Applications of Planar Newtonian Four-body Problem to the Central Configurations

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Abstract

The present study deals with the applications of the planar Newtonian four-body problem to the different central configurations. The basic concept of central configuration is that the vector force must be in the direction of the position vector so that the origin may be taken at the centre of mass of the four bodies and the force towards the position vector multiplied by corresponding inverse mass is directly proportional to the position vector relative to the centre of mass. For applying the Newtonian four body problem to the central configuration, the equations of motion of four bodies have been established in inertial frame. By the methods of previous authors, some mathematical tools of the planar problem have been developed and by using them, the Newtonian four-body problem have been reduced to central configuration. All the previously generated mathematical models depend upon the directed areas, weighted directed areas and position vectors of the centre of the bodies. With the help of these models some conditions have been established, which show that the origin is located at the centre of mass of the system. Finally these conditions have been used to some particular cases of concave configuration, convex configuration and symmetric configuration to generate some tools which will be helpful for further researches of this field.

Keywords: Newtonian four-body problem; Directed area; Weighted directed area; Flat-tetrahedron; Concave central configuration; Convex central configuration; Symmetric central configuration

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1. Introduction

The planar Newtonian four-body problem is to determine the motion of four point masses that interacts each other only by Newton’s law of gravitation. In other words, they behave gravitationally like four particles at a distance apart equal to the distance between their centers. The four masses will be assumed to be sufficiently isolated from each other in the universe so that only force acting; is the inverse square force of their mutual attraction along the line joining their centers. Dziobek (1900) for the first time, discussed central configuration with relative distances as coordinates. Moulton (1910) focused on the straight line solutions in the problem of N bodies. MacMillan and Bartky (1932) studied permanent configurations in the problem of four bodies. Brumberg (1957) discussed permanent configurations in the problem of four bodies and their stability also. Cabral (1973) derived integral manifolds of the N–body problem.


In the present work, we have proposed to study the concave central configuration, convex central configuration and symmetric central configuration in the planar Newtonian four-body problem in the light of Pinna and Lonngi (2010). Also, we have applied planar Newtonian four-body problem to the specific configurations so that some mathematical tools can be generated for future research.

2. Equations of Motion

Let the system consists of four particles of masses \( m_1, m_2, m_3 \) and \( m_4 \) situated at \( P_1, P_2, P_3 \) and \( P_4 \) respectively. Let \( \vec{r}_i \) \((i = 1, 2, 3, 4)\) be the position vectors of the four particles respectively with respect to an inertial frame with \( O \) as the origin. Considering the masses spherically symmetrical with homogenous layers so that they attract one another like point masses. The only forces acting are the mutual Newtonian gravitational attractions between the bodies. The equations of motion of the four particles given by Pina and Lonngi (2010) can be written as

\[
\begin{align*}
\ddot{\vec{r}}_1 &= \frac{Gm_1m_2}{\rho_{12}^3} (\vec{r}_2 - \vec{r}_1) + \frac{Gm_1m_3}{\rho_{13}^3} (\vec{r}_3 - \vec{r}_1) + \frac{Gm_1m_4}{\rho_{14}^3} (\vec{r}_4 - \vec{r}_1), \\
\ddot{\vec{r}}_2 &= \frac{Gm_2m_1}{\rho_{21}^3} (\vec{r}_1 - \vec{r}_2) + \frac{Gm_2m_3}{\rho_{23}^3} (\vec{r}_3 - \vec{r}_2) + \frac{Gm_2m_4}{\rho_{24}^3} (\vec{r}_4 - \vec{r}_2), \\
\ddot{\vec{r}}_3 &= \frac{Gm_3m_1}{\rho_{31}^3} (\vec{r}_1 - \vec{r}_3) + \frac{Gm_3m_2}{\rho_{32}^3} (\vec{r}_2 - \vec{r}_3) + \frac{Gm_3m_4}{\rho_{34}^3} (\vec{r}_4 - \vec{r}_3), \\
\ddot{\vec{r}}_4 &= \frac{Gm_4m_1}{\rho_{41}^3} (\vec{r}_1 - \vec{r}_4) + \frac{Gm_4m_2}{\rho_{42}^3} (\vec{r}_2 - \vec{r}_4) + \frac{Gm_4m_3}{\rho_{43}^3} (\vec{r}_3 - \vec{r}_4),
\end{align*}
\]

where \( \rho_{ij} \) \((i \neq j = 1, 2, 3, 4)\) are the distances between the \( i^{th} \) and \( j^{th} \) particles, \( m_j \) \((j = 1, 2, 3, 4)\) denotes the masses of four particles and \( G \) is the gravitational constant. The right hand sides of the above equations are the gravitational forces which are derived from the potential energy given by

\[
U = \frac{Gm_1m_2}{\rho_{12}} - \frac{Gm_1m_3}{\rho_{13}} - \frac{Gm_1m_4}{\rho_{14}} - \frac{Gm_2m_3}{\rho_{23}} - \frac{Gm_2m_4}{\rho_{24}} - \frac{Gm_3m_4}{\rho_{34}}.
\]

3. The Planar Problem

Following Pina and Lonngi (2010) the directed areas \( (\bar{D}_1, \bar{D}_2, \bar{D}_3, \bar{D}_4) \) representing twice the area of each of the four faces of a tetrahedron with vertices \( \bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4 \) have been derived in detail as follows:

In triangle \( P_4 P_3 P_2 \),

\[
\Delta P_4 P_3 P_2 = \frac{1}{2} \left( P_4 P_2 \times P_3 P_2 \right) = \frac{1}{2} \left( (\bar{\rho}_2 - \bar{\rho}_3) \times (\bar{\rho}_3 - \bar{\rho}_4) \right) 
\]

\[
\Rightarrow 2\Delta P_4 P_3 P_2 = \bar{D}_1 = \bar{\rho}_2 \times \bar{\rho}_3 + \bar{\rho}_3 \times \bar{\rho}_4 + \bar{\rho}_4 \times \bar{\rho}_2 .
\]

Similarly in the triangle \( P_4 P_3 P_4 \)

\[
\bar{D}_2 = \bar{\rho}_1 \times \bar{\rho}_4 + \bar{\rho}_4 \times \bar{\rho}_3 + \bar{\rho}_3 \times \bar{\rho}_1 .
\]

In triangle \( P_2 P_3 P_4 \)

\[
\bar{D}_3 = \bar{\rho}_1 \times \bar{\rho}_2 + \bar{\rho}_2 \times \bar{\rho}_4 + \bar{\rho}_4 \times \bar{\rho}_1 .
\]

and in triangle \( P_1 P_2 P_3 \)

\[
\bar{D}_4 = \bar{\rho}_2 \times \bar{\rho}_3 + \bar{\rho}_3 \times \bar{\rho}_1 + \bar{\rho}_1 \times \bar{\rho}_2 .
\]

Adding the above equations of System (3), we get

\[
\bar{D}_1 + \bar{D}_2 + \bar{D}_3 + \bar{D}_4 = \hat{0} .
\]

i.e., the vectorial sum of the areas of four faces of the tetrahedron is the zero vector.

The concept of particles to be taken at the vertices of flat quadrilateral \( P_1 P_2 P_3 P_4 \) is important but old because the four vector directed areas are all parallel to unit vector \( \hat{\kappa} \), i.e., \( \bar{D}_j = \hat{k} \bar{D}_j \), where \( \hat{D}_j \) is positive or negative according as \( \bar{D}_j \) is parallel towards \( \hat{\kappa} \) or opposite to \( \hat{\kappa} \).

Let us choose the third component of the Cartesian co-ordinates of the four particles as zero and form the matrix of Cartesian co-ordinates as
The four directed areas are written in terms of these co-ordinates as

\[
\begin{align*}
D_1 &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_2 & x_3 & x_4 & 0 \\ y_2 & y_3 & y_4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \\
D_2 &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_4 & x_3 & 0 \\ y_1 & y_3 & y_4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \\
D_3 &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_2 & x_3 & x_4 & 0 \\ x_1 & x_2 & x_4 & 0 \\ y_1 & y_2 & y_4 & 0 \end{vmatrix}, \\
D_4 &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_3 & x_2 & 0 \\ y_1 & y_2 & y_3 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},
\end{align*}
\]

which are the four signed $3 \times 3$ minors forming the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 \\
y_1 & y_2 & y_3 & y_4 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Addition to the previous matrix of a row equal to any of its three rows; produces a square matrix with zero determinant, implying that the necessary and sufficient conditions to have a flat tetrahedron are

\[
D_1 + D_2 + D_3 + D_4 = 0, \\
D_1 x_1 + D_2 x_2 + D_3 x_3 + D_4 x_4 = 0, \\
D_1 y_1 + D_2 y_2 + D_3 y_3 + D_4 y_4 = 0.
\]

The Equations (7) and (8) may be grouped in the zero-vector condition as

\[
D_1 \vec{\rho}_1 + D_2 \vec{\rho}_2 + D_3 \vec{\rho}_3 + D_4 \vec{\rho}_4 = \vec{0}.
\]

From these properties, we deduce a necessary condition for flat solutions in terms of distances as
\[
\rho_2 D_2 = (\bar{\rho}_1 + \bar{\rho}_2 - 2 \bar{\rho}_3) D_2, \\
(a) \Rightarrow \rho_{12} D_2 = (D_2)(\bar{\rho}_1 D_1) + (D_1)(\bar{\rho}_2 D_2) - 2(\bar{\rho}_3 D_2)(\bar{\rho}_3 D_3), \\
\rho_{23} D_2 D_3 = (\rho_3^2 + \rho_3^2 - 2 \rho_4 \rho_3) D_2 D_3, \\
(b) \Rightarrow \rho_{23} D_2 D_3 = (D_2)(\bar{\rho}_2 D_2) + (D_3)(\bar{\rho}_3 D_3) - 2(\bar{\rho}_2 D_2)(\bar{\rho}_3 D_3), \\
\rho_{34} D_3 D_4 = (\rho_3^2 + \rho_3^2 - 2 \bar{\rho}_4 \bar{\rho}_3) D_3 D_4, \\
(c) \Rightarrow \rho_{34} D_3 D_4 = (D_3)(\bar{\rho}_3 D_3) + (D_4)(\bar{\rho}_4 D_4) - 2(\bar{\rho}_3 D_3)(\bar{\rho}_4 D_4), \\
\rho_{24} D_2 D_4 = (\rho_2^2 + \rho_2^2 - 2 \bar{\rho}_4 \bar{\rho}_2) D_2 D_4, \\
(d) \Rightarrow \rho_{24} D_2 D_4 = (D_2)(\bar{\rho}_2 D_2) + (D_4)(\bar{\rho}_4 D_4) - 2(\bar{\rho}_2 D_2)(\bar{\rho}_4 D_4), \\
\rho_{34} D_3 D_4 = (\rho_3^2 + \rho_3^2 - 2 \bar{\rho}_4 \bar{\rho}_3) D_3 D_4, \\
(e) \Rightarrow \rho_{34} D_3 D_4 = (D_3)(\bar{\rho}_3 D_3) + (D_4)(\bar{\rho}_4 D_4) - 2(\bar{\rho}_3 D_3)(\bar{\rho}_4 D_4), \\
(f) \Rightarrow \rho_{34} D_3 D_4 = (D_4)(\bar{\rho}_4 D_4) + (D_4)(\bar{\rho}_4 D_4) - 2(\bar{\rho}_3 D_3)(\bar{\rho}_4 D_4). \\
(10)
\]

Adding all equations of System (10) and using Equations (4) and (9), we get

\[
\sum_{i=1}^{4} \sum_{j=1}^{4} D_{ij} \rho_{ij}^2 = 2 \left( \sum_{i=1}^{4} D_{ii} \right) \left( \sum_{j=1}^{4} D_{ij} \bar{\rho}_j \right) - \left( \sum_{i=1}^{4} D_{ii} \bar{\rho}_i \right) \left( \sum_{j=1}^{4} D_{ij} \bar{\rho}_j \right) = 0. \\
(11)
\]

Here, we observed that Equations (7), (8), (9), (10) and (11) are purely geometrical, independent of the origin of coordinates and independent of the masses. Besides the condition of the sum of directed areas equal to zero. Equations (7) and (9) were considered by Dziobek and are currently connected to affine geometry and the planar four-body problem.

In triangle

\[
P_2 P_3 P_4, \quad D = \frac{\rho_{23} + \rho_{34} + \rho_{24}}{2},
\]

then by Heron’s formula

\[
D_{ij} = \sqrt{D(D - \rho_{23})(D - \rho_{34})(D - \rho_{24})},
\]

By putting the value of \( D \) in the above equation, one can find
\[(4D_1)^2 = (\rho_{23} + \rho_{34} + \rho_{24})(-\rho_{23} + \rho_{34} + \rho_{24}) \times (\rho_{23} - \rho_{34} + \rho_{24})(\rho_{23} + \rho_{34} - \rho_{24})\]
\[\Rightarrow (4D_1)^2 = -(\rho_{23}^4 + \rho_{34}^4 + \rho_{24}^4) + 2(\rho_{23}^2\rho_{34}^2 + \rho_{24}^2\rho_{34}^2 + \rho_{23}^2\rho_{24}^2), \quad (12)\]

Similarly in triangle \(PP_1P_4\),
\[
D = \frac{\rho_{13} + \rho_{34} + \rho_{14}}{2} \quad \text{and} \quad D_2 = \sqrt{D(D - \rho_{13})(D - \rho_{34})(D - \rho_{14})}, \text{ then}
\[(4D_2)^2 = (\rho_{13} + \rho_{34} + \rho_{14})(-\rho_{13} + \rho_{34} + \rho_{14}) \times (\rho_{13} - \rho_{34} + \rho_{14})(\rho_{13} + \rho_{34} - \rho_{14})\]
\[\Rightarrow (4D_2)^2 = -(\rho_{13}^4 + \rho_{34}^4 + \rho_{14}^4) + 2(\rho_{13}^2\rho_{34}^2 + \rho_{14}^2\rho_{34}^2 + \rho_{13}^2\rho_{14}^2). \quad (13)\]

In triangle \(PP_1P_3\) and \(PP_2P_3\) respectively, we have,
\[
(4D_3)^2 = -(\rho_{12}^4 + \rho_{23}^4 + \rho_{14}^4) + 2(\rho_{12}^2\rho_{23}^2 + \rho_{14}^2\rho_{23}^2 + \rho_{12}^2\rho_{14}^2), \quad (14)
\[
(4D_4)^2 = -(\rho_{12}^4 + \rho_{23}^4 + \rho_{14}^4) + 2(\rho_{12}^2\rho_{23}^2 + \rho_{14}^2\rho_{23}^2 + \rho_{12}^2\rho_{14}^2). \quad (15)
\]

The expressions given in Equations (12), (13), (14) and (15) are the required four areas of triangles in the form of square roots of a function of the length of three sides. All the formulas established in Equations (11), (12), (13), (14) and (15) are the tools for further higher research in planar Newtonian four-body problem.

4. Central Configurations

Central configurations are assumed under the condition that the force divided by the corresponding mass is proportional to the position vector from the center of mass (as the origin) i.e., \(\vec{\rho} \propto \vec{\rho} \Rightarrow \vec{\rho} = A\vec{\rho},\) where \(A\) is the proportionality constant to be found.

Thus from Equation (1), we have
\[
A\vec{\rho}_1 = \frac{m_2(\rho_2 - \rho_1)}{\rho_{12}^3} + \frac{m_3(\rho_3 - \rho_1)}{\rho_{13}^3} + \frac{m_4(\rho_4 - \rho_1)}{\rho_{14}^3}, \quad (16)
\]
\[
A\vec{\rho}_2 = -\frac{m_1(\rho_1 - \rho_2)}{\rho_{21}^3} + \frac{m_3(\rho_3 - \rho_2)}{\rho_{23}^3} + \frac{m_4(\rho_4 - \rho_2)}{\rho_{24}^3}, \quad (17)
\]
\[
A\vec{\rho}_3 = -\frac{m_1(\rho_1 - \rho_3)}{\rho_{31}^3} - \frac{m_2(\rho_2 - \rho_3)}{\rho_{32}^3} + \frac{m_4(\rho_4 - \rho_3)}{\rho_{34}^3}, \quad (18)
\]
\[
A\vec{\rho}_4 = -\frac{m_1(\rho_1 - \rho_4)}{\rho_{41}^3} - \frac{m_2(\rho_2 - \rho_4)}{\rho_{42}^3} - \frac{m_3(\rho_3 - \rho_4)}{\rho_{43}^3}. \quad (19)
\]

Taking cross-product to the left of both sides of Equation (16) and (17) by \((\vec{\rho}_1 - \vec{\rho}_2)\) and equating the results, we get
\[ \frac{m_3}{\rho_{13}} (\vec{\rho}_2 \times \vec{\rho}_1 + \vec{\rho}_3 \times \vec{\rho}_1 + \vec{\rho}_3 \times \vec{\rho}_2) - \frac{m_2}{\rho_{14}} (\vec{\rho}_1 \times \vec{\rho}_2 + \vec{\rho}_3 \times \vec{\rho}_1 + \vec{\rho}_3 \times \vec{\rho}_2) = \frac{m_1}{\rho_{23}} (\vec{\rho}_2 \times \vec{\rho}_1 + \vec{\rho}_3 \times \vec{\rho}_2) - \frac{m_4}{\rho_{24}} (\vec{\rho}_1 \times \vec{\rho}_2 + \vec{\rho}_4 \times \vec{\rho}_1 + \vec{\rho}_4 \times \vec{\rho}_2), \]
i.e.,
\[
m_3 (\vec{\rho}_2 \times \vec{\rho}_1 + \vec{\rho}_3 \times \vec{\rho}_1 + \vec{\rho}_3 \times \vec{\rho}_2) \left( \frac{1}{\rho_{23}} - \frac{1}{\rho_{13}} \right) = m_4 (\vec{\rho}_1 \times \vec{\rho}_2 + \vec{\rho}_4 \times \vec{\rho}_1 + \vec{\rho}_4 \times \vec{\rho}_2) \left( \frac{1}{\rho_{24}} - \frac{1}{\rho_{14}} \right). \tag{20} \]

The directed areas \((\vec{D}_1, \vec{D}_2, \vec{D}_3, \vec{D}_4)\) and weighted directed areas \((\vec{W}_1, \vec{W}_2, \vec{W}_3, \vec{W}_4)\) are correlated by the equations
\[
\vec{W}_j = \frac{\vec{D}_j}{m_j} (j = 1, 2, 3, 4), \tag{21} \]
where by Laura and Andoyer (2008), the weighted directed areas are given by
\[
\begin{align*}
\vec{W}_1 &= \frac{1}{m_1} (\vec{\rho}_2 \times \vec{\rho}_3 + \vec{\rho}_3 \times \vec{\rho}_4 + \vec{\rho}_4 \times \vec{\rho}_2), \\
\vec{W}_2 &= \frac{1}{m_2} (\vec{\rho}_1 \times \vec{\rho}_3 + \vec{\rho}_4 \times \vec{\rho}_1 + \vec{\rho}_1 \times \vec{\rho}_3), \\
\vec{W}_3 &= \frac{1}{m_3} (\vec{\rho}_1 \times \vec{\rho}_2 + \vec{\rho}_4 \times \vec{\rho}_1 + \vec{\rho}_4 \times \vec{\rho}_2), \\
\vec{W}_4 &= \frac{1}{m_4} (\vec{\rho}_1 \times \vec{\rho}_3 + \vec{\rho}_4 \times \vec{\rho}_1 + \vec{\rho}_4 \times \vec{\rho}_2). \\
\end{align*} \tag{22} \]
and \(\vec{D}_j\) is the directed area of the triangle with the three particles different from \(m_j\) at its vertices.

By using third and fourth equations of System (22) in (20) and accordingly for other equations also, we get
\[
\begin{align*}
\text{(a)} \quad & \vec{W}_1 \left( \frac{1}{\rho_{31}} - \frac{1}{\rho_{32}} \right) = \vec{W}_4 \left( \frac{1}{\rho_{31}} - \frac{1}{\rho_{32}} \right), \\
\text{(b)} \quad & \vec{W}_2 \left( \frac{1}{\rho_{31}} - \frac{1}{\rho_{32}} \right) = \vec{W}_4 \left( \frac{1}{\rho_{31}} - \frac{1}{\rho_{32}} \right), \\
\text{(c)} \quad & \vec{W}_3 \left( \frac{1}{\rho_{32}} - \frac{1}{\rho_{31}} \right) = \vec{W}_4 \left( \frac{1}{\rho_{32}} - \frac{1}{\rho_{31}} \right), \\
\text{(d)} \quad & \vec{W}_1 \left( \frac{1}{\rho_{32}} - \frac{1}{\rho_{31}} \right) = \vec{W}_4 \left( \frac{1}{\rho_{32}} - \frac{1}{\rho_{31}} \right), \\
\text{(e)} \quad & \vec{W}_2 \left( \frac{1}{\rho_{32}} - \frac{1}{\rho_{31}} \right) = \vec{W}_4 \left( \frac{1}{\rho_{32}} - \frac{1}{\rho_{31}} \right). \tag{23} \\
\end{align*} \]

It is obvious that in the non-planar case, no pair of these four vectors is parallel for arbitrary masses and position and the unique three-dimensional central configuration satisfying the equations of System (23) requires all the six distances to be same, so that the masses lie at the vertices of an equilateral tetrahedron. This configuration has the only solution of straight
paths towards or away from the center of mass. The planar solutions with zero enclosed volume but finite area are obtained by taking into account that the four vectors in Equation (21) are now parallel. They are written in terms of the weighted directed areas \( W_j = W_j \vec{k} \) with

\[
W_j = \frac{D_j}{m_j}.
\]

Suppression of vector \( \vec{k} \) from Equations (23a), (23b), (23c), (23d), (23e) and (23f) leads to the homogeneous system

\[
WX = 0,
\]

where

\[
W = \begin{bmatrix}
0 & W_4 & -W_4 & 0 & W_1 & -W_1 \\
-W_4 & 0 & W_4 & -W_2 & 0 & W_2 \\
W_4 & -W_4 & 0 & W_3 & -W_3 & 0 \\
0 & W_2 & -W_3 & 0 & W_3 & -W_2 \\
-W_1 & 0 & W_3 & -W_3 & 0 & W_1 \\
W_1 & -W_2 & 0 & W_2 & -W_1 & 0
\end{bmatrix}, \quad X = \begin{bmatrix}
1/\rho_{23}^1 \\
1/\rho_{31}^1 \\
1/\rho_{12}^3 \\
1/\rho_{41}^3 \\
1/\rho_{42}^3 \\
1/\rho_{43}^3
\end{bmatrix}.
\]

Since the coefficient matrix of this system is anti-symmetric hence a non-trivial solution exists that is the linear combination of its two eigenvectors with eigenvalue zero, namely \((1 \, 1 \, 1 \, 1 \, 1 \, 1)\) and \((W_2W_3W_4W_1W_2W_3W_4W_2W_3W_1)\) exist.

The first eigenvector is evident to considering the structure of the rows of the matrix and the equilateral solution. The existence of the second eigenvector is secured because the matrix is anti-symmetric of order six. This is the consequence of the fact that a real anti-symmetric matrix becomes Hermitian by multiplying its elements by the imaginary unit \(i\). The eigenvalues of the real anti-symmetric matrix are purely imaginary numbers forming complex conjugate pairs or zero. If the order of the square matrix is an odd number we have at least one zero eigenvalue. If the order of the anti-symmetric matrix is an even number, the zero eigenvalue has even multiplicity. The existence of the second eigenvector for the degenerate zero eigenvalue is a property of Hermitian matrices or anti-symmetric matrices. Also in order to show that there is no other pair of eigenvectors corresponding to the eigenvalue 0. From Equation (24), the characteristic polynomial of the anti-symmetric matrix can be written as
\[ |W - \lambda I| = \begin{vmatrix} -\lambda & W_4 & -W_4 & 0 & W_1 & -W_1 \\ -W_4 & -\lambda & W_4 & -W_2 & 0 & W_2 \\ W_4 & -W_4 & -\lambda & W_3 & -W_3 & 0 \\ 0 & W_2 & -W_3 & -\lambda & W_3 & -W_2 \\ -W_1 & 0 & W_3 & -W_3 & -\lambda & W_1 \\ W_1 & -W_2 & 0 & W_2 & -W_1 & -\lambda \end{vmatrix} \]

\[ = \lambda^6 + 3\lambda^4W_4^2 + 3\lambda^4W_2^2 + 5\lambda^2W_1^2W_4^2 - 2\lambda^2W_1^2W_3^2 - 2\lambda^2W_1^2W_2^2W_4^2 + 3\lambda^4W_2^4 \\
+ 5\lambda^2W_1^2W_3^2 - 2\lambda^2W_1^2W_3^2W_4^2 - 2\lambda^2W_1^2W_4^2 - 2\lambda^2W_1^2W_2^2W_3^2W_4 \\
- 2\lambda^2W_1^2W_2^2W_3^2W_4^2 - 6\lambda W_1^2W_2W_3W_4 - 2\lambda^2W_2^2W_3^2W_4 - 2\lambda^2W_2^2W_3^2W_4 \\
+ 3\lambda^4W_4^2 + 5\lambda^2W_1^2W_3^2 - 2\lambda^2W_1^2W_3^2W_4^2 + 5\lambda^2W_2^2W_3^2 - 2\lambda^2W_1^2W_2^2W_3^2W_4^2 \\
+ 5\lambda^2W_2^2W_3^2W_4^2, \]

\[ |W - \lambda I| = \lambda^2 \left[ \lambda^4 + 3\left(W_1^2 + W_2^2 + W_3^2 + W_4^2\right)\lambda^2 + C \right], \]

where

\[ C = 5W_3^2W_4^2 - 2W_2W_3W_4 \left(W_3 + W_4\right) + W_2^2 \left(5W_3^2 - 2W_1W_4 + 5W_4^2\right) \\
+ W_1 \left\{5W_2^2 + 5W_2^2 - 2W_3W_4 + 5W_4^2 - 2W_2 \left(W_3 + W_4\right)\right\} \\
- 2W_1 \left(W_2^2 \left(W_3 + W_4\right) + W_3W_4 \left(W_3 + W_4\right) + W_2 \left(W_3^2 + 3W_1W_4 + W_4^2\right)\right), \]

\[ = \left(W_2W_3 - W_1W_4\right)^2 + \left(W_1W_2 - W_3W_4\right)^2 + \left(W_1W_3 - W_2W_4\right)^2 + \left(W_1W_4 - W_2W_3\right)^2 \\
+ \left(W_2W_3 - W_1W_4\right)^2 + \left(W_1W_2 - W_3W_4\right)^2 + \left(W_1W_3 - W_2W_4\right)^2 + \left(W_1W_4 - W_2W_3\right)^2 \\
+ \left(W_2W_3 - W_1W_4\right)^2 + \left(W_1W_2 - W_3W_4\right)^2 + \left(W_1W_3 - W_2W_4\right)^2 + \left(W_1W_4 - W_2W_3\right)^2 \geq 0, \]

i.e., \( C \) is positive definite because the Schwarz-inequality holds.

The eigenvector \( X = \left[\rho_{23}^3, \rho_{31}^3, \rho_{12}^3, \rho_{41}^3, \rho_{42}^3, \rho_{43}^3\right]^T \) of \( W \) corresponding to the eigenvalue 0, are given by the non-zero solutions of the Equation (24) as

\[
\begin{bmatrix}
0 & W_4 & -W_4 & 0 & W_1 & -W_1 \\
-W_4 & 0 & W_4 & -W_2 & 0 & W_2 \\
W_4 & -W_4 & 0 & W_3 & -W_3 & 0 \\
0 & W_2 & -W_3 & 0 & W_3 & -W_2 \\
-W_1 & 0 & W_3 & -W_3 & 0 & W_1 \\
W_1 & -W_2 & 0 & W_2 & -W_1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
\rho_{23}^3 \\
\rho_{31}^3 \\
\rho_{12}^3 \\
\rho_{41}^3 \\
\rho_{42}^3 \\
\rho_{43}^3 \\
\end{bmatrix} = 0.
\]
By applying the row transformations to the above matrix equation, one can find

\[
\begin{bmatrix}
1 & 0 & 0 & -\frac{W_1W_3 + W_1W_4}{W_2 - W_3} & \frac{W_1W_3 - W_4W_4}{W_2 - W_3} \\
0 & 1 & 0 & \frac{W_1}{W_4} + M & -\frac{W_1}{W_4} + N \\
0 & 0 & 1 & M & N \\
0 & 0 & 0 & \frac{-W_1 + W_3}{W_2 - W_3} & \frac{W_1 - W_4}{W_2 - W_3} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\rho_{23} \\
\rho_{31} \\
\rho_{32} \\
\rho_{41} \\
\rho_{42} \\
\rho_{43}
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{bmatrix},
\]

(26)

where

\[
M = -\frac{W_1W_2 + W_1W_4}{W_2 - W_3} W_4 \quad \text{and} \quad N = \frac{W_1W_2 - W_2W_4}{W_2 - W_3} W_4.
\]

The coefficient matrix of the Equation (26) is of rank 4, so these equations have 6 - 4 = 2 linearly independent solutions. Thus there is only two linearly independent eigenvector corresponding to the eigenvalue 0. These equations can be written as

\[
\begin{align*}
\rho_{23}^3 + \frac{W_1W_4 - W_2W_1}{W_2 - W_3} W_4 \rho_{42}^3 + \frac{W_2W_3 - W_4W_4}{W_2 - W_3} \rho_{43}^3 &= 0, \\
\rho_{31}^3 + \frac{W_1}{W_4} + \frac{W_1W_4 - W_2W_2}{W_2 - W_3} W_4 \rho_{42}^3 + \left(\frac{W_2W_4 - W_2W_4}{W_2 - W_3} W_4 - \frac{W_1}{W_4}\right) \rho_{43}^3 &= 0, \\
\rho_{32}^3 + \frac{W_4W_4 - W_2W_2}{W_2 - W_3} W_4 \rho_{42}^3 + \frac{W_2W_2 - W_2W_4}{W_2 - W_3} W_4 \rho_{43}^3 &= 0, \\
\rho_{41}^3 + \frac{W_1 - W_1}{W_2 - W_3} \rho_{42}^3 + \frac{W_1 - W_2}{W_2 - W_3} \rho_{43}^3 &= 0.
\end{align*}
\]

(27)

Solving the above equations by taking two linearly independent eigenvectors, one can find the solutions in terms of two parameters \( \lambda \) and \( \mu \) as

\[
\begin{align*}
(a) \quad & \rho_{12}^3 - \mu = \lambda W_1 W_2, \\
(b) \quad & \rho_{23}^3 - \mu = \lambda W_2 W_3, \\
(c) \quad & \rho_{31}^3 - \mu = \lambda W_3 W_1, \\
(d) \quad & \rho_{41}^3 - \mu = \lambda W_4 W_1, \\
(e) \quad & \rho_{42}^3 - \mu = \lambda W_4 W_2, \\
(f) \quad & \rho_{43}^3 - \mu = \lambda W_4 W_3.
\end{align*}
\]

(28)

One way to take into account the zero volume, condition in taking the derivative with respect to \( \rho_{ij}^3 \) of the Cayley-Manager determinant (which is proportional to the square of the volume of the tetrahedron). These derivatives are proportional to the product \( D_i D_j \) if the planar
restrictions are taken into account. An obvious difficulty of that deduction is that the nontrivial entries of the Cayley-Manager determinant are just the squares \( \rho_{ij}^2 \), to obtain the square roots in the expression of the directed areas \( D_j \) as function of \( \rho_{ij}^2 \).

In Equations (12), (13), (14) and (15), one needs to use explicitly the restrictions (7) and (9) connecting these areas. Multiplying both sides of equations in system (28) by \( m_im_2 \), \( m_2m_3, m_3m_1, m_4m_1 \) and \( m_4m_3 \) respectively, one can find

\[
\begin{align*}
\text{(a)} & \quad m_im_2\rho_{12}^3 - m_im_2\mu = \lambda D_iD_2, \\
\text{(b)} & \quad m_2m_3\rho_{23}^3 - m_2m_3\mu = \lambda D_2D_3, \\
\text{(c)} & \quad m_3m_1\rho_{31}^3 - m_3m_1\mu = \lambda D_3D_1, \\
\text{(d)} & \quad m_4m_1\rho_{41}^3 - m_4m_1\mu = \lambda D_4D_1, \\
\text{(e)} & \quad m_2m_4\rho_{24}^3 - m_2m_4\mu = \lambda D_2D_4, \\
\text{(f)} & \quad m_4m_3\rho_{43}^3 - m_4m_3\mu = \lambda D_4D_3.
\end{align*}
\]

The system of Equations (29) can be written in compact form as

\[
m_{ij}\rho_{jk}^3 = m_{ij}\mu + \lambda D_jD_k \quad (i \neq j = 1, 2, 3, 4)
\]

Multiplying both sides of Equations (29) by \( \rho_{12}^2, \rho_{23}^2, \rho_{31}^2, \rho_{14}, \rho_{24}^2 \) and \( \rho_{43}^2 \) respectively, sum over \( j \) and \( k \) on both sides and using the geometric planar constraint (11), one can find

\[
\mu = \frac{\sum_{j<k}(m_jm_k)\rho_{jk}}{\sum_{j<k}m_jm_k\rho_{jk}^2}.
\]

The above expression is positive definite. Either in Equation (28) or (31), it is seen that \( \mu \) has dimensions of the reciprocal of a cubed length.

The combination of Equations (16), (28a), (28c) and (28d) yields

\[
A\tilde{\rho}_1 = \mu \left\{ m_2 \left( \tilde{\rho}_2 - \tilde{\rho}_1 \right) + m_3 \left( \tilde{\rho}_3 - \tilde{\rho}_1 \right) + m_4 \left( \tilde{\rho}_4 - \tilde{\rho}_1 \right) \right\} + \lambda A_1 \left\{ A_2m_2 \left( \tilde{\rho}_2 - \tilde{\rho}_1 \right) + A_3m_3 \left( \tilde{\rho}_3 - \tilde{\rho}_1 \right) + A_4m_4 \left( \tilde{\rho}_4 - \tilde{\rho}_1 \right) \right\},
\]

\[
A\tilde{\rho}_3 = \mu \left( m_2\tilde{\rho}_2 + m_3\tilde{\rho}_3 + m_4\tilde{\rho}_4 \right) - \mu \tilde{\rho}_1 \left( m_2 + m_3 + m_4 \right)
\]

\[
+ \lambda A_1 \left\{ A_2m_2\tilde{\rho}_2 + A_3m_3\tilde{\rho}_3 + A_4m_4\tilde{\rho}_4 \right\} - \lambda A_1\tilde{\rho}_1 \left( A_2m_2 + A_3m_3 + A_4m_4 \right).
\]

Similarly the combination of Equations (17), (28a), (28b) and (28e) gives
\[ A\tilde{\rho}_2 = \mu (m_i\tilde{\rho}_1 + m_i\tilde{\rho}_3 + m_i\tilde{\rho}_4) - \mu\tilde{\rho}_2 (m_1 + m_3 + m_4) \]
\[ + \lambda A_1 (A_1 m_i\tilde{\rho}_1 + A_1 m_i\tilde{\rho}_3 + A_1 m_i\tilde{\rho}_4) - \lambda A_2 \tilde{\rho}_2 (A_3 m_1 + A_3 m_3 + A_3 m_4), \]

the combination of Equations (18), (28b), (28c) and (28f) gives

\[ A\tilde{\rho}_3 = \mu (m_i\tilde{\rho}_1 + m_i\tilde{\rho}_2 + m_i\tilde{\rho}_4) - \mu\tilde{\rho}_3 (m_1 + m_2 + m_4) \]
\[ + \lambda A_1 (A_1 m_i\tilde{\rho}_1 + A_1 m_i\tilde{\rho}_2 + A_1 m_i\tilde{\rho}_4) - \lambda A_3 \tilde{\rho}_3 (A_3 m_1 + A_3 m_2 + A_3 m_4), \]

and the combination of Equations (19), (28d), (28e) and (28f) gives

\[ A\tilde{\rho}_4 = \mu (m_i\tilde{\rho}_1 + m_i\tilde{\rho}_2 + m_i\tilde{\rho}_3) - \mu\tilde{\rho}_4 (m_1 + m_2 + m_3) \]
\[ + \lambda A_4 (A_4 m_i\tilde{\rho}_1 + A_4 m_i\tilde{\rho}_2 + A_4 m_i\tilde{\rho}_3) - \lambda A_4 \tilde{\rho}_4 (A_4 m_1 + A_4 m_2 + A_4 m_3). \]

Adding Equations (32), (33), (34) & (35) and using constraints (7) and (9), one can find

\[ A = -\mu \sum_{i=1}^{4} m_i, \]

which is the required condition that the origin of coordinates is located at the center of mass.

A different proof for \( \lambda \) to be negative was obtained by Albouy et al. (2008). The sign of \( \lambda \) is important because numerical solutions of Equation (24) of the form of Equations (28a), (28b), (28c), (28d), (28e) and (28f) may be found with \( \lambda \) positive that obey restriction (7) but are not in planar central configurations since they do not satisfy Equation (11). Additional relations between distances are obtained from the quotient of Equations (28a), (28b), (28c), (28d), (28e) and (28f) as

\[
\begin{align*}
\text{(a)} \quad & \frac{W_1}{W_2} = \frac{\rho_{31}^3 - \mu}{\rho_{23}^3 - \mu} = \frac{\rho_{31}^3 - \mu}{\rho_{41}^3 - \mu} = \frac{\rho_{31}^3 - \mu}{\rho_{41}^3 - \mu}, \\
\text{(b)} \quad & \frac{W_1}{W_3} = \frac{\rho_{12}^3 - \mu}{\rho_{23}^3 - \mu} = \frac{\rho_{43}^3 - \mu}{\rho_{34}^3 - \mu} = \frac{\rho_{23}^3 - \mu}{\rho_{34}^3 - \mu}, \\
\text{(c)} \quad & \frac{W_1}{W_4} = \frac{\rho_{42}^3 - \mu}{\rho_{31}^3 - \mu} = \frac{\rho_{42}^3 - \mu}{\rho_{31}^3 - \mu} = \frac{\rho_{42}^3 - \mu}{\rho_{31}^3 - \mu}, \\
\text{(d)} \quad & \frac{W_1}{W_5} = \frac{\rho_{23}^3 - \mu}{\rho_{31}^3 - \mu} = \frac{\rho_{34}^3 - \mu}{\rho_{31}^3 - \mu} = \frac{\rho_{23}^3 - \mu}{\rho_{31}^3 - \mu}, \\
\text{(e)} \quad & \frac{W_1}{W_6} = \frac{\rho_{43}^3 - \mu}{\rho_{12}^3 - \mu} = \frac{\rho_{43}^3 - \mu}{\rho_{12}^3 - \mu} = \frac{\rho_{43}^3 - \mu}{\rho_{12}^3 - \mu}, \\
\text{(f)} \quad & \frac{W_1}{W_7} = \frac{\rho_{34}^3 - \mu}{\rho_{12}^3 - \mu} = \frac{\rho_{34}^3 - \mu}{\rho_{12}^3 - \mu} = \frac{\rho_{34}^3 - \mu}{\rho_{12}^3 - \mu},
\end{align*}
\]

which is independent of the parameter \( \lambda \).

The last expression on the right hand side of these equations comes from the previous two, assuming that they are different. The middle terms imply restrictions that may also be obtained from the original equations, namely...
\[
\left\{ \left( \rho_{ab}^3 - \mu \right) \left( \rho_{cd}^3 - \mu \right) \right\} = \left\{ \left( \rho_{ac}^3 - \mu \right) \left( \rho_{bd}^3 - \mu \right) \right\} = \left\{ \left( \rho_{ad}^3 - \mu \right) \left( \rho_{bc}^3 - \mu \right) \right\} = \lambda^2 W_W W_W.
\]

(38)

From these equations one also obtains the parameter \( \mu \) in terms of the distances as
\[
\mu = \frac{\rho_{ab}^3 \rho_{cd}^3 - \rho_{ac}^3 \rho_{bd}^3}{\rho_{ab}^3 \rho_{cd}^3 + \rho_{ac}^3 \rho_{bd}^3} = \frac{\rho_{ad}^3 \rho_{bc}^3 - \rho_{bd}^3 \rho_{ac}^3}{\rho_{ad}^3 \rho_{bc}^3 + \rho_{bd}^3 \rho_{ac}^3} = \frac{\rho_{bc}^3 \rho_{ad}^3 - \rho_{bd}^3 \rho_{ac}^3}{\rho_{bc}^3 \rho_{ad}^3 + \rho_{bd}^3 \rho_{ac}^3}.
\]

(39)

Indeed, this result is an identity substituting the reciprocal cubed distances of Equations (28a), (28b), (28c) and (28d) in the right hand side of this equation. Also in order to show that the central configurations of the four-body problem with equal masses classified according to their dimensions. The dimension of the configuration was deduced by using Cayley-Manager determinants.

For instance, the Cayley-Manager determinant for five-dimensional volume \( V \) is given by
\[
460800V^2 = \begin{vmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & \rho_{12}^2 & \rho_{13}^2 & \rho_{14}^2 & \rho_{15}^2 \\
1 & \rho_{12}^2 & 0 & \rho_{23}^2 & \rho_{24}^2 & \rho_{25}^2 \\
1 & \rho_{13}^2 & \rho_{23}^2 & 0 & \rho_{34}^2 & \rho_{35}^2 \\
1 & \rho_{14}^2 & \rho_{24}^2 & \rho_{34}^2 & 0 & \rho_{45}^2 \\
1 & \rho_{15}^2 & \rho_{25}^2 & \rho_{35}^2 & \rho_{45}^2 & 0
\end{vmatrix}
\]

(40)

Four-dimensional configurations was found by setting the above determinant to zero as
\[
-9216V^2 = \begin{vmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & \rho_{12}^2 & \rho_{13}^2 & \rho_{14}^2 & \rho_{15}^2 \\
1 & \rho_{12}^2 & 0 & \rho_{23}^2 & \rho_{24}^2 & \rho_{25}^2 \\
1 & \rho_{13}^2 & \rho_{23}^2 & 0 & \rho_{34}^2 & \rho_{35}^2 \\
1 & \rho_{14}^2 & \rho_{24}^2 & \rho_{34}^2 & 0 & \rho_{45}^2 \\
1 & \rho_{15}^2 & \rho_{25}^2 & \rho_{35}^2 & \rho_{45}^2 & 0
\end{vmatrix}.
\]

(41)

Three-dimensional configurations was found by setting the above determinant to zero as
\[
288V^2 = \begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & \rho_{12}^2 & \rho_{13}^2 & \rho_{14}^2 \\
1 & \rho_{12}^2 & 0 & \rho_{23}^2 & \rho_{24}^2 \\
1 & \rho_{13}^2 & \rho_{23}^2 & 0 & \rho_{34}^2 \\
1 & \rho_{14}^2 & \rho_{24}^2 & \rho_{34}^2 & 0
\end{vmatrix}.
\]

(42)

The right hand side of the above determinant is the Cayley-Manger determinant for the tetrahedron formed by the masses \( m_1, m_2, m_3 \) and \( m_4 \).

Two-dimensional configurations was found by setting the above determinant to zero as
\[-16\Delta^2 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \rho_{12}^2 & \rho_{13}^2 \\ 1 & \rho_{12}^2 & 0 & \rho_{23}^2 \\ 1 & \rho_{13}^2 & \rho_{23}^2 & 0 \end{vmatrix}. \quad (43)\]

One-dimensional configurations was found by setting the above determinant to zero as

\[2\Delta^2 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & \rho_{12}^2 \\ 1 & \rho_{12}^2 & 0 \end{vmatrix}. \quad (44)\]

Figure 2. Planar central configurations of 4BP with equal masses

5. Applications of Central Configuration to Particular Cases

Let us discuss the Equations of the system (28), for \( \lambda \) negative (Albouy et al. 2008) with the help of central configuration. The concave case occurs when one of the particles is in the convex hull of the other three. In the convex case no particle is in the convex hull of the other three. We consider separately concave and convex cases.

Figure 3. Arrangement of four particles with three equal masses in a central concave configuration

Figure 4. Arrangement of four particles with two equal masses in a central convex configuration
5.1. Concave Configuration with Three Equal Masses

Without loss of generality, we may assume that in the concave case

\[ W_1 = W_2 = W_3 > 0 > W_4. \]  (45)

From the above inequality it is seen that first three Equations (28a), (28b) and (28c) have the positive sign of \( \lambda \) and the last three Equations (28d), (28e) and (28f) have the negative sign of \( \lambda \). By assuming \( \rho_j \) as the largest side of the set \( (\rho_{12}, \rho_{23}, \rho_{31}) \), the equilateral triangle based on \( \rho_j \) overlaps triangle \( P_1 P_2 P_3 \). Therefore \( m_4 \) is in the convex hull of the other three, distances \( \rho_{4i} < \rho_j \) and hence \( \rho_{4i} - \mu > \rho_j^3 - \mu \). Since one of these is positive and the other negative i.e., \( \rho_{4i}^3 - \mu > 0, \rho_j^3 - \mu < 0 \) and hence \( \lambda < 0 \).

5.2. Convex Configuration with Two Equal Masses

Here, the four particles are in the cyclic order \( P_1 P_2 P_3 P_4 \), we have

\[ W_1 = W_3 > 0 > W_4 > W_2. \]  (46)

From the above inequality it is seen that Equations (28c) and (28e) have the positive sign of \( \lambda \) and the other four set of Equations (28a), (28b), (28d) and (28f) have the negative sign of \( \lambda \). Let us assume that \( \rho_j \) is the longest diagonal \( (\rho_{15} \text{ or } \rho_{34}) \) and \( \rho_{ik} \) is the shortest side of the set \( (\rho_{12}, \rho_{23}, \rho_{41}, \rho_{43}) \), then \( \rho_j > \rho_{ik} \). On the contrary, we may assume that the four sides would be longer than \( \rho_j \) and the two triangles with common side \( \rho_j \) could be covered by the quadrangle \( P_1 P_2 P_3 P_4 \) then \( \rho_j \) would not be longest diagonal. Thus \( \rho_j^3 - \mu < \rho_{ik}^3 - \mu \) but they must be of opposite signs i.e., \( \rho_j^3 - \mu < 0, \rho_{ik}^3 - \mu > 0 \) and hence \( \lambda < 0 \).

5.3. Symmetric Configurations

![Figure 5. Kite symmetry](image1)

![Figure 6. Equilateral triangle symmetry](image2)
In this section, we want to discuss some important symmetric cases from Equations (37a), (37b), (37c), (37d), (37e) and (37f) one by one.

(a) Kite Symmetry

The necessary and sufficient condition for kite symmetry is the equality of two weighted area $W_s$ s, i.e., $W_1 = W_3$

$$\Leftrightarrow \rho_{21} = \rho_{23} \text{ and } \rho_{41} = \rho_{43} \text{ and } D_1 = D_3, \; m_1 = m_3. \quad (47)$$

The kite symmetry imposes the above restrictions. The values of the weighted areas $W_2$ and $W_4$ are arbitrary, but at least one of them must have opposite sign to that of $W_1$.

(b) Equilateral Triangle Symmetry

The necessary and sufficient condition for equilateral triangle symmetry is the equality of three weighted areas $W_s$ s, i.e.,

$$W_1 = W_2 = W_3 \Leftrightarrow \rho_{12} = \rho_{23} = \rho_{31} > \rho_{41} = \rho_{42} = \rho_{43}. \quad (48)$$

Since the three particles $P_1, P_2, P_3$ be at vertices of an equilateral triangle and the particle $P_4$ is at the center of this triangle hence by Equations (12), (13), (14) and (15), we get

$$D_1 = D_2 = D_3 = -\frac{D_4}{3}, \; m_1 = m_2 = m_3, \; \rho_{12} = \sqrt{3}\rho_{41}. \quad (49)$$

with $W_4$ as an arbitrary, but satisfying the important condition obtained from the last equation of (37) as

$$\frac{W_4}{W_1} = \frac{3m_1}{m_4}. \quad (50)$$

![Figure 7. Square symmetry configuration](image)

![Figure 8. Rhombus symmetry](image)
(c) Square Symmetry
The necessary and sufficient condition for a square symmetry is the equality of two weighted areas $W$'s i.e.,

$$W_2 = W_4 > 0 > W_1 = W_3 \Rightarrow \rho_{12} = \rho_{23} = \rho_{34} = \rho_{41} \text{ and } D_2 = D_4 = -D_1 = -D_3.$$  \hspace{1cm} (51)

Since the four particles $P_1, P_2, P_3, P_4$ are at vertices of a square hence the diagonal must be equal i.e., $W_1W_3 = W_2W_4$ \hspace{1cm} (52)

(d) Rhombus Symmetry
The necessary and sufficient condition for rhombus symmetry is the equality of two pairs of weighted areas $W$'s

i.e., $W_1 = W_3 > 0 > W_2 = W_4 \Leftrightarrow \rho_{12} = \rho_{23} = \rho_{34} = \rho_{41} \text{ and } D_1 = D_3 = -D_2 = -D_4.$ \hspace{1cm} (53)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{Isosceles trapezium symmetry}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Parallelogram symmetry}
\end{figure}

(e) Isosceles Trapezium Symmetry
The necessary and sufficient condition for isosceles trapezium symmetry is that

$$W_1 = -W_2 \text{ and } W_3 = -W_4 \Rightarrow \rho_{13} = \rho_{24} \text{ and } \rho_{23} = \rho_{14}.$$  \hspace{1cm} (54)

Since from Equation (54) $W_1W_3 = W_2W_4 \text{ and } W_2W_3 = W_1W_4$, hence in such a case of isosceles trapezium symmetry, we also find \begin{align*}
D_1 & = -D_2 \text{ and } D_3 = -D_4 \Rightarrow m_1 = m_2 \text{ and } m_3 = m_4. \hspace{1cm} (55)
\end{align*}

(f) Parallelogram Symmetry
The necessary and sufficient condition for parallelogram symmetry is the equality of two pairs of weighted areas with opposite signs $W$'s i.e.,

$$W_1 = -W_3 \text{ and } W_2 = -W_4 \Leftrightarrow \rho_{12} = \rho_{34} \text{ and } \rho_{23} = \rho_{41}. \hspace{1cm} (56)$$

Since from Equation (56), $W_1W_3 = W_2W_4 \text{ and } W_2W_3 = W_1W_4$, hence in such a case of parallelogram symmetry, we also find
\[ D_1 = -D_2 \text{ and } D_2 = -D_4 \Rightarrow m_1 = m_2 \text{ and } m_2 = m_4 \] (57)

6. Discussions and Conclusion

First, we find out the equations of motion of the four point masses. Next we have studied the planar problem and established some formulas by using Heron’s formula relating the absolute value of the area of a triangle to be square root of a function of the three sides. The sign of the directed areas \( D_j \) is inherited from the sign of the corresponding constraint \( W_j \). Restriction (11) determines the values of \( \lambda \) for a planar solution and from it, the value of the six distances and the four masses are found. This is an implicit way to deduce planar central configurations with four masses. Furthermore we have studied central configurations and find six equations in terms of \( \mu \) and \( \lambda \) to discuss the convex and concave case of the configurations. These equations give all the distances as function of the unknown parameters \( \lambda \) and the four constraint \( W_j \). Lastly, we have studied the symmetry of different configurations. In many non-symmetric cases, restriction (7) is sufficient to define a central configuration and hence the restriction (11) is automatically satisfied but in few cases restriction (11) is not automatically satisfied and allows one to discriminate non-physical situations (such as negative distances or evidently non-planar solutions or geometrically impossible figures). We stress the fact that to determine \( \lambda \), we choose to find the root of (11) as a function of \( \lambda \). In any case considered below the solution is numerically unique. This approach to the four-body central configurations is similar to considering the Euler collinear configurations of three particles, where instead of giving the ratio of the masses and computing the corresponding ratio of two distances with a quantic equation, the ratio of two distances is first given and the ratio of the masses is computed from this approach, based on the weighted areas \( W_j \). The cases of kite symmetry, equilateral triangle symmetry, square symmetry, rhomboidal symmetry and isosceles trapezium symmetry are the applications of this approach, based on the weighted areas \( W_j \).

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