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Numerical solution of linear time delay systems using Chebyshev-tau spectral method

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Abstract

In this paper, a hybrid method based on method of steps and a Chebyshev-tau spectral method for solving linear time delay systems of differential equations is proposed. The method first converts the time delay system to a system of ordinary differential equations by the method of steps and then employs Chebyshev polynomials to construct an approximate solution for the system. In fact, the solution of the system is expanded in terms of orthogonal Chebyshev polynomials which reduces the solution of the system to the solution of a system of algebraic equations. Also, we transform the coefficient matrix of the algebraic system to a block quasi upper triangular matrix and the latter system can be solved more efficiently than the first one. Furthermore, using orthogonal Chebyshev polynomials enables us to apply fast Fourier transform for calculating matrix-vector multiplications which makes the proposed method to be more efficient. Consistency, stability and convergence analysis of the method are provided. Numerous numerical examples are given to demonstrate efficiency and accuracy of the method. Comparisons are made with available literature.

Keywords: Linear delay systems; commensurate delays; method of steps; Chebyshev-tau spectral method; stability analysis; convergence analysis

MSC 2010 No.: 65L03, 65L06, 65D99

1. Introduction

Delay differential equations (DDEs) have been applied to model real phenomena. These equations often appear in many real-world problems. They occur in a wide variety of physical, chemical, engineering, economic and biological systems and their networks. One can cite many examples where delay plays an important role. As an example, in biological mathematics that delay plays a crucial role is drug therapy for the disease of human immunodeficiency virus (HIV) infection (Kirschner (1996), Nelson and Perelson (2002)). These equations also known as difference-differential equations, are a special class of differential equations called functional differential equations (Hale and Lunel (2013), Elsgolts and Norkin (1973)). Delay differential equations were initially introduced in the 18th century by Laplace and Condorcet (Gorecki et al. (1989)).

Numerically approximating a solution to a DDE has been studied by many authors (see, for example, Bellen and Zennaro (2013), Bellman and Cooke (1965) and Feldstein and Neves (1984)). Currently, most DDE solver are generated using adapted continuous numerical methods for solving ODE. To obtain a numerical solution of a DDE by means of a discrete method often we need to know the solution on a set of points that differ from the grid. To calculate the approximate solutions on non-grid points, a local interpolation is needed. In such situation, stability of discrete method may be lost. Bellen and Zennaro (1988), introduced the concept of stable interpolant for Runge-Kutta methods. In other words, a stable interpolant maintains the stability properties of the Runge-Kutta method itself. Hermite interpolation can be also used to solve a DDE numerically. Oberle and Pesch (1981), combined Runge-Kutta methods and Hermite interpolation for the numerical solution of a DDE. In fact, they have been shown that the combination has high order of accuracy and it is reliable.

Nevertheless, the most popular discrete methods are Runge-Kutta methods that use interpolation to obtain the numerical solution of a DDE. Wherever the step size must be reduced often to produce answers at specified points, a Runge-Kutta formula becomes inefficient. Bogacki and Shampine (1990), have been presented a structural way to handle this issue. When a high order method for solving a DDE is applied, the amount of computations for the interpolation process increase considerably. To overcome this difficulty, Zennaro (1985a), presented a one-step collocation method. Also, for super convergence rate, Zennaro (1985b), considered one-step collocation method at Gaussian points. Wang and Wang (2010), proposed the single-step and multiple-domain Legendre-Gauss collocation integration processes for nonlinear DDEs. One of the most important advantages of their method is its accuracy and efficiency.

Another class of numerical methods that can be considered are Spline methods. Spline collocation methods are used by Banks and Kappel (1979), Kappel and Kunisch (1981) and Kemper (1975). So far, all of the mentioned methods for solving a DDE have finite order of accuracy. To get infinite order of accuracy for a smooth problem usually spectral methods are applied. For more details about these methods, see for example Gottlieb and Orszag (1977) and Canuto et al. (1987). Ito et al. (1991), have been employed shifted Legendre polynomials to construct a spectral approximate solution for a DDE. More precisely, they considered Legendre-tau method to construct an approximate solution for a DDE with one constant delay. Sedaghat et al. (2012) proposed a numerical scheme to solve delay differential equations of pantograph type. Their method consists of expanding the required approximate solution as the elements of the shifted Chebyshev polynomials. Aziz and Amin (2016) applied Haar wavelet collocation method to obtain the numerical solution of DDEs. Behroozifar and Yousefi (2013) introduced hybrid of block-pulse functions and Bernstein polynomials to construct an approximate solution for DDEs. Ghasemi and Tavassoli Kajani (2011) employed Chebyshev wavelets to obtain a numerical scheme for solving DDEs. They introduced operational matrix of delay and utilize it to reduce the solution of time-varying delay systems to the solution of algebraic equations.

In this paper, we propose a method for solving linear time delay systems of differential equations. At the first step, the proposed method uses the method of steps (Bellen and Zennaro (2013)) to convert the linear system of DDE on a given interval to an ODE over that interval, by employing the known history function, say $\phi(t)$, for that interval. Then, the approximate solution for this system is obtained by tau-Galerkin method (Gottlieb and Orszag (1977), Canuto et al. (1987)). We construct an approximate solution by Chebyshev polynomials expansion (Fox and Parker (1968)). More precisely, we consider the solution of the ODE system as a truncated series of Chebyshev polynomials. This leads us to solve a linear system of algebraic equations. Furthermore, we transform the coefficient matrix of the algebraic system to a block quasi upper triangular form. Therefore, the algebraic system can be solved more efficiently than the original one. In addition, properties of the Chebyshev polynomials, enriches the proposed method to apply fast Fourier transform for computing matrix-vector multiplications. At the next step, the newly found approximate solution is used as the history function for the next interval and the above process is carried out iteratively to obtain the approximate solution for the whole time interval. Also, the consistency, stability and convergence of the proposed method are provided. Numerical results are presented to demonstrate the accuracy and efficiency of the proposed method. In the following subsection we explain notations, preliminaries and the problem statement.

1.1. *Notations, preliminaries and problem statement*

We adopt the following notations throughout this paper.

$\mathbb{R}^n$ is Euclidean space with corresponding norm $|\cdot|$, $\mathbb{R}^{n\times n}$ is space of real square matrices, $L^2([-1,1];\mathbb{R}^n)$ or for short $L^2(-1,1)$ is the Hilbert space of square integrable functions that map $[-1,1]$ to a member of $\mathbb{R}^n$ with inner product

$$(f,g)_w = \int_{-1}^{1} f(x)g(x)w(x)dx,$$

where

$$w(x) = (1 - x^2)^{-1/2}$$

(that for simplicity we may drop the subscript w), and corresponding norm $||\cdot||_{L^2_w}$, $H^1([-1,1];\mathbb{R}^n)$ or for convenience $H^1(-1,1)$ is the Sobolov space of absolutely continuous functions with square integrable derivative with norm $||\cdot||_{1,w}$.

Let Ω be an open bounded domain in $\mathbb{R}$, with piecewise smooth boundary Γ. We assume that we want to approximate the boundary-value problem

$$\mathcal{L}x = f \quad \text{in } \Omega, \tag{1}$$

$$\mathcal{B}x = 0 \quad \text{on } \Gamma, \tag{2}$$

where $\mathcal{L}$ is a linear differential operator in Ω, and $\mathcal{B}$ is a set of linear boundary differential operators on Γ. Let X to be a space of real functions defined in Ω that are square integrable with respect to $w(x)$. The domain of definition of $\mathcal{L}$, i.e., the subset $D(\mathcal{L})$ of those functions x of X for which $\mathcal{L}x$ is still an element of X, is supposed to be a dense subspace of X. Thus, $\mathcal{L}$ is a linear operator from $D(\mathcal{L})$ to X. Prescribing the boundary conditions (2) amounts to restricting the domain of $\mathcal{L}$ to the subspace $D_B(\mathcal{L})$ of $D(\mathcal{L})$ defined by

$$D_B(\mathcal{L}) = \{x \in D(\mathcal{L}) |\, \mathcal{B}x = 0 \,\text{on}\,\ \Gamma\}.$$

Suppose that X_N and Y_N are finite-dimensional subspaces of X having the same dimension. In the tau method, we have

$$x^N \in X_N,$$
$$(\mathcal{L}x^N, v) = (f, v) \ \ \forall\, v \in Y_N.$$

Let $\mathcal{P}^N$ be the orthogonal projection of L^2_w onto the space spanned by $\{p_k\}_{k=0}^N$, in which p_k is a polynomial of degree k and we denote this set by $\mathbb{P}_\mathbb{N}$.

Lemma 1.1. Canuto et al. (1987)

Suppose that

$$\mathcal{P}^N x(t) = \sum_{k=0}^{N} \hat{a}_k T_k(t),$$

where $T_k(t)$ is the Chebyshev polynomial of degree k. Then, for all $x(t) \in H^1(-1,1)$ we have

$$||x(t) - \mathcal{P}^N x(t)||_{L^2_w} \le CN^{-1}||x(t)||_{1,w}, \tag{3}$$

where the positive constant C is independent of the function x, the integer N, and the diameter of the domain.

Theorem 1.2. Lax-Richtmyer equivalence theorem, Gottlieb and Orszag (1977)

A consistent approximation to a well-posed linear problem is stable if and only if it is convergent.

Lemma 1.3. Babuška inf-sup condition, Canuto et al. (1987)

Let $X_N \subset F$ and $Y_N \subset G$ for all positive N. If there exists a constant $\alpha > 0$ independent of N such that

$$\alpha||x||_F \leq \sup_{\substack{v \in Y_N \\ v \neq 0}} \frac{(\mathcal{L}x, v)}{||v||_G} \qquad \forall x \in X_N,$$

then the following estimate holds

$$||x^N||_F \leq \frac{C}{\alpha}||f||,$$

where the positive constant C is independent of N.

One important class of delay differential equations is linear time invariant systems (LTI) or briefly time-delayed LTI systems. General representation of a time-delayed LTI systems can be given by

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - \tau_k) + u(t), \\ x(t) = \phi(t), \quad t \in [-\tau, 0), \\ x(0) = x_0, \end{cases} \tag{4}$$

where

$$0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_N = \tau$$

are positive delays, $x(t), u(t) \in \mathbb{R}^n$ are the state and input vectors respectively, $A_k(k = 0, 1, 2, \cdots, N) \in \mathbb{R}^{n \times n}$ are system matrices which are given and $\phi \in L^2([-\tau, 0]; \mathbb{R}^n)$.

Theorem 1.4. Delfour and Mitter (1975)

Let $(x_0, \phi) \in M^2 = \mathbb{R}^n \times L^2([-\tau, 0]; \mathbb{R}^n)$. Then, the system (4) for any $T \geq 0$ admits a unique solution $x(t) \in L^2([-\tau, T]; \mathbb{R}^n) \cap H^1([-\tau, T]; \mathbb{R}^n)$, which depends continuously on the initial data $(x_0, \phi) \in M^2$.

In this paper, we consider the system (4) with two discrete commensurate delays. However the method can be applied to systems with more than two commensurate delays. It can be also extended to solve nonlinear time delay problems.

The remaining sections of this paper is organized as follows. In Section 2, we recall some properties of Chebyshev polynomials, while Section 3 describes the Chebyshev-tau spectral approach for solving the system (4) with two commensurate delays. Section 4 provides the stability and convergence analysis of the method while Section 5 presents illustrative examples. Finally, Section 6 contains some concluding remarks.

2. Properties of Chebyshev polynomials

Here, we review some properties of Chebyshev polynomials (Fox and Parker (1968)). The Chebyshev polynomials of the first kind, $T_k(x)$, $k = 0, 1, \dots,$ $-1 \le x \le 1$, can be defined as the solution of the following ordinary differential equation

$$\frac{d}{dx}\Big(\sqrt{1-x^2}\frac{dT_k}{dx}(x)\Big) + \frac{k^2}{\sqrt{1-x^2}}T_k(x) = 0, \tag{5}$$

which satisfy $T_k(1) = 1$. Thus, $T_0(x) = 1$, $T_1(x) = x$, and so on.

Chebyshev polynomials $\{T_k(x)\}_{k=0}^{\infty}$ satisfy the orthogonality relation

$$(T_k, T_l)_w = 0, \quad \text{for} \quad k \neq l, \tag{6}$$

and they form a basis for $L^2(-1, 1)$. More precisely, any $u \in L^2(-1, 1)$ can be written as

$$u = \sum_{k=0}^{\infty} \hat{u}_k T_k,$$

where

$$\hat{u}_k = \frac{2}{\pi c_k}\int_{-1}^{1} u(x)T_k(x)w(x)dx, \quad k \ge 0,$$

and

$$c_k = \begin{cases} 2, & k = 0, \\ 1, & k \ge 1, \end{cases} \tag{7}$$

with

$$||u||^2_{L^2_w} = \sum_{k=0}^{\infty} \frac{c_k \pi}{2} \hat{u}_k^2.$$

These polynomials also satisfy the recursion relation

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x). \tag{8}$$

If u is represented as

$$u = \sum_{k=0}^{N} \hat{u}_k T_k(x),$$

then

$$\dot{u} = \frac{du}{dx} = \sum_{k=0}^{N-1} \hat{u}_k^{(1)} T_k(x), \tag{9}$$

where

$$\hat{u}_k^{(1)} = \frac{2}{c_k} \sum_{\substack{j=k+1 \\ j+k \text{ odd}}}^{N} j\hat{u}_j, \quad k = 0, 1, \cdots, N-1. \tag{10}$$

One important property that we will use it for efficient implementation of the method is the following relation (Canuto et al. (1987))

$$2T_k(x) = \frac{1}{k+1} T'_{k+1}(x) - \frac{1}{k-1} T'_{k-1}(x), \quad k \geq 1. \tag{11}$$

From (11), we have

$$2k\hat{u}_k = c_{k-1}\hat{u}_{k-1}^{(1)} - \hat{u}_{k+1}^{(1)}, \quad k \geq 1,$$

and so

$$c_k \hat{u}_k^{(1)} = \hat{u}_{k+2}^{(1)} + 2(k+1)\hat{u}_{k+1}, \quad 0 \leq k \leq N-1. \tag{12}$$

The generalization of (12) is given by (Canuto et al. (1987))

$$c_k \hat{u}_k^{(q)} = \hat{u}_{k+2}^{(q)} + 2(k+1)\hat{u}_{k+1}^{(q-1)}, \quad k \geq 0. \tag{13}$$

3. Chebyshev-tau approximation

Now, we focus on the Chebyshev approximation of solutions to the system (4) with two discrete point delays which are commensurate. The proposed method uses the method of steps (Bellen and Zennaro (2013)) and Chebyshev-tau spectral method (Gottlieb and Orszag (1977), Canuto et al. (1987)). We begin our discussion by considering the following system

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2) + u(t) \\ x(t) = \phi(t), \quad t \in [-\tau_2, 0) \\ x(0) = x_0, \end{cases} \tag{14}$$

where $0 < \tau_1 \leq \tau_2, \frac{\tau_2}{\tau_1} = k \in \mathbb{N}$ ($\mathbb{N}$ stands for natural number set). It is clear that for $t \in [0, \tau_1]$, the linear delay system (14) is equivalent to the following initial value problem

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 \phi(t-\tau_1) + A_2 \phi(t-\tau_2) + u(t), \qquad t \in [0, \tau_1], \\ x(0) = x_0. \end{cases} \tag{15}$$

It is assumed that the approximate solution $x^N(t)$ to be expanded in terms of Chebyshev polynomials, i.e.,

$$x^N(t) = \sum_{k=0}^{N} \hat{a}_k T_k(\frac{2t - \tau_1}{\tau_1}), \qquad t \in [0, \tau_1], \tag{16}$$

where $T_k(\theta)$ are the Chebyshev polynomials, $\hat{a}_k \in \mathbb{R}^n$, $k = 0, 1, 2, \ldots, N$, are the unknowns to be computed. On the other hand, we also expand $\phi(t-\tau_1)$, $\phi(t-\tau_2)$ and u in terms of a truncated Chebyshev series as follows

$$\phi^N(t-\tau_1) = \sum_{k=0}^{N-1} \hat{\phi}_k^{(1)} T_k(\frac{2t-\tau_1}{\tau_1}), \tag{17}$$

$$\phi^N(t-\tau_2) = \sum_{k=0}^{N-1} \hat{\phi}_k^{(2)} T_k(\frac{2t-\tau_1}{\tau_1}), \tag{18}$$

$$u^N(t) = \sum_{k=0}^{N-1} \hat{u}_k T_k(\frac{2t-\tau_1}{\tau_1}), \qquad t \in [0, \tau_1], \tag{19}$$

where

$$\hat{\phi}_k^{(1)} = \frac{2}{\pi c_k}\int_{-1}^{1}\phi\big(\frac{\tau_1}{2}(s-1)\big)T_k(s)\frac{1}{\sqrt{1-s^2}}\mathrm{d}s,$$
$$\hat{\phi}_k^{(2)} = \frac{2}{\pi c_k}\int_{-1}^{1}\phi\big(\frac{\tau_1(s+1)-2\tau_2}{2}\big)T_k(s)\frac{1}{\sqrt{1-s^2}}\mathrm{d}s,$$
$$\hat{u}_k = \frac{2}{\pi c_k}\int_{-1}^{1}u\big(\frac{\tau_1}{2}(s+1)\big)T_k(s)\frac{1}{\sqrt{1-s^2}}\mathrm{d}s,$$

and the c_k are given by (7).

Now, by using the method of weighted residuals (Canuto et al. (1987)), we obtain

$$\big(\dot{x}^N(t) - A_0x^N(t) - A_1\phi^N(t-\tau_1) - A_2\phi^N(t-\tau_2) - u^N(t), T_k(\frac{2t-\tau_1}{\tau_1})\big)_\omega = 0, \tag{20}$$

for $0 \le k \le N-1$. From (16), (17), (18), (19), the orthogonality of the Chebyshev polynomials and

$$\dot{x}^N(t) = \frac{2}{\tau_1}\sum_{k=0}^{N-1} b_k T_k(\frac{2t-\tau_1}{\tau_1}),$$

where b_k are defined to be the same as $\hat{u}_k^{(1)}$ in (10), we obtain

$$\frac{2}{\tau_1}b_k = A_0\hat{a}_k + A_1\hat{\phi}_k^{(1)} + A_2\hat{\phi}_k^{(2)} + \hat{u}_k, \qquad 0 \le k \le N-1. \tag{21}$$

Linear algebraic system (21) has $N+1$ unknowns but only N equations. To obtain a unique solution for the system, we need an additional equation that deals with the initial condition. We note that the important difference between the tau approximation (Gottlieb and Orszag (1977), Canuto et al. (1987)), and the Galerkin approximation is that the basis functions $T_k(\theta)$ are not required to satisfy the initial condition in (15). The additional equation is

$$x^N(0) = \sum_{k=0}^{N}\hat{a}_k T_k(-1) = \sum_{k=0}^{N}\hat{a}_k(-1)^k = x_0. \tag{22}$$

Rewriting (21) and (22) in matrix form, leads to the following algebraic system for the $N+1$ unknowns $\hat{a}_k$

$$V^N\vec{a}^N = W^N\vec{a}^N + \vec{b}^N, \qquad t \in [0,\tau_1], \tag{23}$$

where

$$\vec{a}^N = (\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_N)^T,$$
$$\vec{b}^N = (A_1\hat{\phi}_0^{(1)} + A_2\hat{\phi}_0^{(2)} + \hat{u}_0, A_1\hat{\phi}_1^{(1)} + A_2\hat{\phi}_1^{(2)} + \hat{u}_1, \ldots, A_1\hat{\phi}_{N-1}^{(1)} + A_2\hat{\phi}_{N-1}^{(2)} + \hat{u}_{N-1}, x_0)^T,$$

and

$$W^N = \mathrm{diag}(A_0, A_0, \ldots, A_0, 0).$$

For N even (for N odd, only the last column of V^N is different), V^N is given by

$$V^N = \frac{2}{\tau_1}\begin{bmatrix} 0 & 1 & 0 & 3\,0 \ldots & N-1 & 0 \\ 0 & 0 & 4 & 0\,8 \ldots & 0 & 2N \\ 0 & 0 & 0 & 6\,0 \ldots & 2(N-1) & 0 \\ \vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots & \vdots \\ 0 & 0 & 0 & & 2(N-1) & 0 \\ 0 & 0 & 0 & & 0 & 2N \\ -\frac{\tau_1}{2} & \frac{\tau_1}{2} & -\frac{\tau_1}{2} & & -\frac{\tau_1}{2} & \frac{\tau_1}{2} \end{bmatrix} \bigotimes I, \tag{24}$$

where I is $n \times n$ identity matrix and $\bigotimes$ denotes the Kronecker product. We can repeat the above procedure to solve (14) on $[\tau_1, 2\tau_1]$, $[2\tau_1, 3\tau_1]$, and so on. We note that for each time interval $[i\tau_1, (i+1)\tau_1]$, $i = 0, 1, \cdots$, only the right hand side of Equation (23) changes, but the coefficient matrix $V^N - W^N$ remains unchange.

Now, we discuss some implementation issues. First, computation of the right hand side of Equation (23) needs to compute the Chebyshev coefficients $\hat{\phi}_k^{(1)}$, $\hat{\phi}_k^{(2)}$ of the initial function and Chebyshev coefficients $\hat{u}_k$ of the forcing function on each time interval $[i\tau_1, (i+1)\tau_1]$, $i = 0, 1, \cdots$. In fact we use the matrix $C_{jk} = \cos\frac{\pi jk}{N}$, $k =, 1, \ldots, N-1$, to transform Chebyshev space to physical space and inverse transform, i.e., transform from physical space to Chebyshev space that done by

$$(C^{-1})_{jk} = \frac{2}{N\bar{c}_j\bar{c}_k}\cos\frac{\pi jk}{N},$$

where

$$\bar{c}_j = \begin{cases} 2, & j = 0, N, \\ 1, & j = 1, \ldots, N-1. \end{cases}$$

Both transforms may be evaluated by the fast Fourier transform (Canuto et al. (1987)).

Second, the linear system of Equation (23) can be reduced to a block quasi upper triangular form with two off-diagonal band. If ordinary Gauss elimination were used to solve the linear system of Equations (23) with $n(N+1)$ equations and unknowns, it would be require about $(n(N+1))^3$ multiplications for the $L^N U^N$ decomposition of $V^N - W^N$.

However, by taking into consideration the spectral structures of matrix V^N and W^N the LU-factorization can be carried out in only $c_1 n(N+1)$ multiplications and about $c_2 n(N+1)$ multiplications for each right hand side, where $c_2 < c_1$ and $c_j \ll N, j = 1, 2$. From (21), we have

$$b_k = \frac{\tau_1}{2}\big(A_0 \hat{a}_k + A_1 \hat{\phi}_k^{(1)} + A_2 \hat{\phi}_k^{(2)} + \hat{u}_k\big). \tag{25}$$

If we invoke the recursion relation(13) with $q = 2$, then

$$\hat{a}_{k+1} = \frac{1}{2(k+1)}\big(c_k b_k - b_{k+2}\big). \tag{26}$$

Now, from (25) we obtain

$$\begin{aligned}\hat{a}_{k+1} = \frac{1}{2(k+1)} \Big(& c_k \big(\frac{\tau_1}{2}(A_0 \hat{a}_k + A_1 \hat{\phi}_k^{(1)} + A_2 \hat{\phi}_k^{(2)} + \hat{u}_k)\big) \\ & -\frac{\tau_1}{2}\big(A_0 \hat{a}_{k+2} + A_1 \hat{\phi}_{k+2}^{(1)} + A_2 \hat{\phi}_{k+2}^{(2)} + \hat{u}_{k+2})\Big).\end{aligned} \tag{27}$$

By simplifying, for $0 \le k \le N-1$ we obtain

$$\begin{aligned}2(k+1)\hat{a}_{k+1} - \frac{\tau_1 c_k}{2} A_0 \hat{a}_k + \frac{\tau_1}{2} A_0 \hat{a}_{k+2} = & \frac{\tau_1 c_k}{2} A_1 \hat{\phi}_k^{(1)} + \frac{\tau_1 c_k}{2} A_2 \hat{\phi}_k^{(2)} \\ & + \frac{\tau_1 c_k}{2}\hat{u}_k - \frac{\tau_1}{2} A_1 \hat{\phi}_{k+2}^{(1)} - \frac{\tau_1}{2} A_2 \hat{\phi}_{k+2}^{(2)} - \frac{\tau_1}{2}\hat{u}_{k+2}.\end{aligned}$$

If we put

$$g_k = \frac{\tau_1}{2}\left(c_k A_1 \hat{\phi}_k^{(1)} + c_k A_2 \hat{\phi}_k^{(2)} + c_k \hat{u}_k - \hat{u}_{k+2} - A_1 \hat{\phi}_{k+2}^{(1)} - A_2 \hat{\phi}_{k+2}^{(2)}\right),$$

we arrive at a linear system of equations with the following coefficient matrix

$$A^N = \begin{bmatrix} -\tau_1 A_0 & 2 & \frac{\tau_1}{2} A_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -\frac{\tau_1}{2} A_0 & 4 & \frac{\tau_1}{2} A_0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -\frac{\tau_1}{2} A_0 & 6 & \frac{\tau_1}{2} A_0 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & & -\frac{\tau_1}{2} A_0 & 2(N-1) & \frac{\tau_1}{2} A_0 & 0 \\ 0 & 0 & 0 & & & -\frac{\tau_1}{2} A_0 & 2N & \frac{\tau_1}{2} A_0 \\ -\frac{\tau_1}{2} I & \frac{\tau_1}{2} I & -\frac{\tau_1}{2} I & & & & -\frac{\tau_1}{2} I & \frac{\tau_1}{2} I \end{bmatrix}, \tag{28}$$

for the right hand side $\vec{g} = (g_0, g_1, \dots, g_{N-1}, x_0)^T$ and the unknown vector $\vec{a}^N = (\hat{a}_0, \hat{a}_1, \dots, \hat{a}_N)^T$. Therefore, the following linear system of equations should be solved

$$A^N \vec{a}^N = \vec{g}. \tag{29}$$

The linear system of Equations (23) and (29) are equivalent. However, the system matrix in (29) is of block quasi upper triangular form and therefore, Equation (29) can be solved more efficiently than Equation (23). One way is to apply Gaussian elimination procedure to a block at a time (Isaacson and Keller (1996)). An operation count reveals that block elimination for a block quasi upper triangular matrix of the form (29) requires about $2n^3N$ multiplications and about as many additions (or subtractions) for the LU-factorization plus about $3n^2N$ for solving (29) for each right-hand side (Ito et al. (1991)).

4. Consistency, stability and convergence analysis

In this section, we show that the approximate solution x^N converges to the exact solution x of the system (14). In the following discussions, we put

$$x^{N,i}(s) = x^N((i-1)\tau_1 + s), \quad s \in [0, \tau_1],\ i \geq 1, \tag{30}$$

to denote the approximate solution x^N on the interval $[(i-1)\tau_1, i\tau_1]$. First, we prove the consistency and then the stability of the method will be proved. For the proof of the consistency let us to define the operator $\mathcal{L}$ as following

$$\mathcal{L}x^i \equiv \dot{x}^i - A_0 x^i, \tag{31}$$

where

$$x^i(s) = x((i-1)\tau_1 + s), \quad s \in [0, \tau_1], \quad i \geq 1.$$

In this case,

$$D(\mathcal{L}) = \{x^i \in \mathbf{C}^1(0, \tau_1)\}$$

and

$$D_B(\mathcal{L}) = \{x^i \in D(\mathcal{L}) | x^i(0) = x^{i-1}(\tau_1)\}.$$

Then, thanks to Lemma 1.1, we have

$$||x^i - \mathcal{P}^N x^i||_{L^2_w} \to 0, \quad \text{as } N \to \infty, \text{ for all } x^i \in D_B(\mathcal{L}). \tag{32}$$

Therefore, the method is consistent.

Second, we show that the method is stable, i.e., estimates of the form

$$||x^{N,i}|| \leq C||x^{N,i-1}||, \quad i = 1, 2, 3, \cdots, M, \tag{33}$$

can be obtained in some appropriately chosen norm and the constant C is independent of N. For doing this job, let us to define spaces

$$X_N = \{x^i \in \mathbb{P}_N | x^i(0) = x^{i-1}(\tau_1)\}$$

and

$$Y_N = \mathbb{P}_{N-1}.$$

Note that for all $x^i \in X_N$, we have

$$\mathcal{P}^{N-1}\mathcal{L}x^i = \mathcal{L}x^i - A_0(\mathcal{P}_{N-1}x^i - x^i).$$

Therefore,

$$\begin{aligned}(\mathcal{L}x^i, \mathcal{P}^{N-1}\mathcal{L}x^i)_w &= ||\mathcal{L}x^i||^2_{L^2_w} - (\mathcal{L}x^i, A_0(\mathcal{P}^{N-1}x^i - x^i))_w \\ &\geq ||\mathcal{L}x^i||^2_{L^2_w} - ||\mathcal{L}x^i||_{L^2_w}||A_0||||x^i - \mathcal{P}^{N-1}x^i||_{L^2_w}.\end{aligned} \tag{34}$$

According to Lemma 1.1, we have

$$||A_0||||x^i - \mathcal{P}^{N-1}x^i||_{L^2_w} \leq C_0N^{-1}||x^i||_{1,w}.$$

Moreover, it is possible to prove the following a priori estimate

$$||x^i||_{1,w} \leq C_1||\mathcal{L}x^i||_{L^2_w},$$

for a suitable constant C_1. Now, using (34), we get

$$(\mathcal{L}x^i, \mathcal{P}^{N-1}\mathcal{L}x^i)_w \geq (1 - C_0C_1N^{-1})||\mathcal{L}x^i||^2_{L^2_w} \geq (2C_1^2)^{-1}||x^i||^2_{1,w},$$

provided N is so large that

$$(1 - C_0 C_1 N^{-1}) \geq \frac{1}{2}.$$

Since

$$||\mathcal{P}^{N-1}\mathcal{L}x^i||_{L^2_w} \leq C_3||x^i||_{1,w},$$

we conclude the following estimate

$$\frac{(\mathcal{L}x^i, \mathcal{P}^{N-1}\mathcal{L}x^i)_w}{||\mathcal{P}^{N-1}\mathcal{L}x^i||_{L^2_w}} \geq \alpha||x^i||_{1,w}, \tag{35}$$

where $\alpha = 2C_1^2 C_3$.

If we define

$$F = \{x^i \in H^1(0, \tau_1) |\, x^i(0) = x^{i-1}(\tau_1)\}$$

and

$$G = L^2_w(0, \tau_1),$$

then the stability of the method (i.e., inequality (33)) results from Lemma 1.3. Now, the convergence of the method can be established as a consequence of Lax-Richtmyer equivalence Theorem 1.2.

5. Numerical results

In this section, we provide some numerical examples and compare the proposed method with the exact and Runge-Kutta solutions. Runge-Kutta solutions are obtained by using dde23 function of Matlab 2014a which is adapted for solving delay differential equations. The computational codes were conducted on an Intel(R) Core(TM) i7-6700K processor, equipped with 8 GB of RAM.

Example 1.

In this example (Banks and Kappel (1979)), the equation for a damped oscillator with delay restorting force and constant external force is considered

$$\begin{cases} \ddot{x}(t) + \dot{x}(t) + x(t-1) = 10, \\ x(t) = \cos t, \ \dot{x}(t) = -\sin t, \quad t \in [-1, 0]. \end{cases} \tag{36}$$

We can write the above equation in terms of a linear first order system as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \tag{37}$$

where $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$. The exact solution for $0 \le t \le 1$ is

$$\begin{cases} x_1(t) = \frac{1}{2}\sin(t-1) - \frac{1}{2}\cos(t-1) - e^{-t}a + 10t - 9 + \sin(1) - 9, \\ x_2(t) = \frac{1}{2}\cos(t-1) + \frac{1}{2}\sin(t-1) + e^{-t}a + 10, \end{cases}$$

where $a = -10 - \frac{1}{2}\cos(1) + \frac{1}{2}\sin(1)$. The exact solution for $1 \le t \le 2$ is

$$\begin{cases} x_1(t) = 5t^2 - 19t - 0.5\sin(t-2) - e^{-t+1}\left(10 + 10t + 0.5\right)\cos(1) \\ \qquad -0.5\cos(1)te^{-t+1} + \sin(1)\left(0.5e^{-t+1} + 0.5te^{-t+1} + t\right) - 1.36e^{-t}b, \\ x_2(t) = 10t - 19 - 0.5\cos(t-2) + 10te^{-t+1} + 0.5\cos(1)te^{-t+1} \\ \qquad -0.5\sin(1)te^{-t+1} + \sin(1) + 1.36e^{-t}b, \end{cases}$$

where

$$\begin{aligned} b = {} & 19 - \sin(1) - \cos(1)\cosh(1) + \cos(1)\sinh(1) - 20\cosh(1) \\ & +20\sinh(1) + \sin(1)\cosh(1) - \sin(1)\sinh(1). \end{aligned}$$

Numerical and analytical solutions are plotted in Figure 1 and the maximum error between Rune-Kutta and the proposed method solutions is given in Table 1. As it was expected, the proposed method is more accurate than the Runge-Kutta method. In this example, the spectral method only uses 8 nodes to get the results, while Runge-Kutta method gives the results by taking 15 nodes. Note that if one try to obtain the accuracy of the spectral method, needs to refine the mesh and this leads to a heavy computation cost of Runge-Kutta method.

Table 1: Maximum error between the exact solution, Runge-Kutta solution and the Chebyshev-tau spectral solution for $0 \le t \le 2$ of Example 1.

x_i	Runge-Kutta	Current work
x_1	9.8569×10^{-4}	4.6172×10^{-10}
x_2	1.0358×10^{-3}	5.3382×10^{-10}

Example 2.

Consider the scalar equation (Bellen and Zennaro (2013))

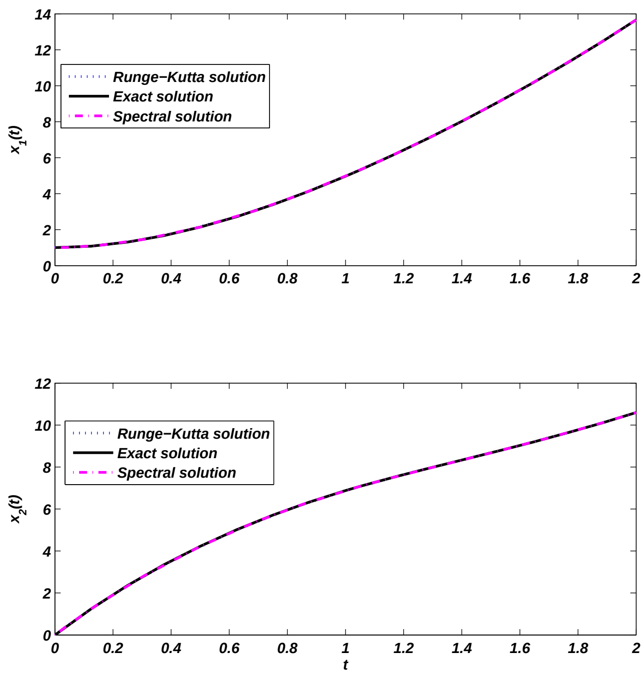

Figure 1: Comparison of state vector ($x_1(t)$ and $x_2(t)$) in the exact solution, Runge-Kutta solution and Chebyshev-tau spectral solution ($N = 8$) of Example 1.

$$\dot{x}(t) = -x(t-1),$$

with the initial function

$$x(t) = 0.5t, \quad t \in [-1, 0].$$

The exact solution for $0 \le t \le 2$ can be expressed as

$$x(t) = \begin{cases} -\frac{1}{4}t^2 + \frac{1}{2}t, & 0 \le t \le 1, \\ \frac{1}{12}(t-1)^3 - \frac{1}{4}t^2 + \frac{1}{2}t, & 1 \le t \le 2. \end{cases}$$

In Figure 2 numerical and exact solutions are demonstrated. For a longer time interval, the spectral and Runge-Kutta solutions are plotted in Figure 3 which confirms that the spectral method produces a smoother solution compared to Runge-Kutta method. In addition, the maximum error for Runge-Kutta method was 1.6574×10^{-1} with 37 nodes, while for the proposed method was 1.1102×10^{-16} which demonstrates the accuracy of the method as expected.

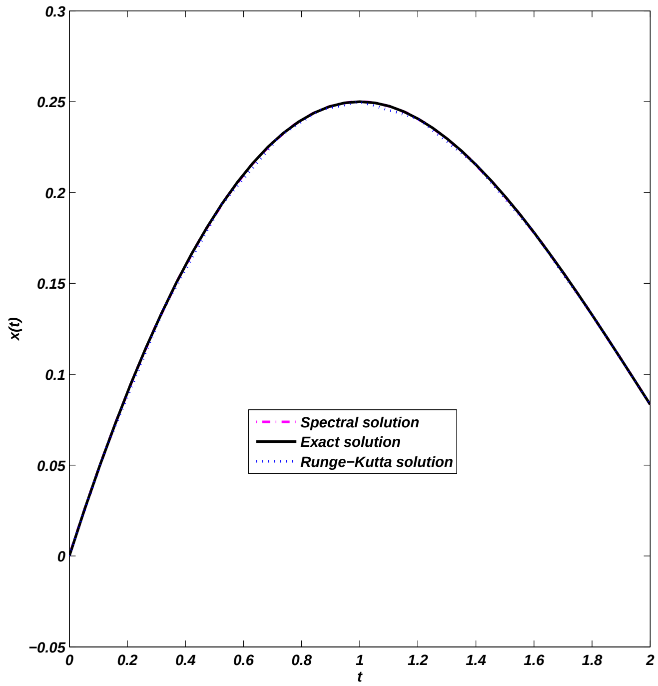

Figure 2: Comparison between the exact solution, Runge-Kutta solution and Chebyshev-tau spectral solution ($N = 8$) of Example 2.

Example 3.

Consider the following system (Banks and Kappel (1979))

$$\dot{x}(t) = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} x(t-1),$$

which is equipped with the initial function

$$x(t) = 1, \quad t \in [-1, 0].$$

The analytical solution for $0 \leq t \leq 1$ is

$$\begin{cases} x_1(t) = -\frac{2}{3}t^3 + 2t + 1, \\ x_2(t) = -t^2 + 1, \\ x_3(t) = 2t + 1, \end{cases}$$

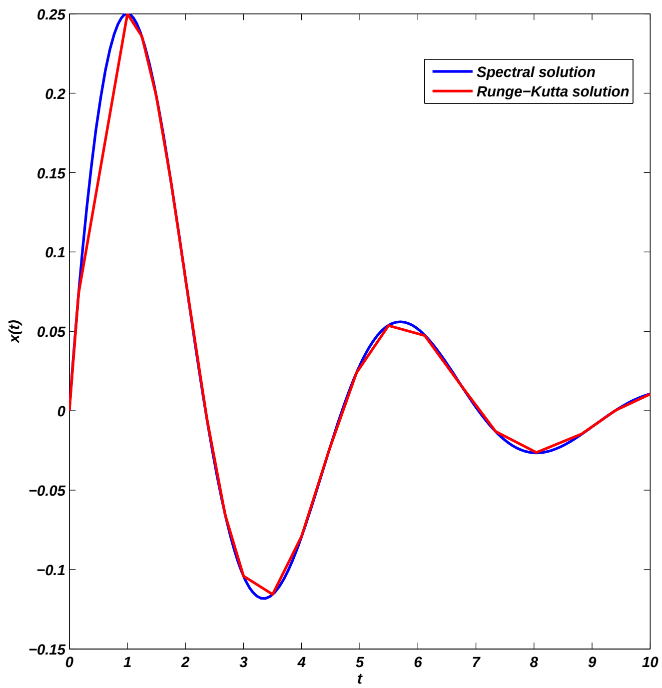

Figure 3: Comparison between Runge-Kutta method and Chebyshev-tau spectral method ($N = 8$) of Example 2.

and for $1 \leq t \leq 2$ is

$$\begin{cases} x_1(t) = -2t^2 + 4t + \frac{1}{3}, \\ x_2(t) = -2t + 2, \\ x_3(t) = -\frac{2}{3}t^3 + 2t^2 + \frac{5}{3}. \end{cases}$$

Numerical simulations for $N = 8$ are demonstrated in Figure 4. In this case, the maximum error between exact and numerical simulations is given in Table 2. The number of nodes for Runge-Kutta method was 39.

Table 2: Maximum error between the exact solution, Runge-Kutta solution and the Chebyshev-tau spectral solution for $0 \leq t \leq 2$ of Example 3.

x_i	Runge-Kutta	Current work
x_1	6.7972×10^{-4}	6.6613×10^{-16}
x_2	1.9309×10^{-3}	2.2204×10^{-16}
x_3	8.8818×10^{-16}	1.7763×10^{-15}

Example 4.

In this example, we consider the following scalar equation with rational delay (Bellen and Zennaro (2013))

$$\dot{x}(t) = -x(t-0.5),$$

with the following initial function

$$x(t) = 0.5t, \quad t \in [-0.5, 0].$$

The exact solution for $0 \leq t \leq 2$ is

$$x(t) = \begin{cases} -0.25(t-0.5)^2 + \frac{1}{16}, & 0 \leq t \leq 1, \\ \frac{1}{12}(t-1)^3 - \frac{1}{16}t + \frac{5}{48}, & 1 \leq t \leq 2. \end{cases}$$

Approximate solution obtained by the Chebyshev-tau spectral method for $N = 8$ is illustrated in Figure 5. Moreover, the maximum error for Runge-Kutta method was 4.7516×10^{-2}, whereas for Chebyshev-tau spectral method was 2.0817×10^{-17}. If the time interval is taken

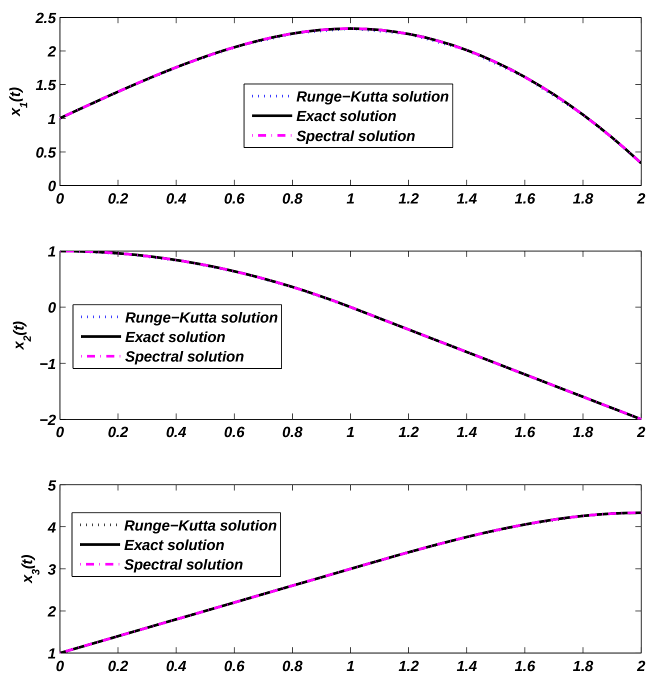

Figure 4: Comparison of state vector $(x_1(t), x_2(t)$ and $x_3(t))$ in the exact solution, Runge-Kutta solution and Chebyshev-tau spectral solution $(N = 8)$ of Example 3.

more longer, one can see that the Chebyshev-tau spectral method has a smooth transient response compared to Runge-Kutta method by using 37 nodes. This event can be observed in Figure 6.

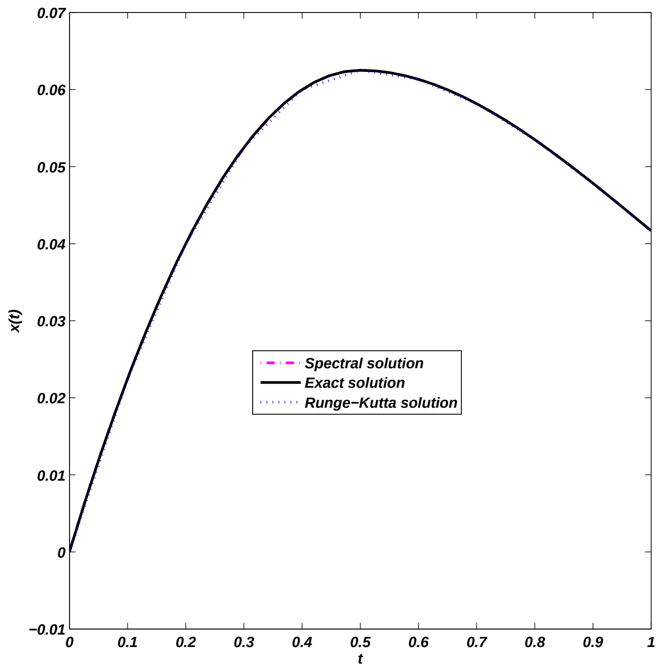

Figure 5: Comparison between the exact solution, Runge-Kutta solution and Chebyshev-tau spectral solution $(N = 8)$ of Example 4.

Example 5.

In this example, we consider the following inhomogeneous system

$$\dot{x}(t) = x(t-1) + t^2,$$

which constrained to the following history

$$x(t) = t, \quad t \in [-1, 0].$$

The exact solution for $0 \leq t \leq 2$ is

$$x(t) = \begin{cases} \frac{1}{3}t^3 + \frac{1}{2}t^2 - t, & 0 \leq t \leq 1, \\ \frac{1}{12}t^4 + \frac{1}{16}t^3 - \frac{1}{2}t^2 + \frac{7}{6}t - \frac{13}{12}, & 1 \leq t \leq 2. \end{cases}$$

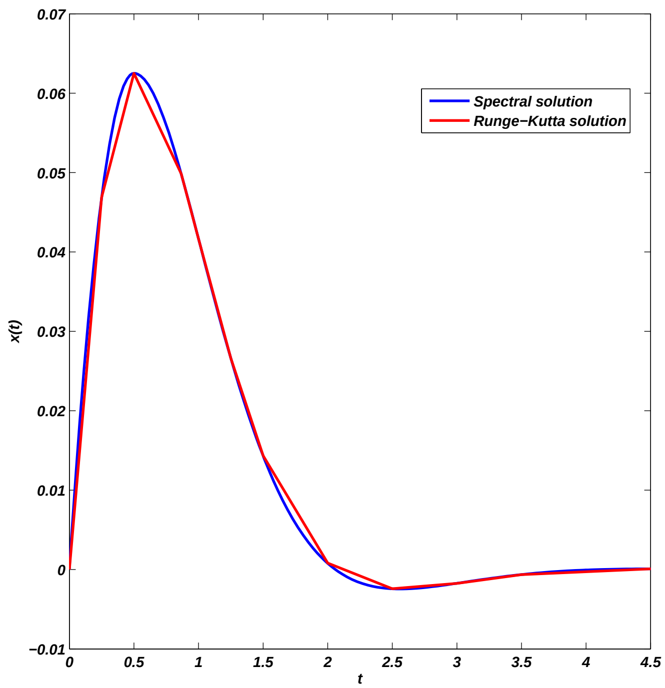

Figure 6: Comparison between Runge-Kutta method and Chebyshev-tau spectral method ($N = 8$) of Example 4.

Numerical and analytical solutions are illustrated in Figure 7. Furthermore, the maximum error for Runge-Kutta method was 1.242×10^{-3} by using 21 nodes, whereas for Chebyshev-tau spectral method was 2.131×10^{-15}.

Example 6.

As a final example, consider the following system with two discrete point delays

$$\dot{x}(t) = x(t - 0.5) + x(t - 1),$$

with the history function as

$$x(t) = 0.5t, \quad t \in [-1, 0].$$

The analytical solution for $0 \leq t \leq 1.5$ is

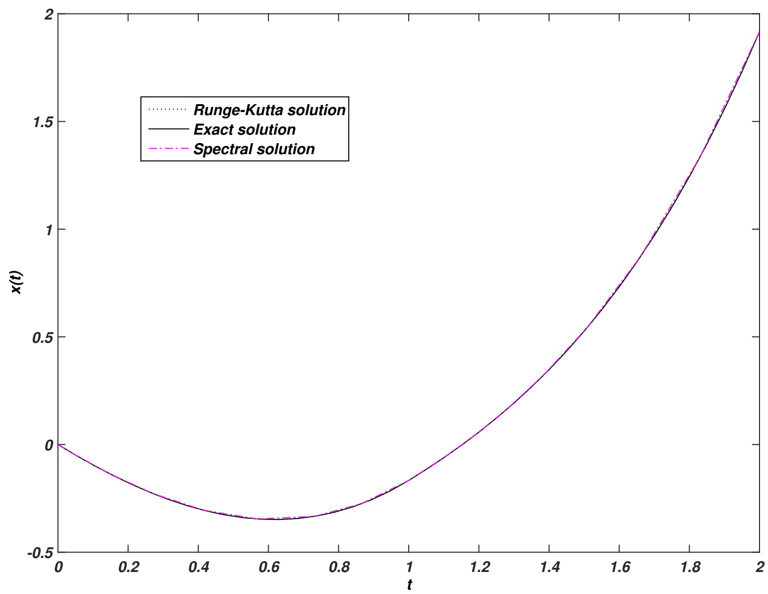

Figure 7: Comparison between the exact solution, Runge-Kutta method and Chebyshev-tau spectral method ($N = 8$) of Example 5.

$$x(t) = \begin{cases} 0.5t^2 - 0.75t, & 0 \le t \le 0.5, \\ \frac{1}{6}(t-0.5)^3 - \frac{3}{8}(t-0.5)^2 + 0.25(t-1)^2 - \frac{5}{16}, & 0.5 \le t \le 1, \\ \frac{1}{24}t(t-1)^3 + \frac{1}{12}(t-\frac{3}{2})^3 - \frac{5}{16}t - \frac{3}{8}(t-1)^2 - \frac{6}{96}, & 1 \le t \le 1.5. \end{cases}$$

Numerical and analytical solutions are depicted in Figure 8. The maximum error for Runge-Kutta method was 6.2042×10^{-3} by employing 37 nodes, while for the proposed method was 1.1303×10^{-16} which confirms the accuracy of the method as expected.

The CPU time of the above examples are given in Table 3. As one observe, the CPU time required by the proposed method using 8 nodes is about one half of that used by Runge-Kutta method.

Table 3: CPU time (s) for the proposed ($N = 8$) and Runge-Kutta method

Example	Current work	Runge-Kutta	No. nodes
1	0.098	0.156	15
2	0.065	0.111	37
3	0.113	0.186	39
4	0.056	0.112	37
5	0.078	0.168	21
6	0.075	0.152	37

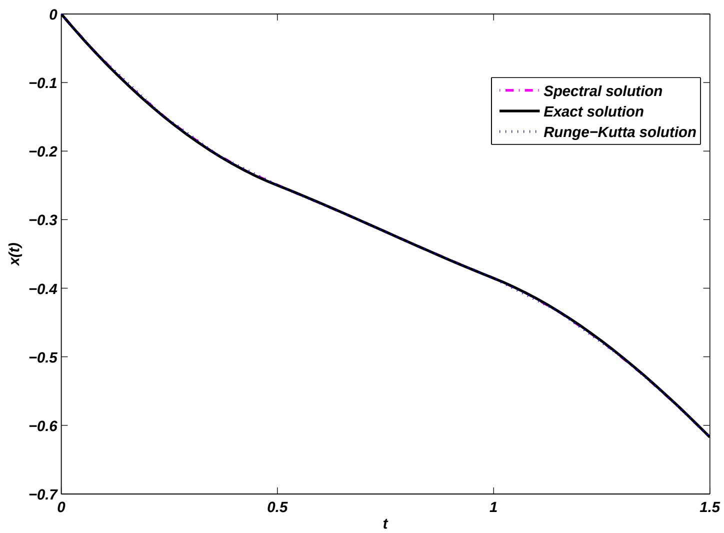

Figure 8: Comparison between the exact solution, Runge-Kutta method and Chebyshev-tau spectral method $(N = 8)$ of Example 6.

6. Conclusion

In this article, the method of steps and Chebyshev-tau spectral method are combined to solve linear time delay systems of differential equations. Usually, Runge-Kutta method is used to solve ODE systems. Here, we used Chebyshev-tau spectral method to solve ODE systems which improve the accuracy of approximate solution dramatically and decrease the computational costs. In fact, we construct the approximate solution using small number of nodes. Implementation of the method leads to a linear system of algebraic equations which has, in general, a dense coefficient matrix. Here we convert the coefficient matrix to a block quasi upper triangular form using Chebyshev polynomials which can be solve more efficiently. Moreover, employing Chebyshev-tau spectral method enables us to apply FFT which reduces the CPU time of calculations. The consistency and stability analysis of the proposed method is provided, and consequently its convergence is established. Numerical results show a very good agreement with available literature. The proposed method can be extended to nonlinear problems and systems with more than two commensurate delays. This will be the subject of further studies.

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