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Two dimensional kinematic surface in Lorentz-Minkowski 5-space with constant scalar curvature

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Abstract

In this paper we analyzed the problem of investigating locally the scalar curvature $\boldsymbol{S}$ of the two dimensional kinematic surfaces foliated by the homothetic motion of an eight curve in Lorentz-Minkowski 5-space $\boldsymbol{L}^5$. We express the scalar curvature $\boldsymbol{S}$ of the corresponding two dimensional kinematic surfaces as the quotient of hyperbolic functions $\{\sinh m\vartheta, \cosh m\vartheta\}$. From that point, we derive the necessary and sufficient conditions that the coefficients of hyperbolic functions vanished identically. Additionally, an example is given to show two dimensional kinematic surfaces with constant scalar curvature.

Keywords: Minkowski space; kinematic surfaces; eight curve; homothetic motion; scalar Curvature; parametric curves

MSC 2010 No.: 53A05, 53A17, 53B30

1. Introduction

The Lemniscate is a figure-eight curve with a simple mechanical construction attributed to Bernoulli, see Gray (1997). Choose two focal points F_1, F_2 at distance L (see Figure 1), then take three rods, one of length L, two of length $L/\sqrt{2}$. The shorter ones can rotate around the focal points and they are connected by the long one with joints which allow rotation. This gives the Cartesian implicit equation:

$$(x^2 + y^2)^2 = x^2 y^2.$$

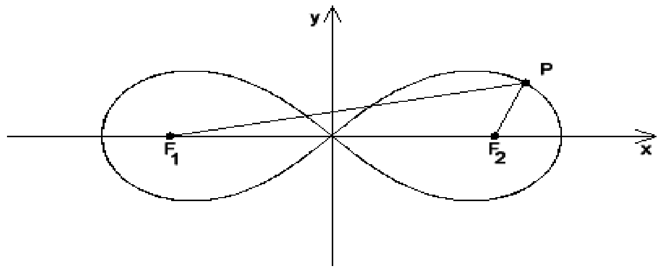

Figure 1: The shape of an eight curve

Eight curve (Gerono lemniscate) is the shape of the movement for many of the micro-organisms and bacteria. When we see some movement of those objects in the microscope we find it moving in the eight curve shape. Also, eight curve shape can be seen in many of the mechanical and dynamic movements and many of the applications in the field of computer aided design and computer graphics. (see Watson and Crick (1953)). From the point of view of differential geometry, eight curve is a geometric curve with non-vanishing constant curvature K, see Barros (1997).

Kinematics is a study of motion apart from the forces producing the motion that is described by position, displacement, rotation, speed, velocity, and acceleration. In kinematics we assume that all the bodies under investigation are rigid bodies; thus, their deformation is negligible and does not play important role, and the only change that is considered in this case is the change in the position Bottema and Roth (1990).

An equiform transformation is an affine transformations whose linear part is composed of an orthogonal transformation and a homothetical transformation. Such an equiform transformation maps points $p \in \mathbb{R}^n$ according to

$$p \mapsto \alpha \, \mathcal{B} \, p + a, \mathcal{B} \in SO(n), \alpha \in \mathbb{R}^+, a \in \mathbb{R}^n. \tag{1}$$

A smooth one-parameter equiform motion moves a point p via $q(t) = \alpha(t)\mathcal{B}(t)p(t) + a(t)$. The kinematics corresponding to this transformation group is called similarity kinematics, see Bottema and Roth (1990), Farin (2002) or Odenhnal (2006). Recently there appeared some articles on differential geometry studying some properties of surfaces obtained by the equiform motions of special curves in Euclidean and Minkowski space-time; see Solouma et al. (2007), Solouma (2012), Solouma (2015) or Solouma and Wageeda (2016) (for a list of references).

In the present paper we shall investigate locally the scalar curvature $\boldsymbol{S}$ of the two dimensional kinematic surfaces foliated by the homothetic motion of an eight curve e_0 in Lorentz-Minkowski space $\boldsymbol{L}^5$ under a one-parameter homothetic motion of moving space Σ^0 with respect to fixed space Σ. Suppose that $e_0 \subset \Sigma^0$ is moved according to homothetic motion. The point paths of an eight curve generate a two dimensional kinematic surfaces Ψ, containing the position of the starting eight curve. At any moment, the infinitesimal transformations of the motion will map the points of eight curve e_0 into the velocity vectors whose end points will form an affine image of e_0 that will be in general an eight curve in the moving space Σ. Both curves are planar and therefore, they span a subspace $V$of $\boldsymbol{L}^n$, with $\dim(V) \leq 5$. This is the reason why we restrict our considerations to dimension $n = 5$.

2. Preliminaries

In this section, we give some definitions and fundamental facts about Minkowski space-time and scalar curvature that will be used throughout the paper.

The Lorentz-Minkowski 5-space $\boldsymbol{L}^5$ is the Euclidean 5-space $\mathbb{R}^5$ provided with the standard flat metric given by

$$\mathcal{G} = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2,$$

where $(x_1, x_2, x_3, x_4, x_5)$ is a rectangular Cartesian coordinate system of $\mathbb{R}^5$. Since $\mathcal{G}$ is an indefinite metric, recall that a nonzero vector $\upsilon \in \boldsymbol{L}^5$ can have one of the three Lorentzian causal characters; it can be spacelike if $\mathcal{G}(\upsilon, \upsilon) > 0$ or $\upsilon = 0$, timelike if $\mathcal{G}(\upsilon, \upsilon) < 0$, and lightlike if $\mathcal{G}(\upsilon, \upsilon) = 0$ and $\upsilon \neq 0$. In particular, the norm (length) of a vector $\upsilon \in \boldsymbol{L}^5$ is given by $\|\upsilon\| = \sqrt{\mathcal{G}(\upsilon, \upsilon)}$ and two vectors u and υ are said to be orthogonal, if $\mathcal{G}(u, \upsilon) = 0$ (for more details see O'Neill (1983) and Weinsten (1995)).

Definition 2.1. Gundogan and Kecilioglu (2006)

Let $\mathcal{A} = (a_{ij}) \in \boldsymbol{L}_n^m$ and $\mathcal{B} = (b_{jk}) \in \boldsymbol{L}_p^n$. Lorentzian matrix multiplication is defined as

$$\mathcal{AB} = \left(-a_{i1}b_{1k} + \sum_{j=1}^{n} a_{ij}b_{jk}\right).$$

Definition 2.2. Gundogan and Kecilioglu (2006)

A matrix $\mathcal{A} \in \boldsymbol{L}_n^n$ is called Lorentzian invertible if there exists an $n \times n$ matrix $\mathcal{B}$ such that $\mathcal{AB} = \mathcal{BA} = I_n$. Then $\mathcal{B}$ is called the Lorentzian inverse of $\mathcal{A}$ and is denoted by $\mathcal{A}^{-1}$.

Definition 2.3. Gundogan and Kecilioglu (2006)

The transpose of a matrix $\mathcal{A} = (a_{ij}) \in \boldsymbol{L}_n^m$ is denoted by $\mathcal{A}^T$ and defined as $\mathcal{A}^T = (a_{ji}) \in \boldsymbol{L}_m^n$.

Definition 2.4. Gundogan and Kecilioglu (2006)

A matrix $\mathcal{A} \in \boldsymbol{L}_n^n$ is called Lorentzian orthogonal matrix if $\mathcal{A}^{-1} = \mathcal{A}^T$. The set of Lorentzian orthogonal matrices is denoted by $O_1(n)$.

Next, recall that an arbitrary curve $\omega = \omega(s)$ in $\boldsymbol{L}^5$ can be locally spacelike, timelike or lightlike if all of its velocity vectors $\omega'(s)$ are spacelike, timelike or lightlike, respectively.

Let M be a smooth surface immersed in $\boldsymbol{L}^5$. We say that M is spacelike, respectively, timelike, if the induced metric on the surface is a positive definite Riemannian metric, respectively, Lorentz metric. Furthermore, the normal vector on the spacelike surface is a timelike vector.

Let $\boldsymbol{X} = \boldsymbol{X}(u,v)$ be a local parametrization of a surface M defined in the (u,v)-domain. The tangent vectors to the parametric curves of M are

$$\boldsymbol{X}_u = \frac{\partial \boldsymbol{X}}{\partial u}\ ,\ \boldsymbol{X}_v = \frac{\partial \boldsymbol{X}}{\partial v}.$$

In each tangent plane, the induced metric $\mathcal{G}$ is determined by the first fundamental form

$$\mathrm{I} = \mathcal{G}(d\boldsymbol{X}, d\boldsymbol{X}) = Edu^2 + 2Fdudv + Gdv^2,$$

with differential coefficients

$$E = \mathcal{G}(\boldsymbol{X}_u, \boldsymbol{X}_u),\ \ F = \mathcal{G}(\boldsymbol{X}_u, \boldsymbol{X}_v),\ \ \ G = \mathcal{G}(\boldsymbol{X}_v, \boldsymbol{X}_v).$$

The Christoffel symbols of the second kind are defined by

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{n=1}^{2} f^{kn}\left(\frac{\partial f_{in}}{\partial u_j} + \frac{\partial f_{jn}}{\partial u_i} - \frac{\partial f_{ij}}{\partial u_n}\right),$$

where $u_i \in \{u, v\}$, $\{i,j,k\}$ are indices that take the value 1 or 2 and (f^{kn}) is the inverse matrix of (f_{ij}). From that point, the scalar curvature of $\boldsymbol{X}(u,v)$ is given by the formula

$$\boldsymbol{S} = \sum_{i,j,k=1}^{2} f^{ij}\left[\frac{\partial \Gamma_{ij}^k}{\partial u_k} - \frac{\partial \Gamma_{ik}^k}{\partial u_j} + \sum_{n=1}^{2}\left(\Gamma_{ij}^k\Gamma_{kn}^n - \Gamma_{ik}^n\Gamma_{jn}^k\right)\right].$$

3. Representation of the motion

In two copies Σ^0, Σ of the Lorentz-Minkowski 5-space $\boldsymbol{L}^5$, we consider a Lorentzian eight curve e_0 in the xy-plane of Σ^0 centered at the origin and described by

$$p(\vartheta) = (\cosh\vartheta, \sinh\vartheta\cosh\vartheta, 0,0,0)^T,\ \ \vartheta \in \mathbb{R}.$$

According to a one-parameter homothetic motion of e_0 in the moving space Σ^0 with respect to fixed space Σ, the position of a point $p(\vartheta) \in \Sigma^0$ at time t can be described in the fixed system as

$$\Psi(t,\vartheta) = \alpha(t)\mathcal{B}(t)p(\vartheta) + a(t),\ \ t \in I \subset \mathbb{R}, \vartheta \in \mathbb{R}, \tag{2}$$

where $a(t) = (a_1(t), a_2(t), a_3(t), a_4(t), a_5(t))^T$ defines the position of the origin of Σ^0 at the time t, $\mathcal{B}(t) = \left(b_{ij}\right)$, $1 \leq i, j \leq 5$, is a semi orthogonal matrix and $\alpha(t)$ provides the scaling factor of the moving system. For varying t and fixed $p(\vartheta)$, $\Psi(t, \vartheta)$ gives a parametric representation of the path (or trajectory) of $p(\vartheta)$. Moreover, we assume that all involved functions are of class C^1. Expanding the two dimensional kinematic surfaces given by Equation. (2) using the Taylor's expansion up to the first order, then we have

$$\Psi(t, \vartheta) = \left[\alpha(0)\mathcal{B}(0) + \left[\dot{\alpha}(0)\mathcal{B}(0) + \alpha(0)\dot{\mathcal{B}}(0)\right]t\right]p(\vartheta) + a(0) + t\dot{a}(0),$$

where $\left(\cdot = \frac{d}{dt}\right)$. As the homothetic motion has an invariant point, we assume that the moving space Σ^0 and the fixed space Σ are coincide at the zero position $t = 0$, this mean that

$$\mathcal{B}(0) = I, \alpha(0) = 1 \text{ and } a(0) = 0.$$

Thus,

$$\Psi(t, \vartheta) = [I + (\dot{\alpha}(0)I + \Theta)t]p(\vartheta) + t\dot{a}(0),$$

where $\Theta = \dot{\mathcal{B}}(0) = (\theta_k)$, $1 \leq k \leq 10$ is a semi skew-symmetric matrix. Throughout this paper all values of α, a_i and their derivatives are computed at $t = 0$ and for simplicity, we write α' and a_i' instead of $\dot{\alpha}(0)$ and $\dot{a}_i(0)$ respectively. In these frames, we can write $\Psi(t, \vartheta)$ in the form

$$\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\\ \Psi_5\end{pmatrix}(t, \vartheta) = \begin{pmatrix} 1+\alpha' t & t\theta_1 & t\theta_2 & t\theta_3 & t\theta_4 \\ t\theta_1 & 1+\alpha' t & t\theta_5 & t\theta_6 & t\theta_7 \\ t\theta_2 & -t\theta_5 & 1+\alpha' t & t\theta_8 & t\theta_9 \\ t\theta_3 & -t\theta_6 & -t\theta_8 & 1+\alpha' t & t\theta_{10} \\ t\theta_4 & -t\theta_7 & -t\theta_9 & -t\theta_{10} & 1+\alpha' t \end{pmatrix} \begin{pmatrix}\cosh\vartheta\\ \sinh\vartheta\cosh\vartheta\\ 0\\ 0\\ 0\end{pmatrix} + t\begin{pmatrix}a_1'\\ a_2'\\ a_3'\\ a_4'\\ a_5'\end{pmatrix},$$

or in the simple form

$$\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\\ \Psi_5\end{pmatrix}(t, \vartheta) = \begin{pmatrix}1+\alpha' t\\ t\theta_1\\ t\theta_2\\ t\theta_3\\ t\theta_4\end{pmatrix}\cosh\vartheta + \begin{pmatrix}t\theta_1\\ 1+\alpha' t\\ -t\theta_5\\ -t\theta_6\\ -t\theta_7\end{pmatrix}\sinh\vartheta\cosh\vartheta + t\begin{pmatrix}a_1'\\ a_2'\\ a_3'\\ a_4'\\ a_5'\end{pmatrix}. \tag{3}$$

For any fixed t in Equation (3), we generally get Lorentzian eight curve centered at the point $t(a_1', a_2', a_3', a_4', a_5')$ subject to the following conditions

$$\begin{gathered}\theta_2\theta_5 + \theta_3\theta_6 + \theta_4\theta_7 = 0,\\ \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 = \theta_1^2 - \theta_5^2 - \theta_6^2 - \theta_7^2 = \ell,\end{gathered} \tag{4}$$

where $\ell \in \mathbb{R}^+$.

4. Scalar curvature of kinematic surfaces

In this section, we compute the scalar curvature $\boldsymbol{S}$ of the two dimensional kinematic surfaces $\Psi(t,\vartheta)$. The proof of our results involves explicit computations of the scalar curvature $\boldsymbol{S}$ of the surface $\Psi(t,\vartheta)$. As we shall see, the equation $\boldsymbol{S} = constant$ reduces to an expression that can be written as a linear combination of the hyperbolic functions $\{\sinh m\vartheta, \cosh m\vartheta\}$, $m \in N$ namely,

$$\sum_{m=0}^{16} (\beta_m(t)\cosh m\vartheta + \gamma_m(t)\sinh m\vartheta) = 0,$$

where β_m and γ_m are two functions depend on the variable t. In particular, the coefficients must vanish.

The work then is to compute explicitly these coefficients β_m and γ_m by successive manipulations. The author was able to obtain the results using the symbolic program Mathematica 9 to check his work. See López (2001) for an example in a similar context.

The tangent vectors to the parametric curves of $\Psi(t,\vartheta)$ are

$$\Psi_t(t,\vartheta) = [\alpha' I + \Theta]p(\vartheta) + a', \quad \Psi_\vartheta(t,\vartheta) = [I + (\alpha' I + \Theta)\mathrm{t}]p'(\vartheta).$$

Under the conditions given in Equation (4), a straightforward computation commanding the coefficients of the first fundamental form are given by

$$E = \zeta_0 + \zeta_3 + \zeta_1\cosh\vartheta + \zeta_2\sinh 2\vartheta + \zeta_3\cosh 2\vartheta - \tfrac{1}{4}\zeta_3\cosh 4\vartheta,$$

$$F = \frac{1}{4}\theta_1(3\cosh\vartheta + \cosh 3\vartheta) + \frac{1}{2}(\zeta_1 t\sinh\vartheta + (2\zeta_3 t - \alpha')\sinh 2\vartheta + (a_2' + \zeta_2 t)\cosh 2\vartheta) + \tfrac{1}{4}(\alpha'(1 + 2\alpha' t) + 2\zeta_3 t), \quad (5)$$

$$G = 1 + 2\alpha' t + t^2(\ell - 2\zeta_3) + \tfrac{1}{2}(1 + 2\alpha' t - \zeta_3 t^2)(\cosh 4\vartheta - \cosh 2\vartheta),$$

and

$$\zeta_0 = \textstyle\sum_{i=2}^{5} {a_i'}^2 - {a_1'}^2 - \tfrac{1}{8}\left({\alpha'}^2 - \ell\right),$$

$$\zeta_1 = 2(-\alpha' a_1' + \theta_1 a_2' + \theta_2 a_3' + \theta_3 a_4' + \theta_4 a_5'), \quad (6)$$

$$\zeta_2 = -2(-\alpha' a_2' + \theta_1 a_1' + \theta_5 a_3' + \theta_6 a_4' + \theta_7 a_5'),$$

$$\zeta_3 = \tfrac{1}{2}\left(-{\alpha'}^2 + \textstyle\sum_{i=1}^{4} {\theta_i}^2\right).$$

The key in our proofs lies that we can write the scalar curvature $\boldsymbol{S}$ in the form

$$\boldsymbol{S}=\frac{N(\cosh m\vartheta,\sinh m\vartheta)}{D(\cosh m\vartheta,\sinh m\vartheta)}=\frac{\sum_{m=0}^{12}(\lambda_m(t)\cosh m\vartheta+\mu_m(t)\sinh m\vartheta)}{\sum_{m=0}^{16}(\xi_m(t)\cosh m\vartheta+\nu_m(t)\sinh m\vartheta)}. \tag{7}$$

The assumption of the constancy of the scalar curvature $\boldsymbol{S}$ indicated that Equation (7) can be converts into

$$\boldsymbol{S}\sum_{m=0}^{16}(\xi_m(t)\cosh m\vartheta+\nu_m(t)\sinh m\vartheta)-\sum_{m=0}^{12}(\lambda_m(t)\cosh m\vartheta+\mu_m(t)\sinh m\vartheta)=0. \tag{8}$$

Equation (8) means that if we write it as a linear combination of the functions $\{\sinh m\vartheta, \cosh m\vartheta\}$ namely, $\sum_{m=0}^{16}(\beta_m(t)\cosh m\vartheta+\gamma_m(t)\sinh m\vartheta)=0$, the corresponding coefficients must vanish. Then, we will delinate all two dimensional kinematic surfaces with constant scalar curvature foliated by homothetic motion of a Lorentzian eight curve e_0.

5. Two dimensional kinematic surfaces with $\boldsymbol{S}=0$

In this section we assume that $\boldsymbol{S}=0$ on the surface $\Psi(t,\vartheta)$. From Equation (7), we have

$$\begin{cases} N(\cosh m\vartheta,\sinh m\vartheta)=\sum_{m=0}^{12}(\lambda_m(t)\cosh m\vartheta+\mu_m(t)\sinh m\vartheta)=0,\\ D(\cosh m\vartheta,\sinh m\vartheta)=\sum_{m=0}^{16}(\xi_m(t)\cosh m\vartheta+\nu_m(t)\sinh m\vartheta)\neq 0.\end{cases} \tag{9}$$

Then, the work consists in the explicit computations of the coefficients λ_m and μ_m. We distinguish all different cases that fill all possible cases (Note that we used the symbolic program Mathematica to have all solutions under the condition $\alpha\neq 0$).

5.1. Case $a_1'=a_2'=0$

By solving the Equation (9), we have $\zeta_1=\zeta_2=0$, $\zeta_3=-\frac{1}{2}\alpha'^2$ and $\theta_1=0$. Then, all coefficients λ_m and μ_m for all $0\leq m\leq 12$ vanish identically. Also, the coefficients ξ_m and $\nu_m\neq 0$ for $0\leq m\leq 16$. For example the coefficient ξ_0 is given by $\xi_0=\frac{189}{512}\alpha'^4\neq 0$. That means the Equation (9) holds (i.e., $D(\cosh m\vartheta,\sinh m\vartheta)=\sum_{m=0}^{16}(\xi_m(t)\cosh m\vartheta+\nu_m(t)\sinh m\vartheta)\neq 0$). From expression (6), we have the following conditions

$$\theta_2a_3'+\theta_3a_4'+\theta_4a_5'=0,$$

$$\theta_5a_3'+\theta_6a_4'+\theta_7a_5'=0,$$

$$\theta_2^2+\theta_3^2+\theta_4^2=0\,.$$

In this case, the Lorentzian eight curve generating the two kinematic surfaces are coaxial.

5.2. Case $a_1'a_2'=0$ but either a_1' or a_2' is not zero

We have two possibilities:

- If $a_1' \neq 0$ and $a_2' = 0$, then we have $\zeta_1 = -2\alpha' a_1'$, $\zeta_2 = 0$, $\zeta_3 = -\frac{1}{2}\alpha'^2$ and $\theta_1 = 0$. Then, the coefficients $\lambda_m = \mu_m = 0$, for $0 \leq m \leq 12$ and at least the coefficient $\nu_4 = \frac{3}{2} {a_1'}^2 \zeta_0 \neq 0$. This implies that the Equation (9) is satisfied and the scalar curvature $\boldsymbol{S} = 0$. Also from expression (6), we have

$$\theta_2 a_3' + \theta_3 a_4' + \theta_4 a_5' = \theta_5 a_3' + \theta_6 a_4' + \theta_7 a_5' = 0,$$

$$\theta_2^2 + \theta_3^2 + \theta_4^2 = 0.$$

- If $a_2' \neq 0$ and $a_1' = 0$, then we have $\zeta_1 = 0$, $\zeta_2 = 2\alpha' a_2'$, $\zeta_3 = -\frac{1}{2}\alpha'^2$ and $\theta_1 = 0$. Then the coefficients $\lambda_m = \mu_m = 0$, for $0 \leq m \leq 12$ and at least the coefficient $\xi_8 = -\frac{3}{32} {a_1'}^2 \zeta_0 \neq 0$. This leads to the Equation (9) is satisfied and from expression (6), have the same result as the previous case.

5.3. Case $a_1' a_2' \neq 0$

At $\zeta_1 = -2\alpha' a_1'$, $\zeta_2 = 2\alpha' a_2'$, $\zeta_3 = -\frac{1}{2}\alpha'^2$ and $\theta_1 = 0$. Then, all coefficients λ_m and μ_m for all $0 \leq m \leq 12$ vanish identically. Additionally, the coefficients ξ_m and $\nu_m \neq 0$ for $0 \leq m \leq 16$. For example $\nu_{12} = \frac{1}{64}\alpha'^2(a_2' - 2a_1') \neq 0$. That means the Equation (9) holds and the scalar curvature $\boldsymbol{S}$ equal zero. From expression (6) we have

$$\theta_2 a_3' + \theta_3 a_4' + \theta_4 a_5' = 0,$$

$$\theta_5 a_3' + \theta_6 a_4' + \theta_7 a_5' = 0,$$

$$\theta_2^2 + \theta_3^2 + \theta_4^2 = 0.$$

As a consequence with the above results in subsections 5.1, 5.2 and 5.3, we conclude the following theorem

Theorem 5.1.

Let $\Psi(t, \vartheta)$be a two dimensional kinematic surfaces in the Lorentz-Minkowski 5-space $\boldsymbol{L}^5$ obtained by the homothetic motion of a Lorentzian eight curve e_0 given by Equation (3) under conditions (4). Assume $a_1', a_2' \geq 0$, then the scalar curvature $\boldsymbol{S}$ vanishes identically on the surface if and only if the following conditions hold:

1. $\theta_i = 0$; $1 \leq i \leq 7$,
2. $a_j' = 0$; $j = 3,4,5$.

Example 5.2.

We assume $\alpha(t) = \sinh \rho t$ such that $\rho \in \mathbb{R} - \{0\}$ and $a(t) = (te^t, 2t, t^2, \cos t\,, 0)^T$. Then $\alpha' = \rho$, $a_1' = 1, a_2' = 2$ and $a_3' = a_4' = a_5' = 0$. Now consider the following orthogonal matrix.

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \sin \rho t \sinh \rho t & 0 & \cos \rho t & -\sin \rho t \cosh \rho t & 0 \\ 0 & 0 & -\sin \rho t & \cos^2 \rho t & \sin \rho t \cos \rho t \\ 0 & \sin^2 \rho t & 0 & -\sin \rho t \cos \rho t & \cos \rho t \end{pmatrix}. \tag{10}$$

Then, we have $\theta_8 = \theta_{10} = \rho, \theta_i = 0, i = 1,2,3,4,5,6,7,9$. Theorem 5.1 says that $\boldsymbol{S} = 0$. In Figure 2, we display a piece of $\Psi(t, \vartheta)$ of Example 5.1 in axonometric viewpoint $\Phi(t, \vartheta)$. For this, the unit vectors $E_4 = (0,0,0,1,0)$ and $E_5 = (0,0,0,0,1)$ are mapped onto the vectors $(1,1,0)$ and $(0, 1, 1)$, respectively see Gordon and Sement Sov (1980). Then,

$$\Psi(t, \vartheta) = \begin{pmatrix} t \\ 2t \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 + \rho t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cosh \vartheta + \begin{pmatrix} 0 \\ 1 + \rho t \\ 0 \\ 0 \\ 0 \end{pmatrix} \sinh \vartheta \cosh \vartheta,$$

$$\Phi(t, \vartheta) = t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 + \rho t \\ 0 \\ 0 \end{pmatrix} \cosh \vartheta + \begin{pmatrix} 0 \\ 1 + \rho t \\ 0 \end{pmatrix} \sinh \vartheta \cosh \vartheta.$$

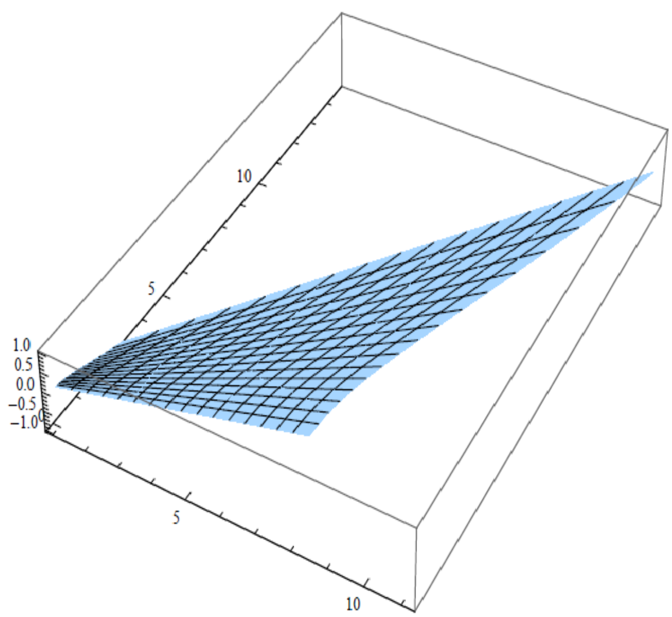

Figure 2. A piece of two dimensional kinematic surfaces in axonometric view $\Phi(t, \vartheta)$ with zero scalar curvature

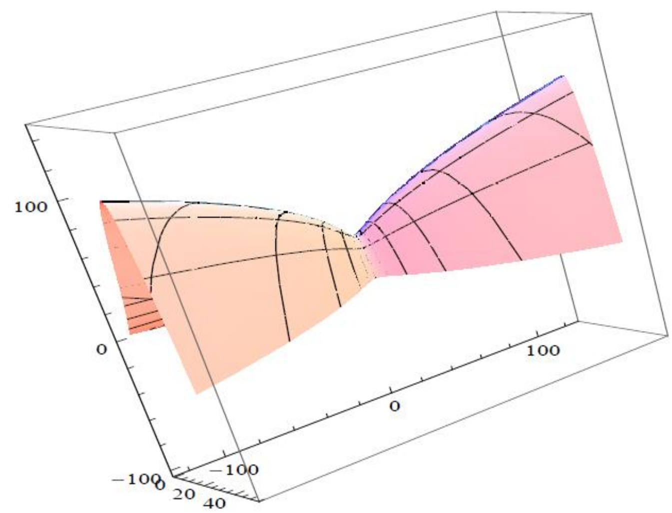

Figure 3. Corresponding two dimensional kinematic surfaces $\Psi(t,\vartheta)$ with Eqn. (2) that approximate

6 . Two dimensional kinematic surfaces with $\boldsymbol{S} \neq 0$

In this section, we assume that the scalar curvature $\boldsymbol{S}$ of the two dimensional kinematic surfaces $\Psi(t,\vartheta)$ defined by Equation (3) does't equal zero and $a_1', a_2' \geq 0$. Equation (8) can be written as

$$\sum_{m=0}^{16}(\beta_m(t)\cosh m\vartheta + \gamma_m(t)\sinh m\vartheta) = 0. \quad (11)$$

Following the same scheme as in the case $\boldsymbol{S} = 0$ studied in Section 5, we begin to compute the coefficients β_m and γ_m. Let us put $t = 0$. The coefficient γ_{14} and β_{13} are

$$\gamma_{14} = \frac{1}{64} a_2' \boldsymbol{S} \alpha'^3,$$

$$\beta_{13} = -\frac{1}{64} \boldsymbol{S} \alpha'^2 (a_1' - a_2' \theta_1).$$

Then, the coefficients $\gamma_{14} = 0$ and $\beta_{13} = 0$ implies that $a_1' = a_2' = 0$. The the coefficient γ_{15} can be written as

$$\gamma_{15} = \frac{1}{256} \boldsymbol{S} \alpha' \theta_1.$$

Then, $\gamma_{15} = 0$, implies that $\theta_1 = 0$, and the coefficients β_2 is given by the formula

$$\beta_2 = -\frac{11}{128} \boldsymbol{S} \alpha'^4.$$

So, the coefficient $\beta_2 = 0$ mean $\alpha' = 0$ or $\boldsymbol{S} = 0,$ which gives a contradiction.

As a conclusion of the above reasoning, we conclude the following theorem.

Theorem 6.1.

There are no two dimensional kinematic surfaces in the Lorentz-Minkowski 5-space $\boldsymbol{L}^5$obtained by the homothetic motion of a Lorentzian eight curve e_0 given by Equation (3) under conditions (4) whose scalar curvature $\boldsymbol{S}$ is a non-zero constant.

Corollary 6.2.

Let $\Psi(t, \vartheta)$be a two dimensional kinematic surfaces in the Lorentz-Minkowski 5-space $\boldsymbol{L}^5$ obtained by the homothetic motion of a Lorentzian eight curve e_0 defined by Equation (3) under conditions (4). If the scalar curvature $\boldsymbol{S}$ is constant, then $\boldsymbol{S} = 0$.

7. Conclusion:

As a conclusion of our results, the two dimensional kinematic surfaces $\Psi(t, \vartheta)$ which is obtained by the homothetic motion of a Lorentzian eight curve e_0 given by Equation (3) have generally zero constant scalar curvature $\boldsymbol{S} = 0$ on the surface in cases such that there is a translation in the plane containing the starting Lorentzian eight curve e_0 or not, as shown by the results in Theorem 5.1. Also, if $\boldsymbol{S}$ is constant, then

$$a_3'^2 + a_4'^2 + a_5'^2 = \theta_2 a_3' + \theta_3 a_4' + \theta_4 a_5' = 0,$$

$$\theta_5 a_3' + \theta_6 a_4' + \theta_7 a_5' = \theta_2^2 + \theta_3^2 + \theta_4^2 = -\theta_5^2 - \theta_6^2 - \theta_7^2 = 0,$$

and the condition $\theta_2\theta_5 + \theta_3\theta_6 + \theta_4\theta_7 = 0$ is now fulfilled everywhere.

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