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A Discrete Time Counterpart of the Black-Scholes Bond Replication Portfolio

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Abstract

We construct a discrete time self-financing portfolio comprised of call options short and stock shares long which is riskless and grows at a fixed rate of return. It is also shown that when shorting periods tend to zero then so devised portfolio turns into the Black-Scholes bond replication. Unlike in standard approach the analysis presented here requires neither Ito Calculus nor solving the Heat Equation for option pricing.

Keywords: Binomial Model, Call Replication, Discrete Time Black-Scholes Hedging

1. Introduction

Black and Scholes (1973) discovered that a portfolio of one call option short and $\frac{\partial C(t,S)}{\partial S}$ stock shares long, where $C(t,S)$ and S is the option and stock price at time t respectively, has no risk and consequently it replicates a bond portfolio earning a fixed rate of return, provided $C(t,S)$ satisfies certain partial differential equation. Since then a great deal of studies circled around discrete time option pricing originated by Cox -Ross-Rubinstein (1979), Rubinstein and Leland (1981), which focused entirely on no-arbitrage option replication portfolio determined from a martingale measure corresponding to the discounted stock price on binomial tree. Meanwhile no attention has been given to the fact that the bond portion of such hedging portfolio, in contrast to its continuous time counterpart, is never riskless! In fact, the bond portion is a unique random process whose distribution at each time step is determined by terminal stock prices under the martingale measure. The above raises a question of whether it is possible to construct riskless portfolio in discrete time analogous to continuous time Black-Scholes bond replication portfolio.

The aim of this article is to answer the question in the positive and explain how the findings bridge the long-standing differences between the two models. Namely, we first construct a self-financing bond replication portfolio by augmenting a combination of suitably chosen number of stock shares long and one call option short with a third position in carry-on cash flow (excess or shortage of funds resulting from creating the stock-option position) and then we convert the cash flow position into the risk free combination of stocks and options. The key feature of this discrete time portfolio lies in the fact that it does not suffer from the well-known shortcomings present in the continuous model: an impossible to implement continuous time trading coupled with prohibitive(infinite) transaction costs associated with it. The model presented here applies to both small and large time period situations and works particularly well in the latter case. For example, when the time intervals are days, months or years and the underlying security can be modeled by assuming only two different outcomes at the end of each time period (e.g., success or failure with a price tag) then over the multi-step period the model provides a hedging strategy that has no risk and for which the transaction costs may in fact be assumed negligible, whereas in the continuous Black-Scholes model the costs cannot be ignored unless the realities of the actual security trading are being rejected!

2. Notation, assumptions and binomial model facts

Let $(\Omega, \mathcal{F}, P)$ be a probability space $\Omega = \{(\omega_1, ..., \omega_N) \mid \omega_i \in \{U, D\}, 1 \le i \le N\}, \mathcal{F} = 2^{\Omega}$, $P((\omega_1, ..., \omega_N)) = p^{\#U's}(1-p)^{\#D's}$, with $0 < p = \dfrac{1+r-d}{u-d} < 1$ subject to $0 < d < 1,\ 1+r < u$ where r is a fixed risk-free rate of return per one time period, $\mathcal{F}_n = \sigma((\omega_1, ..., \omega_n))$ and $\mathcal{F}_o = \{\emptyset, \Omega\},\ n = 1, ..., N$. Then the risk-neutral stock process $S_n \in \mathcal{F}_n$ is defined by P(stock goes up) $= P(S_{n-1} \nearrow S_n = S_{n-1}u) = P(\omega_n = U) = p$ and P(stock goes down) $= P(S_{n-1} \searrow S_n = S_{n-1}d) = P(\omega_n = D) = 1-p$. Here $S_{n-1} = S_{n-1}(\omega_1, ..., \omega_{n-1})$ and $S_n = S_n(\omega_1, ..., \omega_{n-1}, U)$, or $S_n = S_n(\omega_1, ..., \omega_{n-1}, D)$, respectively, and S_n is binomial with $P(S_n = S_0 u^k d^{n-k}) = \binom{n}{k} p^k (1-p)^{n-k}$. We emphasize that the probability distribution of the actual stock process is irrelevant as it does not play any role in determining no-arbitrage option pricing, aside from the fact that it is rarely known! The only thing the two stock processes have in common is the identical sample space $\{S_0, S_1\ ...,\ S_N\}$ which constitutes the binomial binary tree.

We make the following assumptions about discrete time portfolio modeling:

(p1) prices of stocks, options and bonds change at $n = 0, ..., N$ and stay constant throughout $[n-1, n),\ n = 1, ..., N$;

(p2) portfolio values are declared at $n = 0, ..., N$ and stay constant during $[n-1, n),\ n = 1, ..., N$;

(p3) positions (rebalanced holdings) are decided based on time n-1 prices, take effect instantaneously after time n-1 and are held unchanged throughout $(n\text{-}1, n],\ n = 1, ..., N$;

(p4) portfolio is self-financing in the sense that its value at time $n-1$ and after rebalancing are equal, $n = 1, ..., N$; and

(p5) portfolio transaction costs are zero, borrowing along with short and fractional sales are allowed, option pricing assumes no-arbitrage.

The significance of (p3) stems from the fact that different portfolio positions must necessarily be held over non-overlapping time intervals until the securities undergo price change in order to unambiguously define portfolio holdings and value when rebalancing. Moreover, (p3) incorporates the fact that security prices must be known first before the portfolio adjustments can be made and consequently one way to implement this is to consider the discrete time sequence as a subset of the continuous time with a proviso that rebalancing takes effect instantaneously after the prices have changed.

Allowing the portfolio to undergo content and value changes periodically at discrete time instances only, while keeping the portfolio intact at all other times, considerably simplifies the analysis of the portfolio dynamics. In addition, it offers a practical alternative to a continuous time model while retaining its validity for both large and small scales of portfolio inactivity period.

A basic fact from binomial asset pricing model Karatzas & Shreve (1998) or Williams (2001) states that a European call option can be replicated through a hedging strategy by holding a portfolio of uniquely determined combination of stocks and bonds. Since the option replication serves as basis for our bond replication construction we first recall all necessary facts needed for further analysis.

Given a strike price $K > 0$ the value of the call option at time N is defined by

$$C_N = Max(S_N - K, 0) \equiv (S_N - K)^+$$

and we use the following notation:

S_n is the stock price at time n

C_n is the call option price at time n

$B_n = B_0(1+r)^n$ is the bond price at time n

a_n is the number of stock shares held during $(n-1, n]$

b_n is the number of bonds shares held during $(n-1, n]$

r is a fixed risk-free return rate per period

Then by (p3)-(p4) portfolio values $\{X_n, 0 \le n \le N\}$ satisfy

$$X_0 = a_0 S_0 + b_0 B_0 = a_1 S_0 + b_1 B_0$$

$$X_{n-1} = a_{n-1} S_{n-1} + b_{n-1} B_{n-1} = a_n S_{n-1} + b_n B_{n-1} \text{ and } X_n = a_n S_n + b_n B_n, \qquad n = 1, ..., N \tag{1}$$

Remark. Since the initial assets allocation (a_0, b_0) for X_0 is limited to time $n = 0$ and then instantaneously replaced by (a_1, b_1) we may assume without loss of generality that $a_0 = a_1$ and $b_0 = b_1$ because by $X_0 = a_0 S_0 + b_0 B_0 = a_1 S_0 + b_1 B_0$ this has no bearing on the initial portfolio value. Therefore (a_1, b_1) is taken as the initial allocation throughout [0,1] (not just (0,1]) which renders the choice of (a_0, b_0) as a technical assumption and allows identifying X_n allocations with a sequence (a_n, b_n), $n = 1, ..., N$.

A call option replication portfolio $\Pi_c = \{X_n, (a_n, b_n),\ 0 \le n \le N\}$ is determined by

$$X_n = a_n S_n + b_n B_n = C_n = (1+r)^{-(N-n)} E[(S_N - K)^+ \mid \mathcal{F}_n], \qquad n = 0, \ldots, N \tag{2}$$

$$a_n = \frac{C_n(S_{n-1}u) - C_n(S_{n-1}d)}{S_{n-1}u - S_{n-1}d} \in \mathcal{F}_{n-1}, \qquad n = 1, \ldots, N \tag{3}$$

$$b_n = \frac{C_n - a_n S_n}{B_n} = \frac{C_{n-1} - a_n S_{n-1}}{B_{n-1}} \in \mathcal{F}_{n-1}, \qquad n = 1, \ldots, N \tag{4}$$

One checks that $X_n, a_n \ge 0$ but b_n can be negative in which case an investor owes $-b_n B_n$ or equivalently the investor sold b_n shares of bond short. C_n is determined by backward recursion that starts at the terminal time N of the binomial tree according to

$$C_{n-1}(S_{n-1}) = \frac{1}{1+r}[pC_n(S_{n-1}u) + (1-p)C_n(S_{n-1}d)], \qquad n = N, N-1, \ldots, 2, 1 \tag{5}$$

3. Construction of bond replication portfolio

The first step is to built the option replication portfolio $\Pi_c = \{X_n, (a_n, b_n),\ 0 \le n \le N\}$ satisfying (1)-(5). The second step amounts to recursive construction of a portfolio with values $\{Y_n, 0 \le n \le N\}$ subject to (p1)-(p5) which is based on Π_c and grows risk free.

The interval [0,1]. Set $Y_0 = a_0 S_0 - C_0 \equiv a_1 S_0 - C_0$ as the initial capital. Define

$$Y_0 = a_1 S_0 - C_0 = a_1 S_0 - C_0 + (Y_0 + C_0 - a_1 S_0) \equiv a_1 S_0 - C_0 + g_1 \quad \text{during } (0,1).$$

Then

$$Y_1 = a_1 S_1 - C_1 = Y_0(1+r),$$

since $a_1 S_0 - C_0$ becomes $a_1 S_1 - C_1 = -b_1 B_1 = -b_1 B_0(1+r) = (a_1 S_0 - C_0)(1+r) = Y_0(1+r)$ due to $C_0 = X_0 = a_0 S_0 + b_0 B_0 = a_1 S_0 + b_1 B_0$ and $C_1 = X_1 = a_1 S_1 + b_1 B_1$ while $g_1 \equiv 0$.

The interval (1,2]. Use the first step and define

$$Y_1 = Y_0(1+r) = a_2 S_1 - C_1 + (Y_1 + C_1 - a_2 S_1) \equiv a_2 S_1 - C_1 + g_2 \quad \text{during } (1,2)$$

Then

$$Y_2 = a_2 S_2 - C_2 + g_2(1+r) = Y_0(1+r)^2,$$

since $a_2 S_1 - C_1$ becomes $a_2 S_2 - C_2 = -b_2 B_2$ and g_2 earns interest r during (1,2) yielding $g_2(1+r) = Y_1(1+r) + (C_1 - a_2 S_1)(1+r) = Y_1(1+r) + b_2 B_1(1+r) = Y_0(1+r)^2 + b_2 B_2$ due to $C_1 = X_1 = a_1 S_1 + b_1 B_1 = a_2 S_1 + b_2 B_1$.

The interval $(n-1, n]$, $n = 3, \ldots, N$. Use the $n-1$ step and define

$$Y_{n-1} = Y_0(1+r)^{n-1} = a_n S_{n-1} - C_{n-1} + (Y_{n-1} + C_{n-1} - a_n S_{n-1}) \equiv a_n S_{n-1} - C_{n-1} + g_n \text{ on } (n\text{-}1, n)$$

Then

$$Y_n = a_n S_n - C_n + g_n(1+r) = Y_0(1+r)^n, \quad g_n \in \mathcal{F}_{n-1}$$

by an argument similar to the case $n = 1, 2$.

Summing up, given $\{S_{n-1},\ a_n,\ C_{n-1},\ Y_{n-1}\} \in \mathcal{F}_{n-1}$ create portfolio positions for $n = 1, \ldots, N$ as follows:

Open a position taking effect instantaneously after time $n-1$ according to
- sell the entire holdings to receive Y_{n-1}
- sell one option short to receive C_{n-1}
- buy a_n shares of stock to spend $a_n S_{n-1}$
- hold the cash flow $g_n = Y_{n-1} + C_{n-1} - a_n S_{n-1}$ earning interest r over $(n-1, n)$

Close the position at time n
- declare the portfolio value Y_n

Theorem 1:
Let $\Pi_c = \{X_n, (a_n, b_n),\ 0 \le n \le N\}$ be the option replication portfolio. Then there exists a bond replication portfolio $\Pi = \{Y_n, (a_n, g_n),\ 0 \le n \le N\}$ of a_n shares of stock long, one call option short and a carry-on cash flow g_n such that

(i) $Y_{n-1} = a_n S_{n-1} - C_{n-1} + g_n$ during the time period $(n\text{-}1,n)$

(ii) $Y_n = a_n S_n - C_n + g_n(1+r) = Y_0(1+r)^n$ at time n, $n = 1, \ldots, N$ (6)

Proof: By construction, it is a consequence of unique $Y_0 \equiv a_0 S_0 - (1+r)^{-N} E[(S_N - K)^+]$, $a_0 \equiv a_1, a_n, C_n$ found from (3) - (5) and $g_n = Y_{n-1} + C_{n-1} - a_n S_{n-1} = (b_n - b_1) B_{n-1}$, because $a_n S_n - C_n = -b_n B_n$ and $Y_{n-1} = Y_0(1+r)^{n-1} = -b_1 B_0 (1+r)^{n-1} = -b_1 B_{n-1}$ thanks to $C_{n-1} = X_{n-1}$ $= a_{n-1} S_{n-1} + b_{n-1} B_{n-1} = a_n S_{n-1} + b_n B_{n-1}$.

Below we provide concrete calculations in the case $N = 3$.

Example 1. Let $S_0 = K = 40$, $u = \frac{5}{4}$, $d = \frac{4}{5}$, $r = \frac{1}{40}$ which gives $p = \frac{1+r-d}{u-d} = \frac{1}{2}$. A special choice of $ud = 1$, r and consequently $p = \frac{1}{2}$ is only for keeping calculations simple but there are no such restrictions in general, other than as specified in section 2. Solving (5) backward for C_n and using (3) we obtain $C_0 = 7.90$, $a_1 = .636$ and $Y_0 = a_1 S_0 - C_0 = 17.54$.
Then $\{a_2(U) = .826,\ a_2(D) = .338\}, \{a_3(U,U) = 1,\ a_3(U,D) = a_3(D,U) = .555,$ $a_3(D,D) = 0\}, \{C_1(U) = 13.83,\ C_1(D) = 2.38\}, \{C_2(U,U) = 23.47,\ C_2(U,D) = C_2(D,U) =$ $4.88, C_2(D,D) = 0\}, \{C_3(U,U,U) = 38\frac{1}{8},\ C_3(U,U,D) = C_3(U,D,U) = C_3(D,U,U) = 10,$ $C_3(U,D,D) = C_3(D,U,D) = C_3(D,D,U) = C_3(D,D,D) = 0\}$. For the stock process $\{S_1(U) = 50,\ S_1(D) = 32\}, \{S_2(U,U) = 62\frac{1}{2},\ S_2(U,D) = S_2(D,U) = 40,\ S_2(D,D) = 25\frac{3}{5}\},$ $\{S_3(U,U,U) = 78\frac{1}{8},\ S_3(U,U,D) = S_3(U,D,U) = S_3(D,U,U) = 50,\ S_3(U,D,D) = S_3(D,U,D)$

$= S_3(D,D,U) = 32,\ S_3(D,D,D) = 20\tfrac{12}{25}\}$.

Calculations were carried out in several decimal places but only two or three digits are displayed here as needed. We remark that the bond position b_3 in the call option replication portfolio $\Pi_c = \{X_n, (a_n, b_n),\ 0 \le n \le N\}$ is a random variable with a distribution corresponding to binomial probabilities and thus not riskless! Indeed, by $a_3(U,U) = 1$ it follows that

$P(b_3 = \frac{C_3(U,U,U) - S_3(U,U,U)}{(1+r)^3}) = P(S_2 = S_0 u^2) = p^2,\ P(b_3 = \frac{C_3(U,D,U) - S_3(U,D,U)}{(1+r)^3}) =$

$P(S_2 = S_0 ud \cup S_2 = S_0 du) = 2p(1-p),\ P(b_3 = 0) = P(S_2 = S_0 d^2) = (1-p)^2.$

We illustrate the actual portfolio trading and rebalancing on a single stock path which first goes up then down and then up again, i.e., $S_0 = 40,\ S_1(U) = S_0 u,\quad S_2(U,D) = S_0 ud,$ $S_3(U,D,U) = S_0 udu = 50.$ Open a position on (0,1] by selling one option short and receive $C_0 = 7.90$, which combined with initial $Y_0 = 17.54$ buys $a_1 = .636$ shares of stock at the price $S_0 = 40$ and leaves the cash flow $g_1 = 0$. Close the position at time 1 with holdings valued $Y_1 = a_1 S_1(U) - C_1(U) = 17.98 = Y_0(1+r)$. Open a position on (1,2] by selling one call option short to receive $C_1(U) = 13.83,$ which combined with initial $Y_1 = 17.98$ requires to borrow $g_2(U) = Y_1 + C_1(U) - a_2(U)S_1(\mathrm{U}) = -9.49$ to buy $a_2(U) = .826$ shares of stock at the price $S_1(U) = 50$ and arrive at $Y_1 = a_2(U)S_1(U) - C_1(\mathrm{U}) + g_2(U)$. Close the position at time 2 with holdings valued $Y_2 = a_2(U)S_2(U,D) - C_2(U,D) + g_2(U)(1+r) =$ $18.43 = Y_0(1+r)^2$. Open a position on (2,3] by selling one call option short to receive $C_2(U,D) = 4.88,$ which combined with initial Y_2=18.43 buys $a_3(U,D) = .555$ shares of stock at the price $S_2(U,D) = 40$ and leaves $g_3(U,D) = Y_2 + C_2(U,D) - a_3(U,D)S_2(U,D) = 1.11$ to invest at rate r. Close the position at time 3 with holdings valued $Y_3 = a_3(U,D)S_3(U,D,U)$ $-C_3(U,D,U) + g_3(U,D)(1+r) = 18.89$ which again matches the desired $Y_0(1+r)^3$. The analysis of the other seven cases is similar and shows that in each time step the porfolio so devised matches the fixed rate of return given the initial investment Y_0.

Theorem 2: There exists a bond replication portfolio $\Pi_b = \{Y_n^*, (a_n^*, d_n^*),\ 0 \le n \le N\}$ of a_n^* shares of stock long and d_n^* call options short which replicates the bond exactly in each time interval according to

(i) $Y_{n-1}^* = a_n^* S_{n-1} - d_n^* C_{n-1}$ during the time period $(n-1, n)$

(ii) $Y_n^* = a_n^* S_n - d_n^* C_n = Y_0(1+r)^n$ at time n , $n = 1, \dots, N$ (7)

where

$$a_n^* = \frac{b_1}{b_n} a_n \quad , \quad d_n^* = \frac{b_1}{b_n} \quad \text{with } (a_n, b_n) \text{ determined by } \Pi_c = \{X_n, (a_n, b_n)\} \qquad (8)$$

Proof: It follows from *Theorem*1 because by $C_{n-1} = X_{n-1} = a_n S_{n-1} + b_n B_{n-1}$ we have $-b_n B_{n-1} = a_n S_{n-1} - C_{n-1}$ which multiplied by $\frac{b_n - b_1}{-b_n}$ gives $g_n = a_n(\frac{b_1}{b_n} - 1)S_{n-1} - (\frac{b_1}{b_n} - 1)C_{n-1}$ and consequently $Y_{n-1} = a_n S_{n-1} - C_{n-1} \; + \; g_n = a_n \frac{b_1}{b_n} S_{n-1} - \frac{b_1}{b_n} C_{n-1} = Y_{n-1}^*$ as claimed.

To avoid the phrase *instantaneously after time n* the working assumption below is to use the term *at time n* whenever referring to buy or sell in reference to opening or closing any given position.

Example 2. As before we illustrate the hedging for the up-down-up case. Solving for b_1, $b_2 = b_2(U)$, $b_3 = b_3(U, D)$ gives $\frac{b_1}{b_2} = .654$ and $\frac{b_1}{b_3} = 1.0627$, whence $(a_1^*, d_1^*) = (a_1, 1)$, $(a_2^*, d_2^*) = (.5405,\ .654)$, $(a_3^*, d_3^*) = (.5903,\ 1.0627)$. The first step is the same as in Example1 because $g_1 = 0$. That is, start at time 0 with $Y_0 = 17.54$, short one call at $C_0 = 7.90$ and buy $a_1^* = a_1 = .636$ shares of stock at $S_0 = 40$, then at time 1 sell the stock at $S_1(U) = 50$ and buy back call at $C_1(U) = 13.83$ to arrive at $Y_1 = 17.98 = Y_0(1+r)$. At time 1, with $Y_1 = 17.98$, short $d_2^*(U)$ = .654 calls at $C_1(U) = 13.83$ and buy $a_2^*(U)$=.5405 shares of stock at $S_1(U) = 50$, then at time 2 sell the stock at $S_2(U, D) = 40$ and buy back calls at $C_2(U, D) = 4.88$ to arrive at $Y_2 = 18.43 = Y_0(1+r)^2$. At time 2, with $Y_2 = 18.43$, short $d_3^*(U, D)$ = 1.0627 calls at $C_2(U, D) = 4.88$ and buy $a_3^*(U, D) = .5903$ shares of stock at $S_2(U, D) = 40$, then at time 3 sell the stock at $S_3(U, D, U) = 50$ and buy back calls at $C_3(U, D, U) = 10$ to arrive at $Y_3 = 18.89 = Y_0(1+r)^3$.

4. Convergence to Black-Scholes bond replication portfolio

Continuous time modeling assumes the actual stock price to follow exponential Brownian motion process $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ with $ES_t = S_0 e^{\mu t}$. In the special case of $\mu = r$ it turns into the risk-neutral process $S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$ which corresponds to the discrete time risk-neutral process S_n discussed earlier. Here $\{W_t, 0 \le t \le T\}$ is the standard Brownian motion, μ is the average return rate on stock, r is the bond return rate and σ is the stock volatility. Notice that the expected return on the risk-neutral process $ES_t = S_0 e^{rt}$ and when volatility is set to zero then the risk-neutral process turns into the bond process itself. The Black-Scholes no-arbitrage option price $C_t = e^{-r(T-t)} E[(S_T - K)^+ \mid S_t]$, where the conditional expected value is taken with respect to the martingale measure corresponding to the discounted risk-neutral process, says that given the actual stock price $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ the option price at time t is calculated as follows:

$$C_t = e^{-r(T-t)} E[(S_T - K)^+ \mid S_t] = e^{-r(T-t)} E[(S_t e^Z - K) 1_{[\ln[\frac{K}{S}], \infty)}(Z)] \qquad (9)$$

where

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T - W_t)} \equiv S_t e^Z, \ \ Z \text{ is normal} \sim N((r-\tfrac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)) \qquad (10)$$

Furthermore, the Black-Scholes bond replication portfolio $\Pi_{B-S} = \{Y_t, (a_t, 1),\ 0 \le t \le T\}$ of a_t shares of stock long and 1 call option short satisfies the following

$$Y_t = a_t S_t - C_t = Y_0 e^{rt} \ \text{ with } Y_0 = a_0 S_0 - e^{-rT} E[(S_T - K)^+] \qquad (11)$$

with a_t is determined by

$$a_t = \frac{\partial C_t}{\partial S}\Big|_{S=S_t} = e^{-r(T-t)} \frac{d}{S}\int_{\ln[\frac{K}{S}]}^{\infty} (Se^z - K) dF_Z(z)\Big|_{S=S_t} = e^{-r(T-t)} E[e^Z 1_{[\ln[\frac{K}{S_t}],\infty)}(Z)] \qquad (12)$$

Evaluating the above expectations gives $C_t = S_t \Phi(\alpha) - e^{-r(T-t)} K\Phi(\beta)$ and $a_t = \Phi(\alpha)$

where $\alpha = \dfrac{\ln\frac{S_t}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t)}}$, $\beta = \alpha - \sigma\sqrt{T-t}$ and $\Phi(x)$ is the standard normal cumulative distribution.

To see how our previously constructed riskless portfolio Π_b on binomial tree becomes the celebrated Black-Scholes portfolio Π_{B-S} in the limit, additional notation with certain particular choices for underlying variables are in order. Let in the discrete time model $N = n,$ the time period $\Delta t = \dfrac{T}{n},$ $n = m,$ $r\Delta t$ be the return rate per period, $u = e^{\sigma\Delta t},$ $d = \frac{1}{u},$ $p = \dfrac{e^{r\Delta t} - d}{u-d}$. Furthermore, let for any $n = 1, 2, .., U_{n1}, ..., U_{nn}$ be a sequence of independent identically distributed random variables with $P(U_{ni} = u) = p,\ P(U_{ni} = d) = 1-p$. Then for $1 \le m \le n,$ we have $S_m = S_0 U_{n1} \cdots U_{nm} = S_0 e^{\sigma\Delta t(\varepsilon_{n1}+...+\varepsilon_{nm})}$ and ε_{ni}'s are i.i.d. $P(\varepsilon_{ni} = 1) = p,$ $P(\varepsilon_{ni} = -1) = 1-p.$ In the sequel, whenever invoking basic facts about convergence in distribution, Billingsley (1999) is a standard reference without further mention. Denoting by $[x]$ the integer part of x we have the following convergence in distribution

$$Z_1^m = \sigma\Delta t(\varepsilon_{n1} + ... + \varepsilon_{nm}) \xrightarrow{d} Z_1 \sim N((r-\tfrac{1}{2}\sigma^2)t, \sigma^2 t)\ ,\ m = [n\tfrac{t}{T}] \to \infty \qquad (13)$$

and

$$Z_{m+1}^n = \sigma\Delta t(\varepsilon_{nm+1} + ... + \varepsilon_{nn}) \xrightarrow{d} Z \sim N((r-\tfrac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)),\ m = [n\tfrac{t}{T}] \to \infty\,. \ (14)$$

This follows because $Z_1^m = W_1 + ... + W_m + mE\sigma\Delta t\varepsilon_{ni},$ $W_i = \sigma\Delta t\varepsilon_{ni} - E\sigma\Delta t\varepsilon_{ni},$ $EW_i = 0$, $E\sigma\Delta t\varepsilon_{ni} = (r-\frac{1}{2}\sigma^2)\Delta t + o(\Delta t),$ $\mathrm{Var}(W_i) = \sigma^2\Delta t + o(\Delta t),$ $E|W_i|^3 \le \sigma^3\Delta t^3, m\Delta t \to t$ and $m\, o(\Delta t) \to 0$ as $n \ge m \to \infty$. Therefore from Lindeberg's Central Limit Theorem $\dfrac{W_1 + ... + W_m}{\sqrt{Var(W_1 + ... + W_m)}} \xrightarrow{d} N(0,1)$, thanks to Lyapunov's $\dfrac{E|W_1|^3 + ... + E|W_m|^3}{(\sqrt{Var(W_1 + ... + W_m)})^3} \le \dfrac{2\sigma^2\sqrt{T}^3}{n^2} \to 0$ condition, as $n \ge m \to \infty$ concludes the proof of (13). The proof of (14) is similar.

Lemma:
Let $X_n(\omega), Y_n(\omega'),\ n = 1, 2, \ldots,$ be independent defined on $(\Omega \mathrm{x} \Omega', \mathcal{F}\mathrm{x}\mathcal{F}', P\mathrm{x}P')$ such that $\sup_n Ee^{2Y_n} < M < \infty$. If $X_n \xrightarrow{d} X = X(\omega), Y_n \xrightarrow{d} Y$ and $F(y) = P'(Y \le y)$ is continuous then

$$P(\omega \mid \lim_{n\to\infty} Ee^{Y_n} 1_{[X_n(\omega),\infty)}(Y_n) = Ee^Y 1_{[X(\omega),\infty)}(Y)) = 1 \tag{15}$$

Proof: Due to Skorokhod's representation theorem one can assume without loss of generality that X_n, X are chosen in such a way that $P(\omega \mid \lim_{n\to\infty} X_n(\omega) = X(\omega)) = 1$.
By assumption $F_n(y) = P'(Y_n \le y) \to F(y)$ for every real number y. Then denoting by $X_n \wedge X = \min(X_n, X),\ X_n \vee X = \max(X_n, X)$ we have

$$Ee^{Y_n} 1_{[X_n(\omega),\infty)}(Y_n) = \int e^y 1_{[X_n\wedge X, X_n \vee X)}(y) dF_n(y) + \int e^y 1_{[X,\infty)}(y) dF_n(y) \to \int e^y 1_{[X,\infty)}(y) dF(y)$$

because by Cauchy-Schwarz inequality the first integral above is dominated by

$$\int e^{2y} dF_n(y) \int 1_{[X_n\wedge X, X_n\vee X)}(y) dF_n(y) \le M(F_n(X_n \vee X) - F_n(X_n \wedge X - \tfrac{1}{n})) \to 0$$

whence $X_n \vee X \to X, X_n \wedge X - \frac{1}{n} \to X$ and continuity of F implies $F_n(y_n) \to F(y)$, whereas the second integral $Ee^{Y_n} 1_{[X,\infty)}(Y_n)$ converges to $Ee^Y 1_{[X,\infty)}(Y)$ for $X = X(\omega)$ fixed because $e^{Y_n} 1_{[X,\infty)}(Y_n)$ are uniformly integrable by assumption $\sup_n Ee^{2Y_n} < M < \infty$, and $h(y) = e^y 1_{[X,\infty)}(y)$ is continuous except when $y = X$ which has probability zero due to continuity of $F(y)$.

Theorem 3:
The discrete time bond replication portfolio $\Pi_b = \{Y_n^*, (a_n^*, d_n^*),\ 0 \le \mathrm{n} \le \mathrm{N}\}$ with $Y_m^* = a_m^* S_m - d_m^* C_m = Y_0^*(1 + r\Delta t)^m$, converges to the Black-Scholes bond replication portfolio $\Pi_{B-S} = \{Y_t, (a_t, 1),\ 0 \le t \le T\}$ with $Y_t = a_t S_t - C_t = Y_0 e^{rt}, 0 \le t \le T$. The convergence $\Pi_b = \{Y_n^*, (a_n^*, d_n^*)\} \ \to\ \Pi_{B-S} = \{Y_t, (a_t, 1),\ 0 \le t \le T\}$ means

(i) $S_m \to S_t,\quad C_m \to C_t,\quad a_m^* \to a_t,\quad d_m^* \to 1,\quad Y_m^* \to Y_0 e^{rt}$ as $m = [n\frac{t}{T}] \to \infty$

(ii) $S_n \xrightarrow{d} S = \{S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t},\ 0 \le \mathrm{t} \le \mathrm{T}\}$ in $C[0,T]$ as $n \to \infty$

Proof: Since part (ii) is standard, e.g., it follows from Prohorov's generalization of Donsker's Invariance Principle for Brownian motion in Billinglsely (1999), it suffices to show (i). As before, since by (13) $S_m = S_0 e^{Z_1^m} \xrightarrow{d} S_0 e^{Z_1} = S_t$, we may assume without loss of generality that $S_m \to S_t$ with probability one. By (2) with the following substitutions, $N = n,\ n = m,\ r := r\Delta t$ and $m = [n\frac{t}{T}]$, (9)-(10), *Lemma* applied to $Y_n = Z_{m+1}^n \xrightarrow{d} Z,\ X_n = \ln\frac{K}{S_m} \to \ln\frac{K}{S_t}$ and (15) we obtain with probability one

$$C_m = (1+r\Delta t)^{-(n-m)} E[(S_n - K)^+ \mid S_m] \tag{16}$$

$$= (1+r\Delta t)^{-(n-m)} E[(S_m e^{Z^n_{m+1}} - K) 1_{[ln[\frac{K}{S_m}],\infty)}(Z^n_{m+1})] \to e^{-r(T-t)} E[(S_t e^Z - K) 1_{[ln[\frac{K}{S_t}],\infty)}(Z)] = C_t$$

thanks to continuity of Z and the fact that $\sup_n Ee^{2Y_n} < \infty$ which follows from

$$Ee^{2Z^n_{m+1}} = [Ee^{2\sigma\Delta t}]^{n-m} = [e^{2\sigma\Delta t} p + e^{-2\sigma\Delta t})(1-p)]^{n-m} = [1+(2r+\sigma^2)\Delta t + o(\Delta t)]^{n-m}$$
$$\to e^{(2r+\sigma^2)(T-t)}$$

To see that $a^*_m \to a_t$, $d^*_m \to 1$ it suffices to check that $g_m = (b_m - b_1)B_{m-1} \to 0$ and $a_m \to a_t$ because then $\frac{b_1}{b_m} = d^*_m \to 1$. Namely, convergence of portfolio values Y_m of *Theorem* 1 to Y_t implies convergence Y^*_m of *Theorem* 2 to Y_t provided $g_m \to 0$. To show that $g_m \to 0$ it suffices to verify $C_m \to C_t$, $a_m \to a_t$, and

$$Y_m = Y_0(1+r\Delta t)^m = (a_0(Y_n)S_0 - (1+r\Delta t)^{-n} E[(S_n - K)^+])(1+r\Delta t)^m \to Y_t = Y_0 e^{rt}$$
$$= (a_0(Y_T)S_0 - e^{-rT} E[(S_T - K)^+])e^{rt}.$$

The latter convergence follows from $a_m \to a_t$ (i.e., $a_0(Y_n) \to a_0(Y_T)$ for $t=0$), $S_n = S_0 e^{Z_1^m + Z^n_{m+1}} \xrightarrow{d} S_0 e^{Z_1 + Z} = S_T$ by (13)-(14) and the fact that S_n are uniformly integrable. Consequently, by (16), the proof will be concluded if we show that $a_m \to a_t$. Given S_{m-1}, by (2)-(3) applied to $S_m = S_{m-1}u$ and $S_m = S_{m-1}d$ we have

$$a_m = \frac{C_m(S_{m-1}u) - C_m(S_{m-1}d)}{S_{m-1}u - S_{m-1}d} =$$

$$\frac{(1+r\Delta t)^{-(n-m)}\{E[(S_{m-1}ue^{Z^n_{m+1}} - K)1_{[ln[\frac{K}{S_{m-1}u}],\infty)}(Z^n_{m+1})] - E[(S_{m-1}de^{Z^n_{m+1}} - K)1_{[ln[\frac{K}{S_{m-1}d}],\infty)}(Z^n_{m+1})]\}}{S_{m-1}u - S_{m-1}d} =$$

$$\frac{(1+r\Delta t)^{-(n-m)}\{E[(S_{m-1}u - S_{m-1}d)e^{Z^n_{m+1}}1_{[ln[\frac{K}{S_{m-1}u}],\infty)}(Z^n_{m+1})] + E[(S_{m-1}de^{Z^n_{m+1}} - K)1_{[ln[\frac{K}{S_{m-1}u}],ln[\frac{K}{S_{m-1}d}])}(Z^n_{m+1})]\}}{S_{m-1}u - S_{m-1}d}$$

$$= (1+r\Delta t)^{-(n-m)}\{E[e^{Z^n_{m+1}}1_{[ln[\frac{K}{S_{m-1}u}],\infty)}(Z^n_{m+1})] + \frac{(1+r\Delta t)^{-(n-m)}E[(S_{m-1}de^{Z^n_{m+1}} - K)1_{[ln[\frac{K}{S_{m-1}u}],ln[\frac{K}{S_{m-1}d}])}(Z^n_{m+1})]}{S_{m-1}u - S_{m-1}d}$$

$$\to e^{-r(T-t)}E[e^Z 1_{[\ln[\frac{K}{S_t}],\infty)}(Z)] = a_t$$, by *Lemma* and (12), as $m = [n\frac{t}{T}] \to \infty$

since $|(S_{m-1}de^{Z^n_{m+1}} - K)| \le \frac{K}{S_{m-1}u}(S_{m-1}u - S_{m-1}d)$ on the interval $[\ln[\frac{K}{S_{m-1}u}], \ln[\frac{K}{S_{m-1}d}])$ while $E1_{[ln[\frac{K}{S_{m-1}u}],ln[\frac{K}{S_{m-1}d}])}(Z^n_{m+1})] \le F_m(ln[\frac{K}{S_{m-1}d}]) - F_m(ln[\frac{K}{S_{m-1}u}] - \frac{1}{m}) \to 0$, $u = u(m) \to 1$, $F_m \xrightarrow{d} F$ where F_m and F is the distribution function of Z^n_{m+1} and Z respectively.

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