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Private Absorbant of Generalized de Bruijn Digraphs

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Abstract

Let *V* and *A* denote the vertex and arc sets of a digraph *D*. Let *S* be a subset of *V(D)* and let u be an element in *S*. A vertex *v* in *D* is called a private in-neighbor of the vertex u with respect to *S* in *D* if the intersection of closed out-neighborhood of *v* and *S* contains only u . A subset *S* of *V(D)* is called a private absorbant of *D* if every vertex of $V-S$ is a private in-neighbor of some vertex of *S*. The minimum cardinality of a private absorbant is called the private absorbant number of *D*. In this paper, we establish bounds on the private absorbant number of generalized de Bruijn digraphs and we give some sufficient conditions for the private absorbant number of generalized de Bruijn digraphs to achieve the bounds.

Keywords: Private Absorbant; Generalized de Bruijn digraphs; Inter connection; Networks

MSC 2010 No.: 05C69, 05C20

1. Introduction

Domination in graphs has been studied extensively recently, since it has many applications. The book "Fundamentals of domination in graphs" by Haynes et al. (1998), is entirely devoted to this area. Let $G=(V,E)$ be a connected graph. A set $S\subseteq V$ is called a dominating set of *G* if every vertex of $V-S$ is adjacent to some vertex of *S*. The minimum cardinality of a dominating set of *G* is called the domination number of *G* and is denoted by $\gamma(G)$. One variation of domination in graphs called perfect domination was studied by several authors (Bange et al. (1998), Bange, et al. (1987), Biggs, N. (1973) and Livingston et al. (1990)). A set $S\subseteq V$ is called a perfect dominating set of *G* if every vertex of $V-S$ is adjacent to exactly one vertex in *S*. The minimum cardinality of a perfect dominating set of *G* is called the perfect domination number of *G*. It is natural to define the concept of perfect domination in digraphs.

The concept of domination in undirected graphs is naturally extended to digraphs. In fact, domination in digraphs comes up more naturally in modeling real world problems. There is a survey on domination in digraphs written by Ghosal et al. (1998).

In this paper we introduce a new notion called private absorbant in digraphs. Our motivation for studying the notion of private absorbant in digraphs arose from the work involving resource allocation and placement in parallel computers (Livingston et al. (1998)).

A digraph $D=(V,A)$ consists of a finite vertex set V and an arc set $A \subseteq P$, where P is the set of all ordered pairs of distinct vertices of V. The out-neighborhood $N^+(v)$ of a vertex $v \in V(D)$ is $\{w \mid (v,w) \in A(D)\}$ and the in-neighborhood $N^-(v)$ of a vertex $v \in V(D)$ is $\{u \mid (u,v) \in A(D)\}$ The closed out-neighborhood $N^+[v]$ of v is $N^+(v) \cup \{v\}$ and the closed in-neighborhood $N^-[v]$ of v is $N^-(v) \cup \{v\}$. A vertex $v \in V(D)$ out-dominates every vertex in $N^+[v]$. A set $S \subseteq V(D)$ is called a dominating set of D (or out-dominating set) if every vertex of $V-S$ is out-dominated by some vertex of S. The minimum cardinality of an out-dominating set of D is called the out-domination number of D. A set $S \subseteq V(D)$ is called a absorbant of D (or in-dominating set) if every vertex of $V-S$ is in-dominated by some vertex of S. The minimum cardinality of an absorbant of D is called the absorbant number of D. A set $S \subseteq V(D)$ is called a twin dominating set if it is both an out-dominating set and absorbant. The properties of domination number, absorbant number and twin domination number in generalized de Bruijn digraphs have been studied by Kikuchi et al. (2003), Marimuthu et al. (2014) and Shan et al. (2007). Some domination parameters have been studied by Araki (2007, 2008) and Vaidya et al. (2016).

The resource location problem in an interconnection network is one of the facility location problems. Constructing absorbant and dominating sets corresponds to solving two kinds of resource location problems (Ghosal et al. (1998) and Kikuchi et al. (2003)). For example, each vertex in an absorbant or a dominating set provides a service (file-server, and so on) for a network. In this case, every vertex has a direct access to file-servers. Since each file-server may cost a lot, the number of an absorbant or a dominating set has to be minimized. Let m, n be positive integers, then $m|n$ means m divides n.

In 2007, Shan et al., proved the following results which are related to absorbant number of $G_B(n,d)$.

1) $\left\lceil \frac{n}{d+1} \right\rceil \le \gamma_a(G_B(n,d)) \le \left\lceil \frac{n}{d} \right\rceil$.

2) If d =2, 4 and $(d+1) \mid n,$ or $d=3$ and $8 \mid n,$ then $\gamma_a(G_B(n,d)) = \frac{n}{d+1}$.

3) If $d \mid n,$ then $\gamma_a(G_B(n,d)) = \frac{n}{d}$ if and only if $n \le d^2,$ or $n = d(d+l)$ for $1 \le l \le d-2$ and $(l+1) \mid (d-1)$.

4) Let $d \mid n$. If $d(2d-1) \le n \le 2d(d+1)$, or $n = d(d+l)$ for $1 \le l \le d+1$ and $l+1$ does not divide $d-1$, then $\gamma_a(G_B(n,d)) = \dfrac{n}{d-1}$.

In this paper, we discuss the private absorbant in generalized de Bruijn digraphs. First, we present bounds on the private absorbant number and give exact values of private absorbant number for several types of $G_B(n,d)$ by constructing minimum private absorbant. Secondly, we present sharp upper bounds for some special generalized de Bruijn digraphs.

2. Private Absorbant in digraphs

A vertex v in a graph *G* is called a private neighbor of a vertex $u \in S$ with respect to S in *G* if $N[v] \cap S = \{u\}$. The private neighbor set of v in S with respect to S is $P_n[v,S] = N[v] - N[S - \{v\}]$. A vertex v in a digraph *D* is called a private out-neighbor of a vertex $u \in S$ with respect to S in *D* if $N^-[v] \cap S = \{u\}$, and v is called a private in-neighbor of vertex $u \in S$ with respect to S in D if $N^+[v] \cap S = \{u\}$. The private out-neighbor set v in S with respect to S is $P_n^+[v,S] = N^+[v] - N^+[S - \{v\}]$ and private in-neighbor set v in S with respect to S is $P_n^-[v,S] = N^-[v] - N^-[S - \{v\}]$. A subset *S* of $V(D)$ is called a private out-dominating set of *D* if every vertex of *V-S* is a private out-neighbor of some vertex of *S*. The minimum cardinality of a private out-dominating set is called the private out-domination number of *D*. It is denoted by $\gamma_p^+(D)$. The private out-dominating set of cardinality $\gamma_p^+(D)$ is called a minimum private out-dominating set of *D*. A subset *S* of $V(D)$ is called a private absorbant of *D* if every vertex of *V-S* is a private in-neighbor of some vertex of *S*. The minimum cardinality of a private absorbant is called the private absorbant number of *D*. It is denoted by $\gamma_p^-(D)$. The private absorbant of cardinality $\gamma_p^-(D)$ is called a minimum private absorbant of *D*.

The generalized de Bruijn digraph $G_B(n,d)$ is defined by the congruence equations as follows:

$$V(G_B(n,d)) = \{0,1,2,...,n-1\},\ A(G_B(n,d)) = \{(x,y) \mid y \equiv (dx+i)(\mathrm{mod}\, n), 0 \le i \le d-1\}.$$

Consider the example

$$G_B(6,5).\ V(G_B(6,5)) = \{0,1,2,3,4,5\}$$

and

$$A(G_B(6,5)) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (1,5), (1,0), (1,1), (1,2), (1,3), (2,4), (2,5), (2,0), (2,1), (2,2), (3,3), (3,4), (3,5), (3,0), (3,1), (4,2), (4,3), (4,4), (4,5), (4,0), (5,1), (5,2), (5,3), (5,4), (5,5)\}.$$

Table 1. The vertices of $G_B(6, 5)$ and their corresponding in-neighbors

$V(G_B(6, 5))$	In-neighbors				
0	0	1	2	3	4
1	0	1	2	3	5
2	0	1	2	4	5
3	0	1	3	4	5
4	0	2	3	4	5
5	1	2	3	4	5

Let S = {0, 5}. $P_n^-(0, S) = N^-(0) - N^-(5) = \{0\}$ *and* $P_n^-(5, S) = N^-(5) - N^-(0) = \{5\}$. No vertex in *V*-*S* is a private in-neighbor of some vertex in *S*. Therefore, *S* = {0, 5} is a minimum absorbant but it is not a private absorbant and the set *S* = {0, 1, 2, 3, 4, 5} is a minimum private absorbant.

3. The private absorbant number of generalized de Bruijn digraphs

We may assume $d \geq 2$ and $n \geq d$, since $G_B(n, 1)$ is empty graph for the case $d = 1$. We begin by establishing bounds on the private absorbant number of $G_B(n, d)$.

Proposition 3.1.

$$\gamma_p^-(G_B(n,d)) \geq \left\lceil \frac{n}{d+1} \right\rceil.$$

Proof:

Let *S* be a private absorbant of $G_B(n,d)$. Then by the definition of $G_B(n,d)$, $|S| + d|S| \geq n$. This shows that

$$\gamma_p^-(G_B(n,d)) \geq \left\lceil \frac{n}{d+1} \right\rceil.$$

Example 3.2.

The sharpness of the lower bound is satisfied by the digraph $G_B(4,2)$. $V(G_B(4,2))$={0,1,2,3} and $A(G_B(4,2))$={(0,0), (0,1), (1,2), (1,3), (2,0), (2,1), (3,2), (3,3)}.

Table 2. The vertices of $G_B(4, 2)$ and their corresponding in-neighbors

$V(G_B(6, 5))$	In-neighbors	
0	0	2
1	0	2
2	1	3
3	1	3

Let S = {0, 2}. $P_n^-(0, S) = N^-(0) - N^-(2) = \{0\}$ and $P_n^-(2, S) = N^-(2) - N^-(0) = \{1, 3\}$. Every vertex in V-S is a private in-neighbor of some vertex in S. Therefore, S = {0, 2} is a minimum private absorbant.

The following lemma was established by Shibata et al. (1994), and will be used throughout the article.

Lemma 3.3.

Every arc of $G_B(n,d)$ is a loop of a double arc if and only if $d = 1, n-1$ *or n.*

Proposition 3.4.

For any $n \geq 1$,

$$\gamma_p^-(G_B(n,1)) = n\,.$$

Proof:

Let $V = \{0,1,2,3,\ldots,n-1\}$ be the vertex set of $G_B(n,1)$. By Lemma 3.3., we have $N^-(v) = \{v\}$, for any $v \in V$. This implies that for any two distinct vertices $u, v \in V$, $N^-(u) \cap N^-(v) = \{u\} \cap \{v\} = \phi$. This shows that $S = \{0,1,2,3,\ldots,n-1\}$ is a minimum private absorbant. This completes the proof.

It seems to be difficult to determine the private absorbant for generalized de Bruijn digraphs. So we pay our attention to some special generalized de Bruijn digraphs. First, we consider the case that $n = d$.

Proposition 3.5.

For any $n \geq 1$and $n = d$`

$$\gamma_p^-(G_B(n,d)) = 1.$$

Proof:

By the definition of $G_B(n,d)$, every vertex has d in-neighbors. Since n = d. Then for any vertex $v \in V$, $N^-(v) = \{0,1,2,3,\ldots,n-1\}$ and $|N^-(v)| = n = d$. Define $S = \{u\}$, for any $u \in V$. Clearly $P_n^-(u,S) = N^-(u) - N^-(S - \{u\}) = N^-(u)$. This completes the proof.

For $2 \leq d \leq 4$, by giving a method to determine private absorbants of $G_B(n, d)$, we present some sufficient conditions for the private absorbant number of $G_B(n, d)$ to be the lower bound $\left\lceil \dfrac{n}{d+1} \right\rceil$.

Theorem 3.6.

If d =2, 4 and $(d+1)\,|\,n,$ or $d=3$ and $8\,|\,n,$ then

$$\gamma_p^-(G_B(n,d)) = \left\lceil \frac{n}{d+1} \right\rceil.$$

Proof:

Depending on the value of *d*, we distinguish the following cases.

***Case* 1:**

Suppose that d =2. Then clearly

$$V = \bigcup_{i=0}^{\frac{n}{3}-1} \{3i, 3i+1, 3i+2\}.$$

Let

$$S = \bigcup_{i=0}^{\frac{n}{3}-1} \{3i+1\}.$$

Then, S is private absorbant of $G_B(n,2).$ For every vertex $x \in V$, $N^+(x) = \{2x(\bmod n), (2x+1)(\bmod n)\}.$ If $x = 3i \in V - S,$ i = 0 to $\frac{n}{3}-1,$ then $(2(3i)+1)(\bmod n) = (3(2i)+1)(\bmod n) \in N^+(x) \cap S$ and $(2(3i))(\bmod n) = (3(2i))(\bmod n) \notin S,$ while, if $x = 3i+2 \in V - S,$ i = 0 to $\frac{n}{3}-1,$ then $(2(3i+2))(\bmod n) = (3(2i+1)+1)(\bmod n)$ $\in N^+(x) \cap S$ and $(2(3i+2)+1)(\bmod n) =$ $(3(2i+1)+2)(\bmod n) \notin S.$ We get $\left|N^+(x) \cap S\right| = 1,$ for every vertex x in *V-S*. Hence *S* is a private absorbant.

***Case* 2:**

Suppose that d =3 and $8\,|\,n.$ Then clearly

$$V = \bigcup_{i=0}^{\frac{n}{4}-1} \{4i, 4i+1, 4i+2, 4i+3\}.$$

Let

$$S_1 = \bigcup_{j=0}^{\frac{n}{8}-1} \{\, 8j+1\} = \{1, 9, \ldots, 8j+1, \ldots, n-7\}$$

and

$$S_2 = \bigcup_{j=0}^{\frac{n}{8}-1} \{8j+6\} = \{6, 14, \ldots, 8j+6, \ldots, n-2\}.$$

Table 3. The vertices of $G_B(24,3)$. The vertices in bold face are the vertices of S

$V(G_B(24,3))$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
4i	0	4	8	12	16	20
4i+1	**1**	5	**9**	13	**17**	21
4i+2	2	**6**	10	**14**	18	**22**
4i+3	3	7	11	15	19	23

The set S = {1, 6, 9, 14, 17, 22} is a minimum private absorbant for $G_B(24,3.)$

Now show that $S = S_1 \cup S_2$ is a private absorbant. Let u be any vertex in $V - S$. We are going to prove that u has a unique out-neighbor in S. Clearly $N^+(u) = \{3u(\bmod n),$ $(3u+1)(\bmod n), (3u+2)(\bmod n)\}$. If $u = 4i$, for $0 \le i \le \frac{n}{4} - 1$, then $(3(4i)+1)(\bmod n) \in S_1$, when i is even (or) $(3(4i)+2)(\bmod n) \in S_2$, when i is odd. If $u = 4i+1$, for $0 \le i \le \frac{n}{4} - 1$, then $(3(4i+1)+2)(\bmod n) = (12i+5)(\bmod n) \in S_1$, when i is odd (or) $4i+1 = 8k+1 \in S_1$, $k > 0$, when i is even. If $u = 4i+2$, for $0 \le i \le \frac{n}{4} - 1$, then $3(4i+2)(\bmod n) = (12i+6)(\bmod n) \in S_2$, when i is even (or) $4i+2 = 4(2k+1)+2 \ = 8k+6 \in S_2$, $k > 0$, when i is odd. If $u = 4i+3$, for $0 \le i \le \frac{n}{4} - 1$, then either $3(4i+3)(\bmod n) = (12i+9)(\bmod n) \in S_1$, when i is even (or) $(3(4i+3)+1)(\bmod n) = (12i+10)(\bmod n) \in S_2$, when i is odd. Therefore, for any vertex in $u \in V - S$, $|N^+(u) \cap S| = 1.$ Thus, S is a minimum private absorbant.

***Case* 3:**

d =4. Then, clearly

$$V = \bigcup_{i=0}^{\frac{n}{5}-1} \{5i, 5i+1, 5i+2, 5i+3, 5i+4\}.$$

Let

$$S = \bigcup_{i=0}^{\frac{n}{5}-1} \{5i+2\}.$$

For every vertex $u \in V - S$, we can show that $|N^+(u) \cap S| = 1$. Clearly

$$N^+(u) = \{4u(\bmod n), \quad (4u+1)(\bmod n), (4u+2)(\bmod n), (4u+3)(\bmod n)\}.$$

If

$$x = 5i \in V - S, \ i = 0 \text{ to } \frac{n}{5} - 1,$$

then,

$$(4(5i)+2)(\operatorname{mod} n)=(5(4i)+2)(\operatorname{mod} n)\in S.$$

If

$$u=5i+1,\text{ for } 0\le i\le \frac{n}{5}-1,$$

then

$$(4(5i+1)+3)(\operatorname{mod} n)=(5(4i+1)+2)(\operatorname{mod} n)\in S.$$

If

$$u=5i+3,\text{ for } 0\le i\le \frac{n}{5}-1,$$

then

$$4(5i+3)(\operatorname{mod} n)=(5(4i+2)+2)(\operatorname{mod} n)\in S.$$

If

$$u=5i+4,\text{ for } 0\le i\le \frac{n}{5}-1,$$

then $(4(5i+4)+1)(\operatorname{mod} n)=(5(4i+3)+2)(\operatorname{mod} n)\in S.$ Therefore for any vertex in $u\in V-S$ $\left|N^{+}(u)\cap S\right|=1$ and thus S is an private absorbant.

Theorem 3.7.

If $d\mid n$ and $n\le d^{2}$, then

$$\gamma_{p}^{-}(G_{B}(n,d))=\left\lceil \frac{n}{d+1}\right\rceil.$$

Proof:

Define

$$S=\left\{0,d,2d,3d,\ldots,\left(\frac{n}{d}-1\right)d\right\}.$$

We claim that S is a private absorbant. Let u be any vertex in $V-S$. By definition of

$$G_{B}(n,d),\ N^{+}(u)=\{ud(\operatorname{mod} n),\ (ud+1)(\operatorname{mod} n),\ \ldots,\ (ud+d-1)(\operatorname{mod} n)\}.$$

The congruence $kd \equiv (ud+i)(\bmod n),\ k>0\ and\ i=0\ to\ d-1$ has a solution only if $i=0$ and $k<\dfrac{n}{d}-1.$ Since $d \mid n$, $n=md,\ m>0.$ Suppose that $u<m.$ Then $ud(\bmod n)=ud.$ Otherwise $ud(\bmod n)=(n-ud)(\bmod n)=\ (m-u)d(\bmod n)\ and\ \ m-u<\dfrac{n}{d}.$ Then u is a private in-neighbor of ud or $(m-u)d$ in S. This shows that S is a private absorbant of $G_B(n,d)$. Therefore,

$$\gamma_p^-(G_B(n,d)) \le \frac{n}{d} = \left\lceil \frac{n}{d+1} \right\rceil.$$

and the result follows by Proposition 3.1.

An interesting problem is to how the private absorbant number reaches its maximum in some generalized de Bruijn digraphs. We consider $G_B(n,\ d)$ for the special case $n=d+1.$

Theorem 3.8.

If $n=d+1,$ and $n \neq 4$, then

$$\gamma_p^-\left(G_B(n,d)\right)=\begin{cases} 1, & if\ \ d\ \ \ is\ \ \ even, \\ n, & if\ \ d\ \ \ is\ \ \ odd. \end{cases}$$

Proof:

Case* 1*:

d is even. Define $S=\left\{\dfrac{d}{2}\right\}$. Since S contains only one element, we have

$$P_n^-\left(\frac{d}{2},S\right)=N^-\left(\frac{d}{2}\right) \text{ and } N^-\left(\frac{d}{2}\right)=\bigcup_{i=1}^{\frac{d}{2}}\left\{\frac{d}{2}-i,\frac{d}{2}+i\right\}.$$

This set contains d vertices of V. Since $n=d+1,$ all the vertices of $V-S$ are private in-neighbors of $\dfrac{d}{2}$. $\gamma_p^-\left(G_B(n,d)\right)\le 1.$

Case* 2*:

d is odd. Define $S=\{v\}$, for any $v \in V$. Clearly $N^-(v)=P_n^-(v,S)$ and by Lemma 3.3., $\{v\} \subseteq P_n^-(v,S).$ Since $n=d+1$, there is a vertex in $V-S$ and $w \notin P_n^-(v,S).$ Define $S=\{u,v\},$ for any two distinct vertices $u,v \in V$. By Lemma 3.3., $\{u\} \subseteq N^-(u).$ Since $n=d+1$, there is a vertex $x \in V(G_B(n,d))$ but $x \notin N^-(u)$ and also $\{v\} \subseteq N^-(v).$ Since $n=d+1$, there is a vertex $y \in V(G_B(n,d))$ but $y \notin N^-(v).$

Suppose that $x=y.$ Then, $N^-(u)$ contains all the vertices of $G_B(n,d)$ other than x and $N^-(v)$ contains all the vertices of $G_B(n,d)$ other than y. Since $x=y,$ $N^-(u)=N^-(v).$ We get $P_n^-(u,S)=N^-(u)-N^-(v)=\phi$ and $P_n^-(v,S)=N^-(v)-N^-(u)=\phi.$ Therefore,

$P_n^-(z,S) \neq \phi,$ for all $z \in S.$ This shows that every vertex of $V - S$ is not a private in-neighbor of any vertex of S.

Suppose $x \in N^-(v),$ then there exists a vertex $w \in V$ and $w \notin N^-(v).$ Clearly

$$N^-(u) - N^-(S - \{u\}) = \{w\} \text{ and } N^-(v) - N^-(S - \{v\}) = \{x\}.$$

All other vertices in $V - S - \{w, x\}$ are not private in-neighbors of any vertex of S.

Now, define

$$S = \{v_1, v_2, ..., v_k\}, 3 \leq k < n.$$

Since $n = d + 1$ and by Lemma 3.3., the open in-neighborhood of any vertex v of S does not contains one vertex of $V(G_B(n,d))$ other than v. Let x_i be the vertex of $V(G_B(n,d))$, which is not in $N^-(v_i)$, for $i = 1,2,...,k.$

If $x_1 = x_2 = \ldots = x_k$, $k < n,$ then x_1 is not a private out-neighbor of any vertex of S. Suppose that $x_r \neq x_s$ for some $r, s = 0, 1, \ldots, k$. Then,

$$N^-(v_s) \cup N^-(v_r) = V(G_B(n,d))$$

and for any vertex $v_k \in S,$ $k \neq r, s$, we get $P_n^+[v_k, S] = \phi.$ Clearly

$$P_n^-[v_r, S] = \{x_s\} \text{ and } P_n^-[v_s, S] = \{x_r\}.$$

All other vertices $V - S - \{v_r, v_s\}$ are not private out-neighbors of any vertex of S and hence the result follows.

Remark 3.9.

Theorem 3.8 is not true when $n = 4.$

For example. consider the digraph $G_B(4,3)$.

$V(G_B(4,3))=\{0,1,2,3\}$ and $A(G_B(4,3))=\{(0,0), (0,1), (0,2), (1,3), (1,0), (1,1), (2,2), (2,3), (2,0), (3,1), (3,2), (3,3)\}$.

Table 4. The vertices of $G_B(4, 3)$ and their corresponding in-neighbors

$V(G_B(4, 3))$	In-neighbors		
0	0	1	2
1	0	1	3
2	0	2	3
3	1	2	3

Let

$$S = \{0, 2\}.\ P_n^-(0, S) = N^-(0) - N^-(2) = \{1\}\ and\ P_n^-(2, S) = N^-(2) - N^-(0) = \{3\}.$$

Every vertex in *V-S* is a private in-neighbor of some vertex in *S*. Therefore, *S* = {0, 2} is a minimum private absorbant.

We consider $G_B(n, d)$ for the special case $d \mid n$.

Theorem 3.10.

If $d \mid n$, then

$$\gamma_p^-(G_B(n,d)) \le \frac{n}{d}.$$

***Proof*:**

Define

$$S = \left\{0, d, 2d, 3d, \ldots, \left(\frac{n}{d} - 1\right)d\right\}.$$

By a similar argument stated in Theorem 3.7, we can prove that *S* is a minimum private absorbant.

Example 3.11.

The sharpness of the upper bound of Theorem 3.10 is satisfied by the digraph $G_B(12,3)$.

V(G$_B$*(12, 3))* = {0, 1, 2, 3, . . . , 11}
A(G$_B$*(12, 3))* = {(0, 0), (0, 1), (0, 2), (1, 3), (1, 4), (1, 5), (2, 6), (2, 7), (2, 8), (3, 9), (3, 10), (3, 11), (4, 0), (4, 1), (4, 2), (5, 3), (5, 4), (5, 5), (6, 6), (6, 7), (6, 8), (7, 9), (7, 10), (7, 11), (8, 0), (8, 1), (8, 2), (9, 3), (9, 4), (9, 5), (10, 6), (10, 7), (10, 8), (11, 9), (11, 10), (11, 11)}

Let *S* = {0, 3, 6, 9}.

$$P_n^-(0, S) = N^-(0) - N^-(S - \{0\}) = \{4, 8\},\ P_n^-(3, S) = N^-(3) - N^-(S - \{3\}) = \{1, 5, 9\},$$
$$P_n^-(6, S) = N^-(0) - N^-(S - \{6\}) = \{2, 10\},\ P_n^-(9, S) = N^-(9) - N^-(S - \{9\}) = \{7, 11\}.$$

Every vertex in *V-S* is a private in-neighbor of some vertex in *S*. Therefore, *S* = {0, 3, 6, 9} is a minimum private absorbant.

Table 5. The vertices of $G_B(12, 3)$ and their corresponding in-neighbors

$V(G_B(12, 3))$	In-neighbors		
0	0	4	8
1	0	4	8
2	0	4	8
3	1	5	9
4	1	5	9
5	1	5	9
6	2	6	10
7	2	6	10
8	2	6	10
9	3	7	11
10	3	7	11
11	3	7	11

In Shan et al. (2007) proved the following results which are true for private absorbant of $G_B(n, d)$ also.

1) If $d \mid n$ and $n \geq d(2d-1)$, then $\gamma_a(G_B(n,d)) \leq \dfrac{n}{d-1}$.

2) If $n = kd+1$ and $\left\lceil \dfrac{n}{d} \right\rceil \geq d$, then $\gamma_a(G_B(n,d)) \leq \left\lceil \dfrac{n}{d} \right\rceil - 1$.

4. Conclusion

In this paper we investigated the private absorbant problem in generalized de Bruijn digraphs. We established bounds on the private absorbant number for G_B (n, d). In particular, if d divides n, $d = 2$, $d = 3$, $d = 4$, we gave the technique of construction for the minimum private absorbant in G_B (n, d). By somewhat improving the methods in this paper, one can deal with the private absorbant problem for generalized Kautz digraphs. It is interesting to characterize the extremal graphs achieving bounds in Theorems 3.1 and 3.7.

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