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# On Strong Domination Number of Graphs

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**Abstract**

A subset $S$ of a vertex set $V$ is called a dominating set of graph $G$ if every vertex of $V - S$ is dominated by some element of set $S$. If $e$ is an edge with end vertices $u$ and $v$ and degree of $u$ is greater than or equal to degree of $v$ then we say $u$ strongly dominates $v$. If every vertex of $V - S$ is strongly dominated by some vertex of $S$ then $S$ is called strong dominating set. The minimum cardinality of a strong dominating set is called the strong domination number of graph. We investigate strong domination numbers of some graphs and study related parameters.

**Keywords:** Domination number; Independent domination number; Strong domination number

**MSC 2010 No.:** 05C69, 05C76

## 1. Introduction

We consider the simple, finite, connected and undirected graph $G$ with vertex set $V$ and edge set

$E$. For all standard terminology and notations we follow West (2003) while the terms related to the theory of domination in graphs are used in the sense of Haynes et al. (1998). The domination number is a well studied parameter as observed by Hedetniemi and Laskar (1990).

**Definition 1.**

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a $dominating\ set$ if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. A dominating set $S$ is a $minimal\ dominating\ set$ if no proper subset $S^{'} \subset S$ is a dominating set. The set of all minimal dominating sets of a graph $G$ is denoted by $MDS(G)$. The $domination\ number\ \gamma(G)$ of a graph $G$ equals the minimum cardinality of a set in $MDS(G)$.

**Definition 2.**

A set $S \subseteq V$ is an $independent\ set$ of $G$, if $\forall u, v \in S$, $N(u) \cap \{v\} = \phi$. A dominating set which is independent, is called an $independent\ dominating\ set$. The minimum cardinality of an independent dominating set in $G$ is called the $independent\ domination\ number\ i(G)$ of a graph $G$.

The theory of independent domination was formalized by Berge (1962) and Ore (1962). Allan and Laskar (1978) have discussed some results for which $\gamma(G) = i(G)$ whereas bounds on the independent domination number are determined by Goddard and Henning (2013). Vaidya and Pandit (2016) have investigated the exact value of independent domination number of some wheel related graphs.

**Definition 3.**

Let $G$ be a graph and $uv \in E$. Then, $u$ strongly dominates $v$ ($v$ weakly dominates $u$) if $deg(u) \geq deg(v)$. It is obvious that every vertex of $V$ can strongly dominate itself. A set $S$ is a $strong\ dominating\ set$ (weak dominating set) if every vertex $v \in V - S$ is strongly (weakly) dominated by some $u$ in $S$. Strong dominating set and weak dominating set are abbreviated as $sd - set$ and $wd - set$ respectively. The strong domination number of $G$ is denoted by $\gamma_{st}(G)$. The $strong\ domination\ number\ \gamma_{st}(G)$ and the weak domination number $\gamma_w(G)$ of $G$ are the minimum cardinality of an $sd - set$ and a $wd - set$ respectively.

The concepts of strong domination and weak domination were introduced by Sampathkumar and Pushpa Latha (1996). Many researchers like Rautenbach (1999), Meena et al. (2014) and Domke et al. (2002) have explored these concepts. Bounds on strong domination number are also reported by Desai and Gangadharappa (2011) and also by Rautenbach (2000).

**Definition 4.**

The $independent\ strong\ domination\ number\ i_{st}(G)$ of a graph $G$ is the minimum cardinality of a strong dominating set which is independent.

**Definition 5.**

The $center\ C(G)$ of $G$ is the set of vertices of minimum eccentricity, namely, $C(G) = \{v \in V(G) : ecc(v) \leq ecc(u)\ for\ all\ u \in V(G)\}$. The $eccentricity$ of vertex $v$ is $ecc(v) =$

$max\{d(v,w) : w \in V\}$. For any tree $T$, the center $C(T)$ consist of one vertex or two adjacent vertices.

## 2. Main Results

**Theorem 6.**

If $S \subseteq V(G)$ is a strong dominating set and $v \in V(G)$ is the only vertex of maximum degree in $G$ then $v \in S$.

***Proof:***

Let $v$ be the vertex of maximum degree in $G$ and $S$ be a strong dominating set.
To prove : $v \in S$.
Suppose $v \notin S$, which implies that $v \in V - S$. As $v$ is the only vertex with maximum degree it will be strongly dominated by itself only. Therefore, if $v \notin S$ then there is no vertex in $S$ which strongly dominates $v$. That is, $S$ is not an $sd - set$ which contradicts to our assumption that $S$ is a strong dominating set. Hence, $v \in S$. □

**Theorem 7.**

Let $v$ be a vertex with $deg(v) = \Delta(G) = k$ and $v$ is not adjacent to any other vertex of degree $k$ then $v$ must be in $sd - set$.

***Proof:***

Let $v$ be any vertex of maximum degree $k$ in $G$ which is not adjacent to any vertex of the same degree $k$. That is, $d(v) > d(w), \forall w \in N(v)$. So, the vertex $v$ is strongly dominated by itself only. Hence, $v$ must be in an $sd - set$. □

**Theorem 8.**

Let $G$ be a graph of order $n$ such that $\Delta(G) = k$ and there are $r$ mutually non adjacent vertices with degree $k$ such that there is no vertex which is strongly dominated by any two or more vertices of degree $k$ then $r \leq \gamma_{st}(G) \leq n - r\Delta(G)$.

***Proof:***

By Theorem 7 all the mutually nonadjacent vertices of degree $k$ must be in every $sd - set$. Therefore, $r \leq \gamma_{st}(G)$.

Let $v$ be any vertex of maximum degree $k$ in $G$ which is not adjacent to any vertex of the same degree $k$ and there is no vertex which is strongly dominated by any two or more vertices of degree $k$. Therefore, each vertex of degree $k$ strongly dominates $k + 1$ distinct vertices from $V(G)$. If we consider $r$ vertices of maximum degree in an $sd - set$ then $r + r\Delta(G)$ vertices are strongly dominated. If $r + r\Delta(G) \leq n$ then $n - (r + r\Delta(G))$ number of vertices are not strongly dominated. To strongly dominate all the vertices of $V(G)$ we need to consider at least $r + n - (r + r\Delta(G))$ vertices from $V(G)$. Hence, $\gamma_{st}(G) \leq n - r\Delta(G)$. □

**Proposition 9.** (Boutrig and Chellali (2012))

$$\gamma_{st}(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

**Definition 10.**

The $switching\ of\ a\ vertex\ v$ of $G$ means removing all the edges incident to $v$ and adding edges joining $v$ to every vertex which is not adjacent to $v$ in $G$. We denote the resultant graph by $\widetilde{G}$.

**Theorem 11.**

If $\widetilde{P_n}$ is the graph obtained by switching of a pendant vertex $v$ of path $P_n$ then

$$\gamma_{st}(\widetilde{P_n}) = i_{st}(\widetilde{P_n}) = \gamma(\widetilde{P_n}) = i(\widetilde{P_n}) = \begin{cases} 1, & \text{if } n = 3, 4, \\ 2, & \text{otherwise.} \end{cases}$$

***Proof:***

Let $P_n$ be the path with vertices $v_1, v_2, \ldots v_n$ and $\widetilde{P_n}$ be the graph obtained by switching of a pendant vertex of path $P_n$. Without loss of generality we switch the vertex $v_1$.

For $P_2$, the graph $\widetilde{P_2}$ will be a null graph with two vertices. Hence, $\gamma_{st}(\widetilde{P_2}) = 2 = i_{st}(\widetilde{P_2})$ as both the vertices are strongly dominated by themselves and they are independent. We can also observe that since $\widetilde{P_2}$ becomes a null graph with two vertices, $\gamma(\widetilde{P_n}) = 2 = i(\widetilde{P_n})$.

For $P_3$, the graph $\widetilde{P_3}$ has a vertex of degree 2 and remaining two vertices are pendant vertices which are adjacent to the vertex of degree 2. Therefore, it is very clear that the vertex of degree 2 dominate as well as strongly dominates remaining pendant vertices. Thus, $\gamma_{st}(\widetilde{P_3}) = 1 = \gamma(\widetilde{P_3}) = i(\widetilde{P_3}) = i_{st}(\widetilde{P_3})$.

For $P_4$, the graph $\widetilde{P_4}$ has a vertex of degree 3, two vertices of degree 2 and a pendant vertex. So, the vertex of degree 3 dominates as well as strongly dominates remaining vertices of the graph. Hence, $\gamma_{st}(\widetilde{P_4}) = 1 = \gamma(\widetilde{P_4}) = i(\widetilde{P_4}) = i_{st}(\widetilde{P_4})$.

For $P_n(n > 4)$, By the definition of switching of a vertex, the switched vertex of $\widetilde{P_n}$ is adjacent to all the vertices of $P_n$ except the pendant vertex of $\widetilde{P_n}$. Now the degree of the switched vertex is $n-2$ in $\widetilde{P_n}(n \geq 4)$. So, the switched vertex is adjacent to $n-2$ vertices of $\widetilde{P_n}$. But the degree of each vertex which is adjacent to the switched vertex is either 2 or 3 which is less than or equal to the degree of the switched vertex. Therefore, the switched vertex strongly dominates $n-2$ vertices of $\widetilde{P_n}$ other than itself. Now there is only a pendant vertex which is not dominated by the switched vertex. Therefore, we must include a vertex which strongly dominates the pendant vertex. (It may be the pendant vertex of its neighbor). Hence, at least two vertices, namely, the switched vertex and the pendant vertex or its neighbor are enough to strongly dominate all the vertices of $\widetilde{P_n}$ and these two vertices are enough to dominate all the vertices of the graph. Thus, $\gamma_{st}(\widetilde{P_n}) = 2 = \gamma(\widetilde{P_n})$. If we consider the switched vertex and the pendant vertex in $S$ then $S$ will be independent set of minimum cardinality and it will be dominating set as well as strong

dominating set of minimum cardinality. Hence, $i(\widetilde{P_n}) = i_{st}(\widetilde{P_n}) = 2$. □

**Theorem 12.**

If $\widetilde{P_n}$ is the graph obtained by switching of an internal vertex $v$ of path $P_n$ then

$$\gamma_{st}(\widetilde{P_n}) = i_{st}(\widetilde{P_n}) = \gamma(\widetilde{P_n}) = i(\widetilde{P_n}) = \begin{cases} 2 \ or \ 3, & \text{if } 3 \leq n \leq 6, \\ 3, & \text{if } n > 7. \end{cases}$$

***Proof:***

Let $\widetilde{P_n}$ is the graph obtained by switching of an internal vertex $v$ of path $P_n$.

**For $3 \leq n \leq 6$:**

**Case (i):** For $n = 3$:
The graph $\widetilde{P_3}$ is a null graph with three vertices. Hence,

$$\gamma_{st}(\widetilde{P_3}) = i_{st}(\widetilde{P_3}) = \gamma(\widetilde{P_3}) = i(\widetilde{P_3}) = 3.$$

**Case (ii):** For $n = 4$:
In this case as $P_4$ has two pendant vertices and two internal vertices of degree 2, the graph $\widetilde{P_4}$ will have two pendant vertices, an isolated vertex and a vertex of degree 2. The vertex of degree 2 strongly dominates both the pendant vertices while an isolated vertex strongly dominates itself only. Thus, the vertex of degree 2 and the isolated vertex both must be in strong dominating set $S$ and these vertices are independent as well as they form dominating set of minimum cardinality. Hence,

$$\gamma_{st}(\widetilde{P_4}) = i_{st}(\widetilde{P_4}) = \gamma(\widetilde{P_4}) = i(\widetilde{P_4}) = 2.$$

**Case (iii):** For $n = 5$:
In this case $P_5$ has two pendant vertices and three internal vertices of degree 2.

**Subcase (i):** An internal vertex is a central vertex.
In this case the graph $\widetilde{P_5}$ will have the switched vertex which strongly dominates two vertices of degree 2 only (except itself) and to strongly dominate pendant vertices both the pendant vertices or their neighbors (which strongly dominate pendant vertices) must be in strong dominating set $S$. But, if we consider neighbors of pendant vertices in strong dominating set $S$ then the switched vertex is also strongly dominated by them. Therefore, two vertices (neighbors of pendant vertices) are enough to consider in strong dominating set $S$ and these two vertices are independent as well as they form dominating set of minimum cardinality. Hence,

$$\gamma_{st}(\widetilde{P_5}) = i_{st}(\widetilde{P_5}) = \gamma(\widetilde{P_5}) = i(\widetilde{P_5}) = 2.$$

**Subcase (ii):** An internal vertex is not a central vertex.
In this case the graph $\widetilde{P_5}$ will have two vertices of degree 2, a vertex of degree 3, a pendant vertex and an isolated vertex. The vertex of degree 3 strongly dominates both the vertices of

degree 2 and a pendant vertex other than itself. We can observe that the isolated vertex strongly dominated by itself only. Therefore, the vertex of degree 3 and an isolated vertex both must be in strong dominating set $S$ and these two vertices are independent as well as they form dominating set of minimum cardinality. Hence,

$$\gamma_{st}(\widetilde{P_5}) = i_{st}(\widetilde{P_5}) = \gamma(\widetilde{P_5}) = i(\widetilde{P_5}) = 2.$$

**Case (iv):** For $n = 6$:
In this case $P_6$ has two pendant vertices and four internal vertices of degree 2.

**Subcase (i):** Internal vertices are central vertices.
In this case the graph $\widetilde{P_6}$ will have the vertex of degree 3 (not central vertex) which strongly dominates both the central vertices and a vertex of degree 2 except itself. As a pendant vertex and its neighbor are not strongly dominated we must consider one of them in strong dominating set $S$. But we can observe here as the neighbor of pendant vertex strongly dominate both the vertices it must be in strong dominating set $S$. Therefore, we need minimum two vertices to strongly dominate all vertices of $\widetilde{P_6}$ and these two vertices also form strong dominating set of minimum cardinality. Thus, we can observe that if we consider a vertex of degree 3(not central vertex) and another vertex of degree 2 (neighbor of pendant vertex) in $S$ then $S$ will be the independent set also. Hence,

$$\gamma_{st}(\widetilde{P_6}) = i_{st}(\widetilde{P_6}) = \gamma(\widetilde{P_6}) = i(\widetilde{P_6}) = 2.$$

**Subcase (ii):** Internal vertices are not central vertices.
In this case the switched vertex of the graph $\widetilde{P_6}$ strongly dominates all the vertices except a pendant vertex and an isolated vertex. Therefore, to strongly dominate the pendant vertex either the vertex itself or its neighbor must be in strong dominating set $S$. Therefore, we need to include the isolated vertex, the switched vertex and the pendant vertex or its neighbor in strong dominating set $S$ and these vertices form dominating set of minimum cardinality also. We can observe that if we include an isolated vertex, the switched vertex and the pendant vertex in strong dominating set $S$ and these vertices form independent set. Hence,

$$\gamma_{st}(\widetilde{P_6}) = i_{st}(\widetilde{P_6}) = \gamma(\widetilde{P_6}) = i(\widetilde{P_6}) = 3.$$

**For $n > 7$:**

In this case $P_n$ has two pendant vertices and $n - 2$ internal vertices of degree 2.

**Case (i):** The switched vertex is adjacent to a pendant vertex of $P_n$.
In this case the graph $\widetilde{P_n}$ will have an isolated vertex, a pendant vertex, $n-4$ vertices of degree 3 and a vertex of degree 2. Now, the switched vertex is adjacent to $n-4$ vertices of degree 3 and a vertex of degree 2. Now the switched vertex is adjacent to $n-4$ vertices of degree 3 and a vertex of degree 2. Therefore, it is clear that as the degree of the switched vertex is $n - 3$, it strongly dominates all $n-4$ vertices of degree 3 and a vertex of degree 2. We can observe that the pendant vertex strongly dominated by itself and its neighbor only. Therefore, to strongly dominate the

pendant vertex we must include it or its neighbor in $S$. Finally, the isolated vertex, the switched vertex and the pendant vertex or its neighbor must be in strong dominating set $S$. Thus, we can observe that these three vertices form the strong dominating set of minimum cardinality and it will be also a dominating set of minimum cardinality. If we consider the isolated vertex, the switched vertex and the pendant vertex in $S$ then $S$ will be also independent set. Hence,

$$\gamma_{st}(\widetilde{P_n}) = i_{st}(\widetilde{P_n}) = \gamma(\widetilde{P_n}) = i(\widetilde{P_n}) = 3.$$

**Case (ii):** The switched vertex is not adjacent to a pendant vertex.
In this case the graph $\widetilde{P_n}$ will have two pendant vertices and $n-5$ vertices of degree 3. Now, the switched vertex adjacent to $n-5$ vertices of degree 3. Therefore, it is clear that as the degree of the switched vertex is $n-3$, it strongly dominates all the vertices of degree 3. We can observe that both the pendant vertices are not adjacent to each other, to strongly dominate them either both the pendant vertices or the neighbors of pendant vertices (which strongly dominate pendant vertices) must be in . So, we must include at least three vertices in $S$ and these three vertices also form dominating set of minimum cardinality. Hence, $\gamma_{st}(\widetilde{P_n}) = 3 = \gamma(\widetilde{P_n})$. If we consider the switched vertex and both the pendant vertices in $S$ then $S$ will be the strong dominating set of minimum cardinality as well as independent set. Therefore,

$$\gamma_{st}(\widetilde{P_n}) = i_{st}(\widetilde{P_n}) = \gamma(\widetilde{P_n}) = i(\widetilde{P_n}) = 3.$$

**Remark 13.**

If $\widetilde{P_7}$ is the graph obtained by switching of an internal vertex $v$ of path $P_7$ then

1. $\gamma_{st}(\widetilde{P_n}) = i_{st}(\widetilde{P_n}) = \gamma(\widetilde{P_n}) = i(\widetilde{P_n})$ = 3 if $v$ is not a central vertex.
2. $\gamma(\widetilde{P_n}) = i(\widetilde{P_n}) = 2 \neq \gamma_{st}(\widetilde{P_n}) = i_{st}(\widetilde{P_n})$ if $v$ is a central vertex as $\gamma_{st}(\widetilde{P_n}) = i_{st}(\widetilde{P_n})$ = 3, if $v$ is a central vertex.

**Definition 14.** (Sampathkumar (1996))

Let $G = (V, E)$ be a graph and $D \subset V$. Then,

(1) $D$ is full if every $u \in D$ is adjacent to some $v \in V - D$.
(2) $D$ is $s$-full ($w$-full) if every $u \in D$ strongly (weakly) dominates some $v \in V - D$.

**Definition 15.** (Sampathkumar (1996))

A graph $G$ is $domination\ balanced\ (d - balanced)$ if there exists an $sd$-set $D_1$ and a $wd$-set $D_2$ such that $D_1 \cap D_2 = \phi$.

**Proposition 16.** (Sampathkumar (1996))

For a graph $G$, the following statements are equivalent.

1. $G$ is $d$-balanced.
2. There exists an $sd$-set $D$ which is $s$-full.
3. There exists an $wd$-set $D$ which is $w$-full.

**Theorem 17.**

If there exists an isolated vertex in graph $G$ then $G$ is not $d$- balanced.

***Proof:***

By the definition of $d$-balanced graph, $G$ is $d$-balanced if $\exists$ an $sd - set$ $D_1$ and $wd - set$ $D_2$ such that $D_1 \cap D_2 = \phi$. If there is an isolated vertex in $G$ then it must be in every $sd - set$ and every $wd - set$. So, there is neither the set $D_1$ ($sd - set$) nor the set $D_2$ ($wd - set$) such that $D_1 \cap D_2 = \phi$. Therefore, the graph $G$ is not $d$-balanced. Hence, the result. □

**Corollary 18.**

If $\widetilde{P_n}$ is the graph obtained by switching of a neighbor of the pendant vertex of path $P_n$ then, $\widetilde{P_n}$ is not $d$-balanced.

***Proof:***

If $\widetilde{P_n}$ is the graph obtained by switching of a neighbor of the pendant vertex of path $P_n$ then, there is an isolated vertex in $\widetilde{P_n}$. Therefore, by Theorem 17 $\widetilde{P_n}$ is not $d$-balanced graph. □

**Theorem 19.**

If $\widetilde{P_n}$ is the graph obtained by switching of any vertex (except a neighbor of the pendant vertex) of the path $P_n$ for $n \geq 3$ then $\widetilde{P_n}$ is $d$-balanced graph.

***Proof:***

Let $\widetilde{P_n}$ is the graph obtained by switching of any vertex (except a neighbor of the pendant vertex) of the path $P_n$. In Theorem 11 and Theorem 12, we have obtained $sd - set$ for all $\widetilde{P_n}$, $(n \neq 7)$ which is $s$-full. If $n = 7$ then a vertex of degree 4 and two pendant vertices form strong dominating set which is $s$-full. Therefore, by Proposition 16 $\widetilde{P_n}$ is $d$-balanced graph. □

## 3. Concluding Remarks

The concept of strong domination is a variant of usual domination. This concept is useful to deploy the security troops and their transition from one place to another. We have established some fundamental results for path $P_n$ and the larger graph obtained by switching of a vertex in $P_n$. This work can be applied to rearrange the existing security network in the case of high alert situation and to beef up the surveillance.

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