Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 11, Issue 2 (December 2016), pp. 919 - 929

Matching Transversal Edge Domination in Graphs

Anwar Alwardi

Department of Studies in Mathematics
University of Mysore, Manasagangotri
Mysuru - 570 006, India
a_wardi@hotmail.com

Received: November 19, 2015; Accepted: August 3, 2016

Abstract

Let $G=(V,E)$ be a graph. A subset X of E is called an edge dominating set of G if every edge in $E-X$ is adjacent to some edge in X. An edge dominating set which intersects every maximum matching in G is called matching transversal edge dominating set. The minimum cardinality of a matching transversal edge dominating set is called the matching transversal edge domination number of G and is denoted by $\gamma'_{mt}(G)$. In this paper, we begin an investigation of this parameter.

Keywords: Dominating set; Matching set; matching transversal dominating set; Matching transversal domination

MSC 2010 No.: 05C69

1. Introduction

By a graph $G=(V,E)$, we mean a finite and undirected graph with no loops and multiple edges. As usual $p=|V|$ and $q=|E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighbourhood of a vertex v, respectively for any edge $f \in E$. The degree of $f=uv$ in G is defined by $deg(f)=deg(u)+deg(v)-2$. The open neighbourhood and closed neighbourhood of an edge f in G, denoted by $N(f)$ and $N[f]$, respectively, are defined as $N(f)=\{g \in V(G): f \text{ and } g \text{ are adjacent}\}$ and $N[f]=N(f)\cup\{f\}$. The cardinality of $N(f)$ is called the degree of f and denoted by $\deg(f)$. The maximum and minimum degree of edge in G are denoted, respectively by $\Delta^{'}(G)$ and $\delta^{'}(G)$. That is, $\Delta^{'}(G)=\max_{f\in E(G)}|N(f)|$, $\delta^{'}(G)=\min_{f\in E(G)}|N(f)|$. A set D of vertices in a graph G is a dominating set if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

A line graph $L(G)$ (also called an interchange graph or edge graph) of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge, if and only if the corresponding edges of G have a vertex in common. The Corona product $G\circ H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1\le i\le|V(G)|$. A matching M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. The matching number is the maximum cardinality of a matching of G and denoted by $\beta_1(G)$. A vertex is matched (or saturated) if it is an endpoint of one of the edges in the matching; otherwise, the vertex is unmatched. A maximal matching is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching; that is, M is maximal if it is not a proper subset of any other matching in graph G. In other words, a matching M of a graph G is maximal if every edge in G has a non-empty intersection with at least one edge in M.

The independent transversal domination was introduced by Hamid (2012). A dominating set D in a graph G which intersects every maximum independent set in G is called independent transversal dominating set of G. The minimum cardinality of an independent transversal dominating set is called independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. For terminology and notations not specifically defined here we refer reader to Harary (1969). For more details about domination number and its related parameters, we refer to Haynes et al. (1998), Sampathkumar et al. (1985). and Walikar et al. (1979). The concept of edge domination was introduced by Mitchell et al. (1977). Let $G=(V,E)$ be a graph. A subset X of E is called an edge dominating set of G if every edge in $E-X$ is adjacent to some edge in X.

In this paper, we introduce the concept of matching transversal domination in a graph and, exact values for some standard graphs, bounds and some interesting results are established.

2. Matching Transversal Edge Domination Number

Definition 1.

Let $G=(V,E)$ be a graph. An edge dominating set S which intersects every maximum matching in G is called matching transversal edge dominating set. The minimum cardinality of a matching transversal edge dominating set is called the matching transversal edge domination number of G and is denoted by $\gamma'_{mt}(G)$.

Theorem 1.

For any path P_p with $p \geq 3$, we have,

$$\gamma'_{mt}(P_p) = \begin{cases} 2, & \text{if } p = 3 \text{ or } 4; \\ 3, & \text{if } p = 7; \\ \left\lceil \frac{p-1}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof:

Let $E(P_p) = \{v_1v_2, v_2v_3, ..., v_{p-1}v_p\}$. Obviously $\gamma'_{mt}(P_3) = \gamma'_{mt}(P_4) = 2$. Similarly, if $p = 7$, easily we can see that, $\gamma'_{mt}(P_7) = 3$. Now suppose that $p \notin \{3, 4, 7\}$, then we have three cases.

Case 1:
If $p \equiv 0(\text{b}\bmod 3)$, then the set of edges $S = \{v_{3i+1}v_{3i+2} : 0 \leq i \leq \frac{p}{3} - 1\}$ is a minimum edge dominating set of P_p. Also the induced subgraph $\langle E - S\rangle = K_2 \cup \frac{p-3}{3} P_3$ and every matching in $\langle E - S\rangle$ contains at most $\frac{p}{3}$ edges and $\frac{p}{3} < \left\lceil \frac{p-1}{2} \right\rceil = \beta_1(P_p)$. Hence, $\langle E - S\rangle$ contains no maximum matching of P_p, then $\gamma'_{mt}(P_p) = \gamma'(P_p) = \left\lceil \frac{p-1}{3} \right\rceil$.

Case 2:
If $p \equiv 1(\bmod 3)$, then the set of edges $S = \{v_{3i-1}v_{3i} : 1 \leq i \leq \frac{p-1}{3}\}$ is a minimum edge dominating set of P_p. Furthermore, the induced subgraph $\langle E - S\rangle = 2K_2 \bigcup \left\lfloor \frac{p-3}{3} \right\rfloor P_3$ and every matching in $\langle E - S\rangle$ contains at most $\left\lfloor \frac{p-3}{3} \right\rfloor + 2$ edges. Now, since $\left\lfloor \frac{p-3}{3} \right\rfloor + 2 < \left\lceil \frac{p-1}{2} \right\rceil = \beta_1(P_p)$, it follows that $\langle E - S\rangle$ contains no maximum matching of P_p, and then $\gamma'_{mt}(P_p) = \gamma'(P_p) = \left\lceil \frac{p-1}{3} \right\rceil$.

Case 3:
If $p \equiv 2(\bmod 3)$, then the set of edges $S = \{v_{3i+1}v_{3i+2} : 0 \leq i \leq \frac{p-2}{3}\}$ is a minimum edge dominating set of P_p. Also, it is easy to check that the induced subgraph $\langle E - S\rangle = \left\lfloor \frac{p-1}{3} \right\rfloor P_3$ and every matching in $\langle E - S\rangle$ contains at most $\left\lfloor \frac{p-1}{3} \right\rfloor$ edges and $\left\lfloor \frac{p-1}{3} \right\rfloor < \left\lceil \frac{p-1}{2} \right\rceil = \beta_1(P_p)$. So, $\langle E - S\rangle$ contains no maximum matching of P_p. Thus, $\gamma'_{mt}(P_p) = \gamma'(P_p) = \left\lceil \frac{p-1}{3} \right\rceil$.

Theorem 2.

For any cycle C_p of order p, we have $\gamma'_{mt}(C_p)=\begin{cases}3, & \text{p=3,5;}\\ \left\lceil \frac{p}{3}\right\rceil, & \text{otherwise.}\end{cases}$

Proof:

Let $E(C_p)=\{v_1v_2, v_2v_3, ..., v_{p-1}v_p, v_pv_1\}$. It is easy to see that $\gamma'_{mt}(C_3)=\gamma'_{mt}(C_5)=3$. Suppose that $p\notin\{3,5\}$, then we have three cases:

Case 1:
If $p\equiv 0(\text{mod}\,3)$, then the set of edges $S=\{v_{3i+1}v_{3i+2} : 0\le i\le \frac{p-3}{3}\}$ is a minimum edge dominating set of C_p. Furthermore, the induced subgraph $\langle E-S\rangle=\frac{p}{3}P_3$ and every matching in $\langle E-S\rangle$ contains at most $\frac{p}{3}$ edges and $\frac{p}{3}<\left\lceil\frac{p-1}{2}\right\rceil=\beta_1(C_p)$. Hence, $\langle E-S\rangle$ contains no maximum matching of C_p, and then $\gamma'_{mt}(C_p)=\gamma'(C_p)=\left\lceil\frac{p}{3}\right\rceil$.

Case 2:
If $p\equiv 1(\text{mod}\,3)$, then the set of edges $S=\{v_{3i+1}v_{3i+2} : 0\le i\le \frac{p-1}{3}\}$ is a minimum edge dominating set of C_p. Also, it is easy to see that the induced subgraph $\langle E-S\rangle=\frac{p-1}{3}P_3$. Therefore, every matching in $\langle E-S\rangle$ contains at most $\frac{p-1}{3}$ edges and $\frac{p-1}{3}<\beta_1(C_p)$. Hence, $\langle E-S\rangle$ contains no maximum matching of C_p. Thus, $\gamma'_{mt}(C_p)=\gamma'(C_p)=\left\lceil\frac{p}{3}\right\rceil$.

Case 3:
If $p\equiv 2(\text{mod}\,3)$, then the set of edges $S=\{v_{3i+1}v_{3i+2} : 0\le i\le \frac{p-2}{3}\}$ is a minimum edge dominating set of C_p. The induced subgraph $\langle E-S\rangle=K_2\bigcup\frac{p-2}{3}P_3$. Hence, every matching in $\langle E-S\rangle$ contains at most $\frac{p+1}{3}$ edges and $\frac{p+1}{3}<\beta_1(C_p)$. So, $\langle E-S\rangle$ contains no maximum matching of C_p. Then $\gamma'_{mt}(C_p)=\gamma'(C_p)=\left\lceil\frac{p}{3}\right\rceil$.

Proposition 1.

For any graph G, $\gamma'_{mt}(G)\ge\gamma'(G)$.

Observation 1.

For any graph G with at least one edge, $1\le\gamma'_{mt}(G)\le q$.

A bi-star is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendant edges in u and n pendant edges in v and denoted by $B(m,n)$.

The proof of the following proposition is straightforward.

Proposition 2.

For any bi-star graph $B_{m,n}$, $\gamma'_{mt}(B_{m,n}) = \min\{m,n\}+1.$

Remark.

From the definition of $\gamma'(G)$ and $\gamma_{it}(G)$ it is easy to observe that $\gamma'_{mt}(G) = \gamma_{it}(L(G))$, where $L(G)$ is the line graph of G.

Theorem 3.

For any Complete bipartite graph $K_{r,s}$, where $r \leq s$, $\gamma'_{mt}(K_{r,s}) = s.$

Proof:

Let $G = (V_1, V_2, E)$ be any complete bipartite graph $K_{r,s}$, where $|V_1| = r$, $|V_2| = s$. Suppose that $V_1 = \{v_1, v_2, \dots, v_r\}$, $V_2 = \{u_1, u_2, \dots, u_s\}$. Let E_{v_i} , where $i \in \{1, 2, \dots, r\}$ is the set of edges which have v_i as common vertex. It is easy to check that any maximum matching of G contains one edge from each E_{v_i} . Therefore, any E_{v_i} set of edges is the minimum set which intersects each maximum matching of G and also it is an edge dominating set of G . Without loss of generality, let $D = E_{v_1} = \{v_1u_1, v_1u_2, v_1u_3, \dots, v_1u_s\}$, where D is an edge dominating set of G and intersects every maximum matching of G and with minimum size $|D| = s$. Hence $\gamma'_{mt}(K_{r,s}) = s.$

Proposition 3.

For any cycle C_P , $\gamma'_{mt}(C_P \circ K_1) = p$.

Theorem 4.

If $G = (V, E)$ is a graph with at least one isolated edge, then $\gamma'_{mt}(G) = \gamma'(G)$.

Proof:

Let $G = (V, E)$ be a graph with at least one isolated edge say e, and F is a minimum edge dominating set of G. Evidently the edge e must belong to any dominating set and also to any matching of G . Therefore, the minimum dominating set F intersects every maximum matching of G. Hence, F is a minimum matching transversal edge dominating set of G. Thus,

$\gamma'_{mt}(G) = \gamma'(G)$.

Corollary 1.

For any graph $= H \cup K_2$, we have

$$\gamma'_{mt}(G) = \gamma'(G).$$

We can generalize Theorem 3 with the following theorem.

Theorem 3. (generalized)

Let G be a graph with the components $G_1, G_2, \ldots, G_r$. Then,

$$\gamma'_{mt} = min_{1\leq i\leq r}\{\gamma'_{mt}(G_i) + \textstyle\sum_{j=1,j\neq i}^{r} \gamma'(G_j)\} .$$

Proof:

Suppose that without loss of generality

$$\gamma'_{mt}(G_1) + \textstyle\sum_{j=2,\, j\neq i}^{r} \gamma'(G_j) = \min_{1\leq i\leq r}\{\gamma'_{mt}(G_i) + \sum_{j=1,\, j\neq i} \gamma'(G_j)\} .$$

Let S be the minimum matching transversal dominating set of G_1 and let A_j be the minimum edge dominating set of G_j for all $j \geq 2$. It is easy to check that, $S \bigcup (\bigcup_{j=2}^{r} A_j)$ is a matching edge dominating set of G. Hence, $\gamma'_{mt}(G) \leq \gamma'_{mt}(G_1) + \sum_{j=2}^{r} \gamma'(G_j)$.

Conversely, let S be any minimum matching edge dominating set of G . Then, S must intersect all the edge set $E(G_j)$ of each component G_j of G and $S \bigcap E(Gj)$ is edge dominating set of G_j for all $j \geq 1$. Furthermore, for at least one j , the set $S \bigcap E(G_j)$ must be matching edge dominating set of G_j , for otherwise each component G_j will have maximum matching not intersecting the set $S \bigcap E(G_j)$ and hence the union of these maximum matching sets form a maximum matching of G not intersecting S .

Hence, $|S| \geq \min_{1\leq i\leq r}\{\gamma'_{mt}(G_i) + \sum_{j=1,\, j\neq i}^{r} \gamma'(G_j)\}$.

Theorem 4.

For any connected graph G with q edges, $\gamma'_{mt}(G) = q$, if and only if $G \cong K_{1,\, p-1}$ or K_3 .

Proof:

If $G \cong K_{1,p-1}$ or $G \cong K_3$, then it is easy to see that $\gamma'_{mt}(G) = q$.

Conversely, let $\gamma'_{mt}(G) = q$. Then either the minimum edge dominating set of G contains all the edges of G and hence $G = mK_2$ and in this case since G is connected, then $m = 1$ which implies that $G \cong K_{1,1}$, or the maximum matching has cardinality one, which means G is the complete bipartite graph $K_{1,p-1}$ or K_3. Hence, $G \cong K_{1,p-1}$ or $G \cong K_3$.

Theorem 5.

For any connected graph G, $\gamma'_{mt}(G) = 1$, if and only if $G \cong K_2$.

Proof:

If $G = K_2$, then it is clear that $\gamma'_{mt}(G) = 1$. Conversely, if $\gamma'_{mt}(G) = 1$, that means there exists an edge e in G with degree $q-1$ and e must belong to each maximum matching of G. Then $G \cong K_2$.

Theorem 6.

Let G be a graph with q edges. Then $\gamma'_{mt}(G) = q-1$, if and only if $G = P_4$.

Proof:

Let $\gamma'_{mt}(G) = q-1$. Then the matching number is greater than or equal two. If there exist two adjacent edges e and f of degree at least two, then $S = E - \{e, f\}$ is a matching transversal edge dominating set of G and hence $\gamma'_{mt}(G) \leq q-2$ which is a contradiction. Therefore, for any two adjacent edges either e or f pendant edge. Hence $G = P_4$.

Conversely, if $G = P_4$, then it holds that $\gamma'_{mt}(G) = q-1$.

Theorem 7.

Let a and b be any two positive integers with $b \geq 2a-1$. Then there exists a graph G with b edges such that $\gamma'_{mt}(G) = a$.

Proof:

Let $b=2a+r$, $r\geq -1$ and let H be any star graph $K_{1,a}$ with a edges. Suppose $E(H)=\{e_1,e_2,...,e_a\}$. Let G be the graph obtained from H by attaching $r+1$ edges at the pendant vertex of the edge e_1, and one pendant edge at each pendant vertex of the edge e_i for $i\geq 2$. Let f_i , where $i\geq 2$ be the pendant edges in G adjacent to $e_i, i\geq 2$. It is easy to check that $\gamma^{'}(G)=a$ and the set $D=\{e_1, f_2,..., f_a\}$ is the minimum edge dominating set and D intersectd all the maximum matching of G . Hence, $\gamma^{'}_{mt}(G)=a$ and $|E(G)|=b$.

Theorem 8.

Let $G=(V,E)$ be a graph with a unique maximum matching. Then $\gamma^{'}_{mt}(G)\leq \gamma^{'}(G)+1.$

Proof:

Let G be a graph with a unique maximum matching S and let D be a minimum edge dominating set of G . Let e be any edge in S . Then $D\bigcup\{e\}$ is a matching transversal edge dominating set. Hence $\gamma^{'}_{mt}(G)\leq \gamma^{'}(G)+1.$

The converse of Theorem 8, is not true in general, for example for $G=C_4$ we have that $\gamma^{'}_{mt}(G)=\gamma^{'}(G)$, but there are two maximum matching sets.

Theorem 9.

For any graph G, we have $\gamma^{'}_{mt}(G)\leq \gamma^{'}(G)+\delta^{'}(G)$.

Proof:

Let f be an edge with degree $\deg(f)=\delta^{'}(G)$ and let S be a minimum edge dominating set of G. Every maximum matching of G contains an edge of $N[f]$. Hence, $S\bigcup N[f]$ is a matching transversal edge dominating set of G. We have also S intersects $N[f]$, and it follows that $|S\cup N[f]|\leq \gamma^{'}(G)+\delta^{'}(G)$. Hence, $\gamma^{'}_{mt}(G)\leq \gamma^{'}(G)+\delta^{'}(G).$

Let F_1 be the 5-vertex path, F_2 the graph obtained by identifying a vertex of a triangle with one end vertex of the 3-vertex path, F_3 the graph obtained by identifying a vertex of a triangle with a vertex of another triangle. (See Figure 1.)

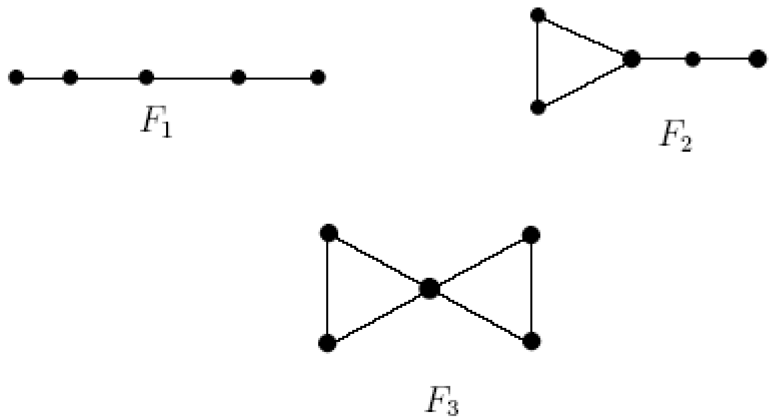

Figure 1: The family F_1, F_2 and F_3

Theorem 10. (Raman et al. (2009))

If $diam(G) \leq 2$ and if none of the three graphs F_1, F_2 and F_3 of Figure l, is an induced subgraph of G, then $diam(L(G)) \leq 2$.

Theorem 11.

Let G be a graph with diameter $diam(G) \leq 2$ and none of the three graphs F_1, F_2 and F_3 in Figure 1 is an induced subgraph of G . Then

$$\gamma_{mt}^{'}(G) \leq \delta^{'}(G) + 1.$$

Proof:

Let f be an edge with $\deg(f) = \delta^{'}(G)$. Since $diam(G) \leq 2$ it follows from Theorem 10 that $diam(L(G)) \leq 2$. Then $N[f]$ is an edge dominating set of G and every maximum matching contains an edge of $N[f]$. Therefore, $N[f]$ is a matching transversal edge dominating set. Hence, $\gamma_{mt}^{'}(G) \leq \delta^{'}(G) + 1$.

Theorem 12.

For any (p,q) graph G , $\gamma_{mt}^{'}(G) = \gamma^{'}(G) = p - q$, if and only if each component of G is isomorphic to K_2 .

Proof:

Let $\gamma_{mt}^{'}(G) = \gamma^{'}(G) = p - q$. Then by (Jayaram, S.R. (1987)), G has $p - q$ components, each of which is isomorphic to a star. Also, since $\gamma'_{mt}(G) = p-q$, then each component contains only one edge. Hence, each component of G is isomorphic to K_2 .

Conversely, if each component of G is isomorphic to K_2 , it is easy to check that $\gamma'_{mt}(G) = \gamma'(G) = p-q$.

3. Conclusion

In this paper, we define a new domination invariant called matching transversal edge domination in graphs which mixes the two important concepts in Graph Theory, domination and matching. Some important properties and exact values for some standard graphs are obtained. Also some bounds for this parameter and some relations with the other domination parameters are established. As this paper is the first to investigate this parameter still there are many things to study in the future, like how to partition the edges into matching transversal edge domination sets with maximum size, how this parameter changes and unchanges by deleting edges, and similar to the standard domination, we can study this new parameter by studying total, connected, independent, split, nonsplit matching transversal edge domination in graphs.

Acknowledgment:

The author is thankful to anonymous referees and the Editor-in-Chief, Professor Aliakbar Montazer Haghighi for useful comments and suggestions towards the improvements of this paper.

REFERENCES

Bondy, J and Murthy U. (1976). Graph Theory with applications, *North Holland, New York.*

Godsil, C and Royle G. (2001). Algebraic graph theory, *Vol. 207 of Graduate Texts in Mathematics, Springer-Verlag, New York.*

Hamid, I.S. (2012). Independent transversal Domination in Graphs, *Graphs,Discussions Mathematicae Graph Theory* **32**, pp. 5-17.

Harary, F. (1969). Graph theory, Reading Mass.

Haynes, T.W, Hedetniemi S.T and Slater, P.J. (1998). Fundamentals of domination in graphs, Marcel Dekker, Inc., New York.

Jayaram, S.R. (1987). Line domination in graphs, *Combin.* **3**, pp. 357-363.

Mitchell, S, Hedetniemi, S.T. (1977). Edge domination in trees, Graph Theory and computing, *Congr. Number,* ***19****, pp. 489-509.*

Raman, H.S, Revankar, D.S, Gutman, I, and Walikar, H.B. (2009). Distance spectra and distance energies of iterated line graphs of regular graphs, *Publ. Inst. Math*. (Beograd),**85**, pp. 39–46.

Sampathkumar, E, and Neeralagi, P.S. (1985). The neighborhood number of a graph, *Indian J.*

Pure and Appl. Math., **16** (2), pp. 126 - 132.

Walikar, H.B, Acharya, B.D, and Sampathkumar, E. (1979). Recent developments in the theory of domination in graphs, *Mehta Research Institute*, *Allahabad*, *MRI Lecture Notes in Math.*,*1*.