steady state (disease-free equilibrium):

$$s^*(a) = c\pi(a), i^*(a) = 0. \tag{4.11}$$

Now, if we suppose that $\displaystyle\int_0^\infty \int_0^a \beta(a)\pi(a)k_1(\sigma)e^{-\lambda^* \int_0^\sigma k_1(\tau)d\tau} d\sigma da \neq 0$ and $\lambda^* \neq 0$, then $i^*(0) = c$, and thus we obtain the following steady state:

$$i^*(a) = c\pi(a), s^*(a) = 0, \lambda^* = c\int_0^\infty k_2(a)\pi(a)da. \tag{4.12}$$

Finally, if $\displaystyle\int_0^\infty \int_0^a \beta(a)\pi(a)k_1(\sigma)e^{-\lambda* \int_0^\sigma k_1(\tau)d\tau} d\sigma da = 0$, and $\lambda^* \neq 0$, then $s^*(0)$ is undetermined and thus for each fixed $s^*(0) \in [0, c)$, and using equations (4.7) and (4.3), we obtain

$$\lambda^* \left[1 - s^*(0) \int_0^\infty \int_0^a k_2(a)\pi(a)k_1(\sigma)e^{-\int_0^\sigma \lambda^* k_1(\tau)d\tau} d\sigma da\right] = (c - s^*(0)) \int_0^\infty k_2(a)\pi(a)da. \tag{4.13}$$

We note that the left-hand side of (4.13) equals zero when $\lambda^* = 0$, and increases to $+\infty$, when $\lambda^* \to +\infty$. Accordingly, for each fixed $s^*(0) \in [0, c)$, we can see that (4.13) gives rise to an endemic equilibrium, and hence problem (2.1) gives rise to a continuum of endemic equilibriums in this special case.

5 Global Stability Results

In this section, we prove the global stability of the endemic equilibriums for problem (2.1).

By integrating (4.9) along characteristic lines $t-a=const.$, we find that $v(a,t)$ satisfies

$$v(a,t)=\begin{cases} v_0(a-t)e^{-\int_0^t k_1(a-t+\tau)\lambda(\tau)d\tau}+\displaystyle\int_0^t k_1(a-t+\sigma)e^{-\int_\sigma^t k_1(a-t+\tau)\lambda(\tau)d\tau}\lambda(\sigma)d\sigma, & a>t,\\ v(0,t-a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}+\displaystyle\int_0^a k_1(\sigma)\lambda(t-a+\sigma)e^{-\int_\sigma^a k_1(\tau)\lambda(t-a+\tau)d\tau}d\sigma, & a<t. \end{cases} \tag{5.1}$$

From (4.9), $v(0,t)=q\displaystyle\int_0^\infty \beta(a)\pi(a)v(a,t)da$, then using (5.1) and changing the order of integration several times and making appropriate changes of variables yields

$$\begin{aligned} v(0,t) &= q\Big\{\int_0^t \beta(a)\pi(a)v(0,t-a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}da \\ &\quad+\int_0^t\int_\sigma^\infty \beta(a)\pi(a)k_1(a-\sigma)\lambda(t-\sigma)e^{-\int_{a-\sigma}^a k_1(\tau)\lambda(t-a+\tau)d\tau}dad\sigma \qquad (5.2)\\ &\quad+\int_t^\infty \beta(a)\pi(a)v_0(a-t)e^{-\int_0^t k_1(a-t+\tau)\lambda(\tau)d\tau}da\Big\}. \end{aligned}$$

Also, from (4.9), $\lambda(t)=c\displaystyle\int_0^\infty k_2(a)\pi(a)v(a,t)da$, then using (5.1) and changing the order of integration several times and making appropriate changes of variables yields

$$\begin{aligned} \lambda(t) &= c\Big\{\int_0^t k_2(a)\pi(a)v(0,t-a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}da \\ &\quad+\int_0^t\int_\sigma^\infty k_2(a)k_1(a-\sigma)\pi(a)\lambda(t-\sigma)e^{-\int_{a-\sigma}^a k_1(\tau)\lambda(t-a+\tau)d\tau}dad\sigma \qquad (5.3)\\ &\quad+\int_t^\infty k_2(a)\pi(a)v_0(a-t)e^{-\int_0^t k_1(a-t+\tau)\lambda(\tau)d\tau}da\Big\}. \end{aligned}$$

Note that by Assumptions 2, 4 and 5 of section 2 and the Dominated Convergence Theorem, we obtain

$$\int_t^\infty k_2(a)\pi(a)v_0(a-t)e^{-\int_0^t k_1(a-t+\tau)\lambda(\tau)d\tau}da\longrightarrow 0,\quad \text{as}\quad t\to\infty.$$

Also, by similar reasoning as above,

$$\int_t^\infty \beta(a)\pi(a)v_0(a-t)e^{-\int_0^t k_1(a-t+\tau)\lambda(\tau)d\tau}da\longrightarrow 0,\quad \text{as}\quad t\to\infty.$$

Therefore, setting $v(0,t) = u(t)$, $u(t)$ and $\lambda(t)$ satisfy the following limiting equations (see Busenberg, et al. (1988)):

$$u(t) = q\Bigg\{\int_0^\infty \beta(a)\pi(a)u(t-a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}\,da + \int_0^\infty\int_\sigma^\infty \beta(a)\pi(a)k_1(a-\sigma)\lambda(t-\sigma)e^{-\int_{a-\sigma}^a k_1(\tau)\lambda(t-a+\tau)d\tau}dad\sigma\Bigg\}, \quad (5.4)$$

$$\lambda(t) = c\Bigg\{\int_0^\infty k_2(a)\pi(a)u(t-a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}da + \int_0^\infty\int_\sigma^\infty k_2(a)k_1(a-\sigma)\pi(a)\lambda(t-\sigma)e^{-\int_{a-\sigma}^a k_1(\tau)\lambda(t-a+\tau)d\tau}dad\sigma\Bigg\}. \quad (5.5)$$

We integrate equations (5.4)-(5.5) to obtain the following:

$$u(t) = q\Bigg\{\int_0^\infty \beta(a)\pi(a)u(t-a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}\,da + \int_0^\infty \beta(a)\pi(a)\left[1-e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}\right]da\Bigg\}, \quad (5.6)$$

$$\lambda(t) = c\Bigg\{\int_0^\infty k_2(a)\pi(a)u(t-a)e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}da + \int_0^\infty k_2(a)\pi(a)\left[1-e^{-\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}\right]da\Bigg\}. \quad (5.7)$$

Now, we set $w(t)$ and $g(t)$ to satisfy the following:

$$w(t) = u(t) - u^*,$$
$$g(t) = \lambda(t) - \lambda^*,$$

where $\lambda^*, u^* = \dfrac{i^*(0)}{c}$, are defined as in section 4. Then after some computations, we obtain that $w(t)$ and $g(t)$ satisfy the following:

$$w(t) = q\Bigg\{\int_0^\infty \beta(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}w(t-a)e^{-\int_0^a k_1(\tau)g(t-a+\tau)d\tau}da + [1-u^*]\int_0^\infty \beta(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}\left[1-e^{-\int_0^a k_1(\tau)g(t-a+\tau)d\tau}\right]da\Bigg\}, \quad (5.8)$$

$$g(t) = c\Bigg\{\int_0^\infty k_2(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}w(t-a)e^{-\int_0^a k_1(\tau)g(t-a+\tau)d\tau}da$$

$$+[1-u^*]\int_0^\infty k_2(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}\left[1-e^{-\int_0^a k_1(\tau)g(t-a+\tau)d\tau}\right]da\Bigg\}. \quad (5.9)$$

In the following theorem, we prove the global stability of the endemic equilibrium when $q \neq 1$.

Theorem 5.1 *If $q \neq 1$, then the unique endemic equilibrium, given by Theorem 4.1, is globally stable.*

Proof. Let $|w|^\infty = \limsup_{t\to\infty} |w(t)|$ and $|g|^\infty = \limsup_{t\to\infty} |g(t)|$. Then if we use the fact that $1-e^{-\int_0^\sigma k_1(\tau)g(t-a+\tau)d\tau} \leq \int_0^\sigma k_1(\tau)g(t-a+\tau)d\tau$, and then use $Fatou's$ Lemma in equations (5.8)-(5.9), we obtain that

$$|w|^\infty \leq \frac{q[1-u^*]|g|^\infty}{[1-q]}\int_0^\infty\int_0^a \beta(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}k_1(\sigma)d\sigma da, \quad (5.10)$$

$$|g|^\infty \leq c|w|^\infty\int_0^\infty k_2(a)\pi(a)da + c[1-u^*]|g|^\infty\int_0^\infty\int_0^a k_2(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}k_1(\sigma)d\sigma da. \quad (5.11)$$

Now, using inequality (5.10) in (5.11), we obtain that

$$|g|^\infty \leq |g|^\infty[1-u^*]\Bigg\{\frac{q}{[1-q]}\int_0^\infty\int_0^a \beta(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}k_1(\sigma)d\sigma da\left[c\int_0^\infty k_2(a)\pi(a)da\right]$$

$$+ c\int_0^\infty\int_0^a k_2(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}k_1(\sigma)d\sigma da\Bigg\}. \quad (5.12)$$

We note that u^* is given by

$$1-u^* = \frac{[1-q]}{\left[1-q\int_0^\infty \beta(a)\pi(a)e^{-\lambda^*\int_0^a k_1(\tau)d\tau}da\right]}. \quad (5.13)$$

Also, λ^* is given by

$$\lambda^* = c\int_0^\infty k_2(a)\pi(a)da + [u^*-1]c\int_0^\infty k_2(a)\pi(a))e^{-\lambda^*\int_0^a k_1(\tau)d\tau}da. \quad (5.14)$$

Now, using equations (5.13)-(5.14) in inequality (5.12) and then using equation (4.5), we obtain

$$|g|^\infty < |g|^\infty\left[(1-u^*)+u^*\right]. \quad (5.15)$$

Accordingly, $|g|^\infty = 0$. And hence from inequality (5.10), we obtain that $|w|^\infty = 0$. Therefore, the endemic equilibrium is globally stable. ■

In the following theorem, we prove the global stability of the endemic equilibrium when $q = 1$. We note that, if $q = 1$, then R_0 is not defined, and in this case either there exists a unique endemic equilibrium, which is globally stable, or problem (2.1) gives rise to a continuum of endemic equilibriums, in the case of non-fertile infectibles (i.e., when the support of $k_1(a)$ lies to the right of the support of $\beta(a)$), see section 4.

Theorem 5.2 *Suppose that* $q = 1$, *and* $\int_0^\infty \int_0^a \beta(a)\pi(a)k_1(\sigma)e^{-\lambda^* \int_0^\sigma k_1(\tau)d\tau} d\sigma da \neq 0$. *Then the endemic steady state* $s^*(a) = 0, i^*(a) = c\pi(a)$, *is globally stable.*

Proof. We note that from (5.13), $[1 - u^*] \to 0$ as $q \to 1$. Also, from (5.14), we find that $\lambda^* \to c\int_0^\infty k_2(a)\pi(a)da$ as $q \to 1$. Therefore, using (5.12), we obtain the following as $q \to 1$:

$$|g|^\infty \leq |g|^\infty \frac{\lambda^* \int_0^\infty \int_0^a \beta(a)\pi(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} k_1(\sigma)d\sigma da}{\left[1 - \int_0^\infty \beta(a)\pi(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da\right]}. \tag{5.16}$$

Now, using equation (4.8), we obtain that $|g|^\infty = 0$, since

$$\frac{\lambda^* \int_0^\infty \int_0^a \beta(a)\pi(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} k_1(\sigma)d\sigma da}{\left[1 - \int_0^\infty \beta(a)\pi(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da\right]} < 1.$$

Using $|g|^\infty = 0$ in equation (5.10), we obtain that $|w|^\infty = 0$. Accordingly, we obtain the global stability for the endemic equilibrium. ■

Conclusion

We studied an S I age-structured epidemic model when the disease is vertically as well as horizontally transmitted and the force of infection of proportionate mixing assumption type. The mortality and fertility rates are age-dependent. We note that herpes and AIDS are examples of SI epidemics.

We established the well posedness of the model equations and proved global stability results for the endemic equilibriums. If $q \neq 1$, then the endemic equilibrium is globally stable. If $q = 1$, then either the endemic equilibrium is the population consisting of infective only, and we proved that this endemic equilibrium is globally stable, or the model gives rise to a continuum of endemic equilibriums, if individuals are susceptible only after the end of their reproductive period.

The global stability results that we obtained are under very general conditions, and, in fact, we did not require any condition other than the existence of a unique endemic equilibrium.

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