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Complex solutions of the time fractional Gross-Pitaevskii (GP) equation with external potential by using a reliable method

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Abstract

In this article, modified $(\frac{G'}{G})$-expansion method is presented to establish the exact complex solutions of the time fractional Gross-Pitaevskii (GP) equation in the sense of the conformable fractional derivative. This method is an effective method in finding exact traveling wave solutions of nonlinear evolution equations (NLEEs) in mathematical physics. The present approach has the potential to be applied to other nonlinear fractional differential equations. Based on two transformations, fractional GP equation can be converted into nonlinear ordinary differential equation of integer orders. In the end, we will discuss the solutions of the fractional GP equation with external potentials.

Keywords: Fractional calculus; Differential equations; The conformable fractional derivative; The modified $(\frac{G'}{G})$-expansion method; The time fractional Gross-Pitaevskii (GP) equation

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1. Introduction

The investigation of exact solutions to nonlinear fractional differential equations plays an important role in various applications in physics, fluid flow, engineering, signal processing, control theory, systems identification, biology, finance and fractional dynamics (see, Kilbas et al., 2006; Mille and Ross, 1993; Podlubny, 1999). Recently, a large amount of literature has been provided to construct the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest.

The Gross-Pitaevskii equation describes the ground state of a quantum system of identical bosons using the HartreeFock approximation and the pseudopotential interaction model. Some numerical methods have been proposed to obtain approximate solutions for fractional Gross-Pitaevskii equation, such as Homotopy analysis method (see, Uzar et al., 2012; Uzar and Ballikaya, 2012). By using the modified $(\frac{G'}{G})$-expansion method (see, Taghizadeh et al., 2012), we find exact and analytical solutions of the time fractional Gross-Pitaevskii (GP) equation with external potential in the sense of the conformable fractional derivative.

There are several definitions for fractional differential equations. These definitions include Grunwald-Letnikov, Riemann-Liouville, Caputo, Weyl, Marchaud, and Riesz fractional derivatives (see Podlubny, 1999). Recently, a new modification of Riemann-Liouville derivative is proposed by Jumarie (Jumarie, 2006),

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, \qquad 0 < \alpha < 1$$

and gave some basic fractional calculus formulae, for example, formulae (4.12) and (4.13) in Jumarie (2006),

$$D_t^{\alpha}(f(t)g(t)) = g(t) D_t^{\alpha} f(t) + f(t) D_t^{\alpha} g(t), \tag{1}$$

$$D_t^{\alpha} f(t)[g(t)] = f(t)'_{g(t)}[g(t)] D_t^{\alpha} g(t) = D_g^{\alpha} f[g(t)](g(t)')^{\alpha}. \tag{2}$$

The last formula (1) has been applied to solve the exact solutions to some nonlinear fractional order differential equations. If this formula were true, then we could take the transformation $\xi = x - \frac{at^{\alpha}}{\Gamma(\alpha+1)}$ and reduce the partial derivative $\frac{\partial^{\alpha} U(x,t)}{\partial\ t^{\alpha}}$ to $U'(\xi)$. Therefore the corresponding fractional differential equations become the ordinary differential equations which are easy to study. But we must point out that Jumarie's basic formulae (1) and (2) are not correct, and therefore the corresponding results on differential equations are not true (see Liu, 2014). Fractional derivative is as old as calculus. The most popular definitions are (see Kilbas et al., 2006; Mille and Ross, 1993; Podlubny, 1999):

(i) [Riemann-Liouville definition] If n is a positive integer and $\alpha \in [n-1, n)$ the αth derivative of f is given by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\xi)}{(t-\xi)^{\alpha-n+1}} d\xi.$$

(ii) [Caputo definition] for $\alpha \in [n-1, n)$ the α derivative of f is

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\alpha-n+1}} d\xi.$$

Now, all definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property. However, the following are the setbacks of one definition or another:

(1) The Riemann-Liouville derivative does not satisfy $D_a^\alpha(1) = 0(D_a^\alpha(1))$ for the Caputo derivative if g is not a natural number.

(2) All fractional derivatives do not satisfy the known product rule:

$$D_a^\alpha(fg) = fD_a^\alpha g + gD_a^\alpha f.$$

(3) All fractional derivatives do not satisfy the known quotient rule:

$$D_a^\alpha\left(\frac{f}{g}\right) = \frac{fD_a^\alpha g - gD_a^\alpha f}{g^2}.$$

(4) All fractional derivatives do not satisfy the chain rule:

$$D_a^\alpha(fog)(t) = f^\alpha(g(t))g^\alpha(t).$$

(5) All fractional derivatives do not satisfy $D^\alpha D^\beta f = D^{\alpha+\beta} f$ in general.

(6) The Caputo definition assumes that the function f is differentiable.

Authors introduced a new definition of fractional derivative as follows (see, Khalil, 2014):

$$D_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for $t > 0, \alpha \in [0, 1)$ and $f : [0, \infty) \longrightarrow R$. $D_t^\alpha f(t)$ is called the conformable fractional derivative of f of order α (see, Abdeljawad et al., 2015; Abdeljawad, 2015).

Using this kind of fractional derivative and some useful formulas, we can convert differential equations into integer-order differential equations.

Some properties for the suggested conformable fractional derivative given in (see, Abdeljawad et al., 2015) are as follows,

$$D_t^\alpha(t^\gamma) = \gamma\, t^{\gamma-\alpha}, \qquad \gamma \in R, \tag{3}$$

$$D_t^\alpha(f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{4}$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t). \tag{5}$$

The rest of this paper is organized as follows: First in Sect. 2, we give the description of the modified $(\frac{G'}{G})$-expansion method. Then in Sect. 3, we apply this method to establish exact solutions for the fractional Gross-Pitaevskii equation. Finally, some results and conclusions are presented.

2. The modified $(\frac{G'}{G})$-expansion method

Let us consider the fractional differential equation with independent variables $x = (x_1, x_2, \ldots, x_n, t)$ and a dependent variable u,

$$F(u, D_t^\alpha u, u_{x_1}, u_{x_2}, u_{x_3}, \ldots, D_t^{2\alpha} u, u_{x_1x_1}, u_{x_2x_2}, u_{x_3x_3}, \ldots) = 0. \tag{6}$$

In the following we give the main steps of the modified $(\frac{G'}{G})$-expansion method.

Step 1. Using the variable transformation

$$u(x_1, x_2, \ldots, x_n, t) = U(\xi), \quad \xi = a_1x_1 + a_2x_2 + \ldots + a_nx_n + \frac{bt^\alpha}{\alpha},$$

where a_i and b are constants to be determined later, the fractional differential equation (6) is reduced to nonlinear ordinary differential equation

$$F(U(\xi), bU'(\xi), a_1U'(\xi), \ldots, a_nU'(\xi), b^2U''(\xi), \ldots) = 0, \tag{7}$$

where "$'$" $= \frac{d}{d\xi}$.

Step 2. Suppose that the solution of ODE (7) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U(\xi) = \sum_{i=0}^{r} \alpha_i (\frac{G'}{G})^i + \sum_{i=1}^{r} \beta_i (\frac{G'}{G})^{-i}, \tag{8}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{9}$$

$\alpha_0, \alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r, \lambda$ and μ are constants to be determined later. The positive integer r can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (7).

Step 3. By substituting (8) into (7) and using second order linear ordinary differential equation (LODE) (9), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of equation (7) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_0, \alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r, \lambda$ and μ.

Step 4. Assuming that the constants $\alpha_0, \alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r, \lambda$ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (9) have been well known for us, then substituting $\alpha_0, \alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r, a_1, a_2, \dots, a_n, b$ and the general solutions of equation (9) into (8) we have more traveling wave solutions of the nonlinear evolution equation (6).

3. Exact solutions to the time fractional Gross-Pitaevskii (GP) equation

Now we seek the time fractional Gross-Pitaevskii (GP) equation (see, Uzar and Ballikaya, 2012),

$$ih\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{h^2}{2m}\frac{\partial^2 u}{\partial x^2} + v(x)u + gu|u|^2, \qquad t > 0, \quad 0 < \alpha \leq 1, \tag{10}$$

where g, m and $v(x)$ are the interacting parameter between particles, mass of the particles and external potential applying to the particle systems, respectively. The interacting parameter, i.e. coupling constant g, is defined as $g = \frac{4\pi h^2 a_s}{m}$ where a_s the scattering length of two interacting bosons. Fractional GP equations are built to investigate boson systems in a more realistic way. One of them is the time-fractional GP equation. The memory effect, long-range interaction and restriction of entropic and ergodic hypotheses (see, Greiner et al., 1995) in BEC can be taken into account with the time-fractional GP equation.

We use the transformation

$$u(x,t) = U(\xi), \qquad \xi = ax + \frac{bt^\alpha}{\alpha}, \tag{11}$$

where a and b are constants. Substituting (11) into equation (10), we obtain

$$D_t^{\alpha} u = D_t^{\alpha} U = U(\xi) = U'(\xi) D_t^{\alpha} \xi = b U_{\xi}, \qquad \frac{\partial^2 u}{\partial x^2} = a^2 U_{\xi\xi}.$$

Then equation (10) is reduced into an ordinary differential equation:

$$ibhU_{\xi} + \frac{a^2 h^2}{2m} U_{\xi\xi} - v(\xi) U - gU|U|^2 = 0. \tag{12}$$

Function U is a complex function so we can write

$$U(\xi) = e^{-\frac{imb}{ha^2}\xi} w(\xi),$$

where $w(\xi)$ is a real function. Then (12) reduced to

$$\left(\frac{mb^2}{2a^2} - v(\xi)\right) w - gw^3 + \frac{a^2 h^2}{2m} w_{\xi\xi} = 0. \tag{13}$$

Suppose that the solution of ODE (13) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$w(\xi) = \sum_{i=0}^{r} k_i \left(\frac{G'}{G}\right)^i + \sum_{i=1}^{r} l_i \left(\frac{G'}{G}\right)^{-i}, \tag{14}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{15}$$

Considering the homogeneous balance between $w_{\xi\xi}$ and w^3 in equation (13) we required that $r + 2 = 3r$ then $r = 1$. So we can write (14) as

$$w(\xi) = k_1 \left(\frac{G'}{G}\right) + k_0 + l_1 \left(\frac{G'}{G}\right)^{-1}. \tag{16}$$

By substituting (16) into equation (13) and collecting all terms with the same power of $(\frac{G'}{G})$ together the left-hand side of ODE (13) is converted into another polynomial to zero, yields a set of simultaneous algebraic equations for k_0, k_1, l_1, a, b, μ and λ as follows:

$$
\begin{aligned}
\left(\frac{G'}{G}\right)^3 &: \quad -gmk_1^3 + a^2h^2k_1 = 0,\\
\left(\frac{G'}{G}\right)^2 &: \quad -2gmk_0k_1^2 + a^2h^2\lambda k_1 = 0,\\
\left(\frac{G'}{G}\right)^1 &: \quad -6gma^2k_1(k_1l_1 + k_0^2) + a^4h^2k_1(\lambda^2 + 2\mu) + m(mb^2 - 2a^2v(\xi))k_1 = 0,\\
\left(\frac{G'}{G}\right)^0 &: \quad -2a^2gmk_0^3 - 12a^2gmk_0k_1l_1 + a^4h^2\lambda\mu k_1 + a^4h^2\lambda l_1 + m(mb^2 - 2a^2v(\xi))k_0 = 0,\\
\left(\frac{G'}{G}\right)^{-1} &: \quad -6gma^2l_1(k_1l_1 + k_0^2) + a^4h^2l_1(\lambda^2 + 2\mu) + m(mb^2 - 2a^2v(\xi))l_1 = 0,\\
\left(\frac{G'}{G}\right)^{-2} &: \quad -2gmk_0l_1^2 + a^2h^2\lambda\mu l_1 = 0,\\
\left(\frac{G'}{G}\right)^{-3} &: \quad -gml_1^3 + a^2h^2\mu^2l_1 = 0.
\end{aligned}
$$

Solving the algebraic equation above with the aid of Maple yields

Case ***A***:

$$
k_1 = \pm\sqrt{\frac{a^2h^2}{gm}}, \quad k_0 = \pm\frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}}, \quad l_1 = 0, \quad \mu = \frac{1}{4}\lambda^2 - \frac{m(mb^2 - 2a^2v(\xi))}{2a^4h^2}. \tag{17}
$$

By using (17) expansion (16) can be written as

$$
w(\xi) = \pm\sqrt{\frac{a^2h^2}{gm}}(\frac{G'}{G}) \pm \frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}}. \tag{18}
$$

Substituting the general solutions of equation (15) into equation (18), we obtain the exact solution to equation (10).

When $\Delta > 0$,

$$
U_{1,2}(\xi) = \pm\frac{1}{2}e^{-\frac{imb}{ha^2}\xi}\ \sqrt{T\Delta}\ \left(\frac{\alpha_1\sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_2\cosh\frac{\sqrt{\Delta}}{2}\xi}{\alpha_2\sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_1\cosh\frac{\sqrt{\Delta}}{2}\xi}\right). \tag{19}
$$

When $\Delta < 0$,

$$
U_{1,2}(\xi) = \pm\frac{1}{2}e^{-\frac{imb}{ha^2}\xi}\ \sqrt{-T\Delta}\ \left(\frac{-\alpha_1\sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_2\cos\frac{\sqrt{-\Delta}}{2}\xi}{\alpha_2\sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_1\cos\frac{\sqrt{-\Delta}}{2}\xi}\right).
$$

When $\Delta = 0$,

$$U_{1,2}(\xi) = \pm e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \left(\frac{\alpha_2}{\alpha_1 + \alpha_2 \xi} \right),$$

where $\xi = ax + \frac{bt^\alpha}{\alpha}, \Delta = \lambda^2 - 4\mu = \frac{2m(mb^2 - 2a^2 v(x))}{a^4 h^2}, T = \frac{a^2 h^2}{gm}$ and a and b are arbitrary constants.

Case ***B***:

$$k_1 = 0, \quad k_0 = \pm \frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}}, \quad l_1 = \pm\sqrt{\frac{a^2h^2\mu^2}{gm}}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{m(mb^2 - 2a^2 v(\xi))}{2a^4h^2}. \tag{20}$$

By using (20) expansion (16) can be written as

$$w(\xi) = \pm \frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}} \pm \sqrt{\frac{a^2h^2\mu^2}{gm}} \left(\frac{G'}{G} \right)^{-1}. \tag{21}$$

Substituting the general solutions of equation (15) into equation (21), we obtain the exact solution to equation (10):

When $\Delta > 0$,

$$U_{3,4}(\xi) = \pm e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{ \frac{\lambda}{2} + \sqrt{\mu^2} [\frac{\sqrt{\Delta}}{2} (\frac{\alpha_1 \sinh \frac{\sqrt{\Delta}}{2}\xi + \alpha_2 \cosh \frac{\sqrt{\Delta}}{2}\xi}{\alpha_2 \sinh \frac{\sqrt{\Delta}}{2}\xi + \alpha_1 \cosh \frac{\sqrt{\Delta}}{2}\xi}) - \frac{\lambda}{2}]^{-1} \}. \tag{22}$$

When $\Delta < 0$,

$$U_{3,4}(\xi) = \pm e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{ \frac{\lambda}{2} + \sqrt{\mu^2} [\frac{\sqrt{-\Delta}}{2} (\frac{-\alpha_1 \sin \frac{\sqrt{-\Delta}}{2}\xi + \alpha_2 \cos \frac{\sqrt{-\Delta}}{2}\xi}{\alpha_2 \sin \frac{\sqrt{-\Delta}}{2}\xi + \alpha_1 \cos \frac{\sqrt{-\Delta}}{2}\xi}) - \frac{\lambda}{2}]^{-1} \}.$$

When $\Delta = 0$,

$$U_{3,4}(\xi) = \pm e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{ \frac{\lambda}{2} + \sqrt{\mu^2} [\frac{\alpha_2}{\alpha_1 + \alpha_2 \xi} - \frac{\lambda}{2}]^{-1} \},$$

where $\xi = ax + \frac{bt^\alpha}{\alpha}, \Delta = \lambda^2 - 4\mu = \frac{2m(mb^2-2a^2v(x))}{a^4h^2}, T = \frac{a^2h^2}{gm}$ and a and b are arbitrary constants.

Case ***C***:

$$k_1 = \pm\sqrt{\frac{a^2h^2}{gm}}, \quad k_0 = \pm\frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}}, \quad l_1 = \pm\sqrt{\frac{a^2h^2\mu^2}{gm}}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{m(mb^2 - 2a^2v(\xi))}{2a^4h^2}. \tag{23}$$

By using (23) expansion (16) can be written as

$$w(\xi) = \pm\sqrt{\frac{a^2h^2}{gm}}(\frac{G'}{G}) \pm \frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}} \pm \sqrt{\frac{a^2h^2\mu^2}{gm}}(\frac{G'}{G})^{-1}. \tag{24}$$

Substituting the general solutions of equation (15) into equation (24), we obtain the exact solution to equation (10):

When $\Delta > 0$,

$$\begin{aligned} U_{5,6}(\xi) = \ & \pm \ e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\sqrt{\Delta}}{2}(\frac{\alpha_1 \sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_2\cosh\frac{\sqrt{\Delta}}{2}\xi}{\alpha_2\sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_1\cosh\frac{\sqrt{\Delta}}{2}\xi}) \\ & + \ \sqrt{\mu^2}[\frac{\sqrt{\Delta}}{2}(\frac{\alpha_1 \sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_2\cosh\frac{\sqrt{\Delta}}{2}\xi}{\alpha_2\sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_1\cosh\frac{\sqrt{\Delta}}{2}\xi}) - \frac{\lambda}{2}]^{-1}\}. \end{aligned} \tag{25}$$

When $\Delta < 0$,

$$\begin{aligned} U_{5,6}(\xi) = \ & \pm \ e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\sqrt{-\Delta}}{2}(\frac{-\alpha_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_2\cos\frac{\sqrt{-\Delta}}{2}\xi}{\alpha_2\sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_1\cos\frac{\sqrt{-\Delta}}{2}\xi}) \\ & + \ \sqrt{\mu^2}[\frac{\sqrt{-\Delta}}{2}(\frac{-\alpha_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_2\cos\frac{\sqrt{-\Delta}}{2}\xi}{\alpha_2\sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_1\cos\frac{\sqrt{-\Delta}}{2}\xi}) - \frac{\lambda}{2}]^{-1}\}. \end{aligned}$$

When $\Delta = 0$,

$$U_{5,6}(\xi) = \pm e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\alpha_2}{\alpha_1 + \alpha_2\xi} + \sqrt{\mu^2}[\frac{\alpha_2}{\alpha_1 + \alpha_2\xi} - \frac{\lambda}{2}]^{-1}\},$$

where $\xi = ax + \frac{bt^\alpha}{\alpha}, \Delta = \lambda^2 - 4\mu = \frac{2m(mb^2-2a^2v(x))}{a^4h^2}, T = \frac{a^2h^2}{gm}$ and a and b are arbitrary constants.

Case ***D***:

$$k_1 = \pm\sqrt{\frac{a^2h^2}{gm}}, \quad k_0 = \pm\frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}}, \quad l_1 = \mp\sqrt{\frac{a^2h^2\mu^2}{gm}}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{m(mb^2-2a^2v(\xi))}{2a^4h^2}. \tag{26}$$

By using (26) expansion (16) can be written as

$$w(\xi) = \pm\sqrt{\frac{a^2h^2}{gm}}(\frac{G'}{G}) \pm \frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}} \mp \sqrt{\frac{a^2h^2\mu^2}{gm}}(\frac{G'}{G})^{-1}. \tag{27}$$

Substituting the general solutions of equation (15) into equation (27), we obtain the exact solution to equation (10):

When $\Delta > 0$,

$$\begin{aligned} U_{7,8}(\xi) = \ & \pm \ e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\sqrt{\Delta}}{2}(\frac{\alpha_1 \sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_2\cosh\frac{\sqrt{\Delta}}{2}\xi}{\alpha_2 \sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_1\cosh\frac{\sqrt{\Delta}}{2}\xi}) \\ & - \ \sqrt{\mu^2}[\frac{\sqrt{\Delta}}{2}(\frac{\alpha_1 \sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_2\cosh\frac{\sqrt{\Delta}}{2}\xi}{\alpha_2 \sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_1\cosh\frac{\sqrt{\Delta}}{2}\xi}) - \frac{\lambda}{2}]^{-1}\}. \end{aligned} \tag{28}$$

When $\Delta < 0$,

$$\begin{aligned} U_{7,8}(\xi) = \ & \pm \ e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\sqrt{-\Delta}}{2}(\frac{-\alpha_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_2\cos\frac{\sqrt{-\Delta}}{2}\xi}{\alpha_2 \sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_1\cos\frac{\sqrt{-\Delta}}{2}\xi}) \\ & - \ \sqrt{\mu^2}[\frac{\sqrt{-\Delta}}{2}(\frac{-\alpha_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_2\cos\frac{\sqrt{-\Delta}}{2}\xi}{\alpha_2 \sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_1\cos\frac{\sqrt{-\Delta}}{2}\xi}) - \frac{\lambda}{2}]^{-1}\}. \end{aligned}$$

When $\Delta = 0$,

$$U_{7,8}(\xi) = \pm e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\alpha_2}{\alpha_1 + \alpha_2\xi} - \sqrt{\mu^2}[\frac{\alpha_2}{\alpha_1 + \alpha_2\xi} - \frac{\lambda}{2}]^{-1}\},$$

where $\xi = ax + \frac{bt^{\alpha}}{\alpha}, \Delta = \lambda^2 - 4\mu = \frac{2m(mb^2 - 2a^2v(x))}{a^4h^2}, T = \frac{a^2h^2}{gm}$ and a and b are arbitrary constants.

Case ***E***:

$$k_1 = \pm\sqrt{\frac{a^2h^2}{gm}}, \quad k_0 = \mp\frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}}, \quad l_1 = \mp\sqrt{\frac{a^2h^2\mu^2}{gm}}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{m(mb^2 - 2a^2v(\xi))}{2a^4h^2}. \tag{29}$$

By using (29) expansion (16) can be written as

$$w(\xi) = \pm\sqrt{\frac{a^2h^2}{gm}}(\frac{G'}{G}) \mp \frac{\lambda}{2}\sqrt{\frac{a^2h^2}{gm}} \mp \sqrt{\frac{a^2h^2\mu^2}{gm}}(\frac{G'}{G})^{-1}. \tag{30}$$

Substituting the general solutions of equation (15) into equation (30), we obtain the exact solution to equation (10):

When $\Delta > 0$,

$$\begin{aligned} U_{9,10}(\xi) = & \pm \ e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\sqrt{\Delta}}{2}(\frac{\alpha_1 \sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_2\cosh\frac{\sqrt{\Delta}}{2}\xi}{\alpha_2\sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_1\cosh\frac{\sqrt{\Delta}}{2}\xi}) - \lambda \\ & - \ \sqrt{\mu^2}[\frac{\sqrt{\Delta}}{2}(\frac{\alpha_1 \sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_2\cosh\frac{\sqrt{\Delta}}{2}\xi}{\alpha_2\sinh\frac{\sqrt{\Delta}}{2}\xi + \alpha_1\cosh\frac{\sqrt{\Delta}}{2}\xi}) - \frac{\lambda}{2}]^{-1}\}. \end{aligned} \tag{31}$$

When $\Delta < 0$,

$$\begin{aligned} U_{9,10}(\xi) = & \pm \ e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\sqrt{-\Delta}}{2}(\frac{-\alpha_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_2\cos\frac{\sqrt{-\Delta}}{2}\xi}{\alpha_2\sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_1\cos\frac{\sqrt{-\Delta}}{2}\xi}) - \lambda \\ & - \ \sqrt{\mu^2}[\frac{\sqrt{-\Delta}}{2}(\frac{-\alpha_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_2\cos\frac{\sqrt{-\Delta}}{2}\xi}{\alpha_2\sin\frac{\sqrt{-\Delta}}{2}\xi + \alpha_1\cos\frac{\sqrt{-\Delta}}{2}\xi}) - \frac{\lambda}{2}]^{-1}\}. \end{aligned}$$

When $\Delta = 0$,

$$U_{9,10}(\xi) = \pm e^{-\frac{imb}{ha^2}\xi} \ \sqrt{T} \ \{\frac{\alpha_2}{\alpha_1 + \alpha_2\xi} - \lambda - \sqrt{\mu^2}[\frac{\alpha_2}{\alpha_1 + \alpha_2\xi} - \frac{\lambda}{2}]^{-1}\},$$

where $\xi = ax + \frac{bt^{\alpha}}{\alpha}, \Delta = \lambda^2 - 4\mu = \frac{2m(mb^2-2a^2v(x))}{a^4h^2}, T = \frac{a^2h^2}{gm}$ and a and b are arbitrary constants.

Remark 1:

- In expression (19), if $\alpha_1 > 0$ and $\alpha_1^2 > \alpha_2^2$, then $U_{1,2}(\xi)$ can be written as:

$$U_{1,2}(\xi) = \pm\frac{1}{2}e^{-\frac{imb}{ha^2}\xi}\ \sqrt{T\Delta}\ \tanh(\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)).$$

- In expression (22), if $\alpha_1 > 0$ and $\alpha_1^2 > \alpha_2^2$, then $U_{3,4}(\xi)$ can be written as:

$$U_{3,4}(\xi) = \pm e^{-\frac{imb}{ha^2}\xi}\ \sqrt{T}\ \{\frac{\lambda}{2} + \sqrt{\mu^2}[\frac{\sqrt{\Delta}}{2}\ \tanh(\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)) - \frac{\lambda}{2}]^{-1}\}.$$

- In expression (25), if $\alpha_1 > 0$ and $\alpha_1^2 > \alpha_2^2$, then $U_{5,6}(\xi)$ can be written as:

$$\begin{aligned} U_{5,6}(\xi) = &\pm\ e^{-\frac{imb}{ha^2}\xi}\ \sqrt{T}\ \{\frac{\sqrt{\Delta}}{2}\tanh(\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)) \\ &+\ \sqrt{\mu^2}[\frac{\sqrt{\Delta}}{2}\tanh(\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)) - \frac{\lambda}{2}]^{-1}\}. \end{aligned}$$

- In expression (28), if $\alpha_1 > 0$ and $\alpha_1^2 > \alpha_2^2$, then $U_{7,8}(\xi)$ can be written as:

$$\begin{aligned} U_{7,8}(\xi) = &\pm\ e^{-\frac{imb}{ha^2}\xi}\ \sqrt{T}\ \{\frac{\sqrt{\Delta}}{2}\tanh(\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)) \\ &-\ \sqrt{\mu^2}[\frac{\sqrt{\Delta}}{2}\tanh(\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)) - \frac{\lambda}{2}]^{-1}\}. \end{aligned}$$

- In expression (31), if $\alpha_1 > 0$ and $\alpha_1^2 > \alpha_2^2$, then $U_{9,10}(\xi)$ can be written as:

$$\begin{aligned} U_{9,10}(\xi) = &\pm\ e^{-\frac{imb}{ha^2}\xi}\ \sqrt{T}\ \{\frac{\sqrt{\Delta}}{2}\tanh(\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)) - \lambda \\ &-\ \sqrt{\mu^2}[\frac{\sqrt{\Delta}}{2}\tanh(\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)) - \frac{\lambda}{2}]^{-1}\}. \end{aligned}$$

• • • In all cases; $\xi = ax + \frac{bt^{\alpha}}{\alpha}, \xi_0 = \tanh^{-1}(\frac{\alpha_2}{\alpha_1}), \Delta = \lambda^2 - 4\mu = \frac{2m(mb^2-2a^2v(x))}{a^4h^2}, T = \frac{a^2h^2}{gm}$ and a and b are arbitrary constants.

Remark 2:

Now we will investigate the time fractional Gross-Pitaevskii equation (10) to discuss the ground state time-dependent dynamic of the Bose-Einstein condensation of weakly interacting system for different potential.

1. For $m = \frac{1}{2}$, $h = 1$ and $v(x) = 0$, GPE of fractional order (10) can be degenerated to the nonlinear fractional Schrödinger equation (Dhaigude and Birajdar, 2013),

$$i\frac{\partial^\alpha u}{\partial t^\alpha} + u_{xx} - gu|u|^2 = 0, \qquad t > 0, \quad 0 < \alpha \leq 1. \tag{32}$$

Then, by using the modified $(\frac{G'}{G})$-expansion method, we can find the exact solution of equation (32).

2. For $v(x) = \frac{1}{2}m\omega^2x^2$, GPE of fractional order (10) can be degenerated to the time-fractional Gross-Pitaevskii equation for harmonic potential (Uzar and Ballikaya, 2012),

$$ih\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{h^2}{2m}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}m\omega^2x^2u + gu|u|^2, \qquad t > 0, \quad 0 < \alpha \leq 1, \tag{33}$$

where ω is the frequency of harmonic potential. Then by using the modified $(\frac{G'}{G})$-expansion method, we can find the exact solution of equation (33).

3. For $v(x) = \pm\sin^2 x$, GPE of fractional order (10) can be degenerated to the time-fractional Gross-Pitaevskii equation for optical lattice potential (Uzar et al., 2012),

$$ih\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{h^2}{2m}\frac{\partial^2 u}{\partial x^2} \pm \sin^2 xu + gu|u|^2, \qquad t > 0, \quad 0 < \alpha \leq 1. \tag{34}$$

Then by using the modified $(\frac{G'}{G})$-expansion method, we can find the exact solution of equation (34).

4. Conclusion

In this paper, the modified $(\frac{G'}{G})$-expansion method has been successfully applied to find the new exact solutions for the time-fractional Gross-Pitaevskii equation. This method is powerful and applicable to many nonlinear fractional partial differential equations.

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