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A Novel Approach for Solving Volterra Integral Equations Involving Local Fractional Operator

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Abstract

The paper presents an approximation method called local fractional variational iteration method (LFVIM) for solving the linear and nonlinear Volterra integral equations of the second kind with local fractional derivative operators. Some illustrative examples are discussed to demonstrate the efficiency and the accuracy of the proposed method. Furthermore, this method does not require spatial discretization or restrictive assumptions and therefore reduces the numerical computation significantly. The results reveal that the local fractional variational iteration method is very effective and convenient to solve linear and nonlinear integral equations within local fractional derivative operators.

Keywords: Volterra integral equations; Local fractional variational iteration method; Local fractional derivative operators

MSC 2010 No.: 26A33; 34A12

1. Introduction

An integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation and partial differential equation can be transformed into problems of solving some approximate integral equations Piaggio (1920), and Rahman (1994). The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations. Engineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. For example, Newton's law, stating that the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics, and transfer of heat in a medium can all be represented mathematically as differential equations. In 1825 Abel, an Italian mathematician first produced an integral equation in connection with the famous tautochrone problem Lovitt (1950) and Tricomi (1957). The problem is connected with the determination of a curve along which a heavy particle, sliding without friction, descends to its lowest position, or more generally, such that the time of descent is a given function of its initial position Rahman (1994).

Some different valid methods for solving integral equation with local fractional derivative operators have been developed in the last years. Yang and Zhang (2012) used Adomian decomposition method to obtain the analytical solution of local fractional Volterra integral equation of the second kind. Yang (2012) applied Picard's approximation method to obtain the approximate solution of local fractional Volterra integral equation. Also, Yang (2012) used local fractional Laplace transform to solve class of local fractional Volterra integral equation. Neamah (2014) applied local fractional variational iteration method for solving Volterra integro-differential equations within the local fractional operators. Jassim (2016) found the approximate solutions of Fredholm integral equation on Cantor sets by using local fractional Adomian decomposition method.

In this paper, our aims are derivative and application of the local fractional variational iteration methods for solving the local fractional Volterra integral equations of the second kind. The general form of this integral equation is given by

$$\Omega(x)=\psi(x)+\frac{1}{\Gamma(1+\upsilon)}\int_a^x \Lambda(x,\kappa)\,F\big(\Omega(\kappa)\big)(d\kappa)^{\upsilon}\ ,0<\upsilon\leq 1\ , \tag{1.1}$$

where $\Lambda(x,\kappa)$ is the kernel of the local fractional integral equation, $\psi(x)$ and $F(\Omega)$ are known functions, and $\Omega(x)$ is the unknown solution of integral equation, which we are going to find, via local fractional variational iteration method. The paper has been organized as follows. In Section 2, we give the concept of local fractional calculus. In Section 3, we give analysis of the methods used. In Section 4, we consider several illustrative examples. Finally, in Section 5, we present our conclusions.

2. Local Fractional Calculus

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper.

Definition 1.

We say that a function $\psi(x)$ is local fractional continuous at $x = x_0$, Yang (2012) if it holds

$$|\psi(x)-\psi(x_0)| < \varepsilon^{\upsilon} \ ,0<\upsilon\le 1 \tag{2.1}$$

with $|x-x_0| < \delta$, for $\varepsilon,\delta > 0$ and $\varepsilon,\delta \in R$. For $x \in (a,b)$, it is so called local fractional continuous on (a,b), denoted by $\psi(x) \in C_{\upsilon}(a,b)$.

Definition 2.

Setting $\psi(x) \in C_{\upsilon}(a,b)$, the local fractional derivative of $\psi(x)$ at $x = x_0$ is defined as Yang (2012), Wang et al. (2014), and Yan et al. (2014):

$$L_x^{(\upsilon)}\psi(x_0) = \psi^{(\upsilon)}(x_0) = \lim_{x\to x_0} \frac{\Delta^{\upsilon}(\psi(x)-\psi(x_0))}{(x-x_0)^{\upsilon}}, \tag{2.2}$$

where $\Delta^{\upsilon}(\psi(x)-\psi(x_0)) \cong \Gamma(\upsilon+1)(\psi(x)-\psi(x_0))$.

The formulas of local fractional derivatives of special functions used in the paper are as follows:

$$L_x^{(\upsilon)} a\psi(x) = aL_x^{(\upsilon)}\psi(x), \tag{2.3}$$

$$L_x^{(\upsilon)}\left(\frac{x^{n\upsilon}}{\Gamma(1+n\upsilon)}\right) = \frac{x^{(n-1)\upsilon}}{\Gamma\left(1+(n-1)\upsilon\right)}, \ n \in N. \tag{2.4}$$

Definition 3.

Let a partition of the interval $[a,b]$ be denoted as (t_j, t_{j+1}), $j = 0, \dots, N-1$, and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \dots\}$. The local fractional integral of $\psi(x)$ in the interval $[a,b]$ is given by Yang (2012), Wang et al. (2014), and Yan et al. (2014)

$${}_aI_b^{(\upsilon)}\psi(x) = \frac{1}{\Gamma(1+\upsilon)}\int_a^b \psi(t)(dt)^{\upsilon} = \frac{1}{\Gamma(1+\upsilon)} \lim_{\Delta t\to 0}\sum_{j=0}^{N-1}\psi(t_j)(\Delta t_j)^{\upsilon}. \tag{2.5}$$

The formulas of local fractional integrals of special functions used in the paper are as follows:

$${}_0I_x^{(\upsilon)} a\psi(x) = a\ {}_0I_x^{(\upsilon)}\psi(x), \tag{2.6}$$

$${}_0I_x^{(\upsilon)}\left(\frac{x^{n\upsilon}}{\Gamma(1+n\upsilon)}\right) = \frac{x^{(n+1)\upsilon}}{\Gamma\left(1+(n+1)\upsilon\right)}, \ n \in N \tag{2.7}$$

Definition 4.

In fractal space, the Mittage Leffler function is defined as Yang (2012), Wang et al. (2014), and Yan et al. (2014):

$$E_{\upsilon}(x^{\upsilon}) = \sum_{k=0}^{\infty} \frac{x^{k\upsilon}}{\Gamma(1+k\upsilon)}\ ,\ 0 < \upsilon \le 1 \tag{2.8}$$

3. Analysis of the Local Fractional Variational Iteration Method

We consider a general nonlinear local fractional partial differential equation:

$$L_{\upsilon}u(x,\kappa) + R_{\upsilon}u(x,\kappa) + N_{\upsilon}u(x,\kappa) = f(x,\kappa),\ \ 0 < \upsilon \le 1, \tag{3.1}$$

where $L_{\alpha} = \frac{\partial^{\upsilon}}{\partial x^{\upsilon}}$ denotes linear local fractional derivative operator, R_{υ} denotes linear local fractional derivative operator, N_{υ} denotes nonlinear local fractional operator, and $f(x,\kappa)$ is the non-differentiable source term.

According to the rule of local fractional variational iteration method, the correction local fractional functional for (3.1) is constructed as Yang et al. (2013) and Jafari et al. (2015):

$$u_{n+1}(x) = u_n(x) + {}_0I_x^{(\upsilon)}\left(\frac{\lambda(\xi)^{\upsilon}}{\Gamma(1+\upsilon)}\left[L_{\upsilon}u_n(\xi) + R_{\upsilon}\tilde{u}_n(\xi) + N_{\upsilon}\tilde{u}_n(\xi) - f(\xi)\right]\right), \tag{3.2}$$

where $\frac{\lambda(\xi)^{\upsilon}}{\Gamma(1+\upsilon)}$ is a fractal Lagrange multiplier.

Making the local fractional variation of (3.2), we have

$$\delta^{\upsilon}u_{n+1}(x) = \delta^{\upsilon}u_n(x) + {}_0I_x^{(\upsilon)}\delta^{\upsilon}\left(\frac{\lambda(\xi)^{\upsilon}}{\Gamma(1+\upsilon)}\left[L_{\upsilon}u_n(\xi) + R_{\upsilon}\tilde{u}_n(\xi) + N_{\upsilon}\tilde{u}_n(\xi) - f(\xi)\right]\right). \tag{3.3}$$

The extremum condition of u_{n+1} is given by Yang (2012)

$$\delta^{\upsilon}u_{n+1}(x) = 0\ . \tag{3.4}$$

In view of (3.4), we have the following stationary conditions:

$$1 + \left.\frac{\lambda(\xi)^{\upsilon}}{\Gamma(1+\upsilon)}\right|_{\xi=x} = 0\ ,\ \ \left.\left(\frac{\lambda(\xi)^{\upsilon}}{\Gamma(1+\upsilon)}\right)^{(\upsilon)}\right|_{\xi=x} = 0. \tag{3.5}$$

So, from (3.5), we get

$$\frac{\lambda(\xi)^{\upsilon}}{\Gamma(1+\upsilon)}=-1. \tag{3.6}$$

In view of (3.6), we have

$$u_{n+1}(x)=u_n(x)-{}_0I_x^{(\upsilon)}\left(L_\upsilon u_n(\xi)+R_\upsilon u_n(\xi)+N_\upsilon u_n(\xi)-f(\xi)\right). \tag{3.7}$$

Finally, from (3.7), we obtain the solution of (3.1) as follows:

$$u(x,\kappa)=\lim_{n\to\infty}u_n(x,\kappa). \tag{3.8}$$

4. Solution of the Local Fractional Volterra Integral Equation of the Second Kind.

First, we differentiate once from both sides of Equation (1.1) with respect to x:

$$\Omega^{(\upsilon)}(x)=\psi^{(\upsilon)}(x)+\Lambda(x,x)F(\Omega(x))+\frac{1}{\Gamma(1+\upsilon)}\int_a^x\frac{\partial^\upsilon\Lambda(x,\kappa)}{\partial x^\upsilon}F(\Omega(\kappa))(d\kappa)^\upsilon. \tag{4.1}$$

Now, we apply local fractional variational iteration method for Equation (4.1). According to this method correction functional can be written in the following form:

$$\Omega_{n+1}(x)=\Omega_n(x)+{}_0I_x^{(\upsilon)}\left(\frac{\lambda(\zeta)^\upsilon}{\Gamma(1+\upsilon)}\begin{bmatrix}\Omega_n^{(\upsilon)}(\zeta)-\psi^{(\upsilon)}(\zeta)-\Lambda(\xi,\xi)F\left(\tilde{\Omega}_n(\xi)\right)\\ -\dfrac{1}{\Gamma(1+\upsilon)}\displaystyle\int_0^\xi\frac{\partial^\upsilon K(\xi,\kappa)}{\partial\xi^\upsilon}F\left(\tilde{\Omega}_n(\kappa)\right)(d\kappa)^\upsilon\end{bmatrix}\right), \tag{4.2}$$

where $\frac{\lambda(\zeta)^\upsilon}{\Gamma(1+\upsilon)}$ is a general fractal Lagrange's multiplier. To make the above correction functional stationary with respect to Ω_n, we have:

$$\begin{aligned}\delta^\upsilon\Omega_{n+1}(x)&=\delta^\upsilon\Omega_n(x)+{}_0I_x^{(\upsilon)}\delta^\upsilon\left(\frac{\lambda(\zeta)^\upsilon}{\Gamma(1+\upsilon)}\begin{bmatrix}\Omega_n^{(\upsilon)}(\zeta)-\psi^{(\upsilon)}(\zeta)\\ -\Lambda(\xi,\xi)F\left(\tilde{\Omega}_n(\xi)\right)-\dfrac{1}{\Gamma(1+\upsilon)}\displaystyle\int_0^\xi\frac{\partial^\upsilon K(\xi,\kappa)}{\partial\xi^\upsilon}F\left(\tilde{\Omega}_n(\kappa)\right)(d\kappa)^\upsilon\end{bmatrix}\right),\\ &=\delta^\upsilon\Omega_n(x)+{}_0I_x^{(\upsilon)}\left(\frac{\lambda(\zeta)^\upsilon}{\Gamma(1+\upsilon)}\delta^\upsilon\left(\Omega_n^{(\upsilon)}(\xi)\right)\right)\\ &=\delta^\upsilon\Omega_n(x)+\frac{\lambda(x)^\upsilon}{\Gamma(1+\upsilon)}\delta^\upsilon\Omega_n(x)+{}_0I_x^{(\upsilon)}\left(\frac{\lambda^{(\upsilon)}(\zeta)^\upsilon}{\Gamma(1+\upsilon)}\delta^\upsilon\Omega_n(\xi)\right)=0.\end{aligned}$$

From the above relation for any $\delta^\upsilon\Omega_n$, we obtain

$$1+\left.\frac{\lambda(\xi)^{\upsilon}}{\Gamma(1+\upsilon)}\right|_{\xi=x}=0\ ,\quad \left.\left(\frac{\lambda(\xi)^{\upsilon}}{\Gamma(1+\upsilon)}\right)^{(\upsilon)}\right|_{\xi=x}=0.$$

This in turn gives

$$\frac{\lambda(x)^{\upsilon}}{\Gamma(1+\upsilon)}=-1. \tag{4.3}$$

Substituting the identified Lagrange multiplier (4.3) into (4.2), result in the following iterative formula:

$$\Omega_{n+1}(x)=\Omega_n(x)-{}_0I_x^{(\upsilon)}\left(\left[\begin{array}{l}\Omega_n^{(\upsilon)}(\zeta)-\psi^{(\upsilon)}(\zeta)-\Lambda(\xi,\xi)F\left(\tilde{\Omega}_n(\xi)\right)\\ -\dfrac{1}{\Gamma(1+\upsilon)}\displaystyle\int_0^{\xi}\frac{\partial^{\upsilon}K(\xi,\kappa)}{\partial\xi^{\upsilon}}F\left(\tilde{\Omega}_n(\kappa)\right)(d\kappa)^{\upsilon}\end{array}\right]\right). \tag{4.4}$$

Finally, we obtain the exact solution or an approximate solution of the Equation (1.1) as follows:

$$\Omega(x)=\lim_{n\to\infty}\Omega_n(x). \tag{4.5}$$

5. Illustrative Examples

In this section, we give some illustrative examples for solving Volterra integral equations with local fractional derivative operator to demonstrate the efficiency of local fractional variational iteration method.

Example 1.

Let us consider the following linear Volterra integral equation with local fractional operator:

$$\Omega(x)=\frac{x^{\upsilon}}{\Gamma(1+\upsilon)}-\frac{1}{\Gamma(1+\upsilon)}\int_0^x\frac{(x-\kappa)^{\upsilon}}{\Gamma(1+\upsilon)}\Omega(\kappa)(d\kappa)^{\upsilon}. \tag{5.1}$$

The corresponding iterative relation (4.4) for this Equation (5.1) can be constructed as:

$$\Omega_{n+1}(x)=\Omega_n(x)-{}_0I_x^{(\upsilon)}\left(\Omega_n^{(\upsilon)}(\xi)-1+\frac{1}{\Gamma(1+\upsilon)}\int_0^{\xi}\Omega_n(\kappa)(d\kappa)^{\upsilon}\right), \tag{5.2}$$

by taking $\Omega_0(x)=\dfrac{x^{\upsilon}}{\Gamma(1+\upsilon)}$, we derive the following results:

$$\Omega_1(x) = \Omega_0(x) - {}_0I_x^{(\upsilon)}\left(\Omega_0^{(\upsilon)}(\xi) - 1 + \frac{1}{\Gamma(1+\upsilon)}\int_0^{\xi}\Omega_0(\kappa)(d\kappa)^{\upsilon}\right)$$
$$= \frac{x^{\upsilon}}{\Gamma(1+\upsilon)} - \frac{x^{3\upsilon}}{\Gamma(1+3\upsilon)},$$

$$\Omega_2(x) = \Omega_1(x) - {}_0I_x^{(\upsilon)}\left(\Omega_1^{(\upsilon)}(\xi) - 1 + \frac{1}{\Gamma(1+\upsilon)}\int_0^{\xi}\Omega_1(\kappa)(d\kappa)^{\upsilon}\right)$$
$$= \frac{x^{\upsilon}}{\Gamma(1+\upsilon)} - \frac{x^{3\upsilon}}{\Gamma(1+3\upsilon)} + \frac{x^{5\upsilon}}{\Gamma(1+5\upsilon)},$$

$$\Omega_3(x) = \Omega_2(x) - {}_0I_x^{(\upsilon)}\left(\Omega_2^{(\upsilon)}(\xi) - 1 + \frac{1}{\Gamma(1+\upsilon)}\int_0^{\xi}\Omega_2(\kappa)(d\kappa)^{\upsilon}\right)$$
$$= \frac{x^{\upsilon}}{\Gamma(1+\upsilon)} - \frac{x^{3\upsilon}}{\Gamma(1+3\upsilon)} + \frac{x^{5\upsilon}}{\Gamma(1+5\upsilon)} - \frac{x^{7\upsilon}}{\Gamma(1+7\upsilon)},$$
$$\vdots$$
$$\Omega_n(x) = \sum_{\eta=0}^{n}(-1)^{\eta}\frac{x^{(2\eta+1)\upsilon}}{\Gamma\big(1+(2\eta+1)\upsilon\big)}.$$

Thus, we have

$$\begin{aligned}\Omega(x) &= \lim_{n\to\infty}\Omega_n(x)\\ &= \sum_{\eta=0}^{\infty}(-1)^{\eta}\frac{x^{(2\eta+1)\upsilon}}{\Gamma\big(1+(2\eta+1)\upsilon\big)}\\ &= \sin_{\alpha}(x^{\alpha}),\end{aligned} \tag{5.3}$$

which is exactly the same as that obtained by local fractional decomposition method Yang et al. (2012) and local fractional Laplace transform method Yang (2012).

Example 2.

Let us consider the following linear Volterra integral equation with local fractional operator:

$$\Omega(x) = 1 + \frac{x^{\upsilon}}{\Gamma(1+\upsilon)} + \frac{x^{2\upsilon}}{\Gamma(1+2\upsilon)} + \frac{1}{\Gamma(1+\upsilon)}\int_0^{x}\frac{(x-\kappa)^{2\upsilon}}{\Gamma(1+2\upsilon)}\Omega(\kappa)(d\kappa)^{\upsilon}. \tag{5.4}$$

The corresponding iterative relation (4.4) for this Equation (5.4) can be constructed as:

$$\Omega_{n+1}(x) = \Omega_n(x) - {}_0I_x^{(\upsilon)}\left(\Omega_n^{(\upsilon)}(\xi) - 1 - \frac{\xi^{\upsilon}}{\Gamma(1+\upsilon)} - \frac{1}{\Gamma(1+\upsilon)}\int_0^{\xi}\frac{(\xi-\tau)^{\upsilon}}{\Gamma(1+\upsilon)}\Omega_n(\tau)(d\tau)^{\upsilon}\right). \tag{5.5}$$

We can use the initial condition $\Omega(0)=1$ to select $\Omega_0(x)=1$. By using this selection with (5.5) gives the following successive approximations:

$$\Omega_1(x)=\Omega_0(x)-{}_0I_x^{(\upsilon)}\left(\Omega_0^{(\upsilon)}(\xi)-1-\frac{\xi^\upsilon}{\Gamma(1+\upsilon)}-\frac{1}{\Gamma(1+\upsilon)}\int_0^\xi\frac{(\xi-\tau)^\upsilon}{\Gamma(1+\upsilon)}\Omega_0(\tau)(d\tau)^\upsilon\right)$$
$$=1+\frac{x^\upsilon}{\Gamma(1+\upsilon)}+\frac{x^{2\upsilon}}{\Gamma(1+2\upsilon)}+\frac{x^{3\upsilon}}{\Gamma(1+3\upsilon)},$$

$$\Omega_2(x)=\Omega_1(x)-{}_0I_x^{(\upsilon)}\left(\Omega_1^{(\upsilon)}(\xi)-1-\frac{\xi^\upsilon}{\Gamma(1+\upsilon)}-\frac{1}{\Gamma(1+\upsilon)}\int_0^\xi\frac{(\xi-\tau)^\upsilon}{\Gamma(1+\upsilon)}\Omega_1(\tau)(d\tau)^\upsilon\right)$$
$$=1+\frac{x^\upsilon}{\Gamma(1+\upsilon)}+\frac{x^{2\upsilon}}{\Gamma(1+2\upsilon)}+\frac{x^{3\upsilon}}{\Gamma(1+3\upsilon)}+\frac{x^{4\upsilon}}{\Gamma(1+4\upsilon)}+\frac{x^{5\upsilon}}{\Gamma(1+5\upsilon)}+\frac{x^{6\upsilon}}{\Gamma(1+6\upsilon)},$$

$$\vdots$$

$$\Omega_n(x)=\Omega_{n-1}(x)-{}_0I_x^{(\upsilon)}\left(\Omega_{n-1}^{(\upsilon)}(\xi)-1-\frac{\xi^\upsilon}{\Gamma(1+\upsilon)}-\frac{1}{\Gamma(1+\upsilon)}\int_0^\xi\frac{(\xi-\tau)^\upsilon}{\Gamma(1+\upsilon)}\Omega_{n-1}(\tau)(d\tau)^\upsilon\right)$$
$$=1+\frac{x^\upsilon}{\Gamma(1+\upsilon)}+\frac{x^{2\upsilon}}{\Gamma(1+2\upsilon)}+\frac{x^{3\upsilon}}{\Gamma(1+3\upsilon)}+\cdots+\frac{x^{n\upsilon}}{\Gamma(1+n\upsilon)}.$$

Hence, we obtain

$$\begin{aligned}\Omega(x)&=\lim_{n\to\infty}\Omega_n(x)\\&=\sum_{\eta=0}^{\infty}\frac{x^{n\upsilon}}{\Gamma(1+n\upsilon)}\\&=E_\alpha(x^\alpha).\end{aligned}\tag{5.6}$$

Example 3.

Consider the following nonlinear Volterra integral equation of second kind with local fractional derivative operator

$$\Omega(x)=\sec_\upsilon(x^\upsilon)+\tan_\upsilon(x^\upsilon)+\frac{x^\upsilon}{\Gamma(1+\upsilon)}-\frac{1}{\Gamma(1+\upsilon)}\int_0^x\left(1+u^2(\kappa)\right)(d\kappa)^\upsilon.\tag{5.7}$$

The corresponding iterative relation (4.4) for Equation (5.7) can be constructed as:

$$\Omega_{n+1}(x)=\Omega_n(x)-{}_0I_x^{(\upsilon)}\left(\Omega_n^{(\upsilon)}(\xi)+\Omega_n^2(\xi)-\tan_\upsilon(\xi^\upsilon)\sec_\upsilon(\xi^\upsilon)-\tan_\upsilon^2(\xi^\upsilon)-1\right).\tag{5.8}$$

Consider initial approximation $\Omega_0(x)=\sec_\upsilon(x^\upsilon)$ and by the iterative formula (5.8), we get:

$$\Omega_1(x)=\Omega_0(x)-{}_0I_x^{(\upsilon)}\left(\Omega_0^{(\upsilon)}(\xi)+\Omega_0^2(\xi)-\tan_\upsilon(\xi^\upsilon)\sec_\upsilon(\xi^\upsilon)-\tan_\upsilon^2(\xi^\upsilon)-1\right)$$
$$=\sec_\upsilon(x^\upsilon),$$

$$\Omega_2(x) = \Omega_1(x) - {}_0I_x^{(\upsilon)}\left(\Omega_1^{(\upsilon)}(\xi) + \Omega_1^2(\xi) - \tan_\upsilon(\xi^\upsilon)\sec_\upsilon(\xi^\upsilon) - \tan_\upsilon^2(\xi^\upsilon) - 1\right)$$
$$= \sec_\upsilon(x^\upsilon)\,,$$

$$\Omega_3(x) = \Omega_2(x) - {}_0I_x^{(\upsilon)}\left(\Omega_2^{(\upsilon)}(\xi) + \Omega_2^2(\xi) - \tan_\upsilon(\xi^\upsilon)\sec_\upsilon(\xi^\upsilon) - \tan_\upsilon^2(\xi^\upsilon) - 1\right)$$
$$= \sec_\upsilon(x^\upsilon)\,,$$

$$\vdots$$

$$\Omega_n(x) = \Omega_{n-1}(x) - {}_0I_x^{(\upsilon)}\left(\Omega_{n-1}^{(\upsilon)}(\xi) + \Omega_{n-1}^2(\xi) - \tan_\upsilon(\xi^\upsilon)\sec_\upsilon(\xi^\upsilon) - \tan_\upsilon^2(\xi^\upsilon) - 1\right)$$
$$= \sec_\upsilon(x^\upsilon)\,.$$

Thus, we have

$$\Omega(x) = \lim_{n\to\infty} \Omega_n(x)$$
$$= \sec_\upsilon(x^\upsilon)\,. \tag{5.9}$$

6. Conclusion

In this work, the local fractional variational iteration method has been used successfully for solving the linear and nonlinear Volterra integral equations within local fractional derivative operators. The examples analyzed illustrate the ability and reliability of the method presented in this paper and reveals that this one is very simple and effective. Results indicate that the proposed method has small size of computation in comparison with the computational size required in other numerical methods and its rapid convergence shows that method is reliable and introduces a significant improvement in solving linear and nonlinear integral equations with local fractional derivative operators.

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