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Implementation of the matrix differential transform method for obtaining an approximate solution of some nonlinear matrix evolution equations

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Abstract

This article introduces the matrix differential transform method (MDTM) to apply to matrix partial differential equations (MPDEs) and employs it for solving matrix Fisher equations, matrix Burgers equations and matrix KdV equations. We show how the MDTM applies to the linear part and nonlinear part of any MPDE and give various examples of MPDEs to illustrate the efficiency of the method. The results obtained are in excellent agreement with the exact solution and show that the proposed method is powerful, accurate, and easy.

Keywords: Matrix KdV equations; Matrix Burgers equations; Matrix Fisher equations; Matrix differential transform method

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1. Introduction

Most phenomena in the real world are described through nonlinear differential equations and such equations have attracted much attention among scientists (Borhanifar and Abazari (2007), Sweilam et al. (2011)). A large class of nonlinear equations does not have a precise analytic solution, so approximate and numerical methods have largely been used to handle these equations. There are also some analytic techniques for nonlinear equations. Some of the classic analytic methods are the Lyapunov artificial small parameter method, perturbation techniques and δ-expansion method. In the last two decades, some new analytic methods have been proposed to handle functional equations, among them are Adomian decomposition method (ADM), $(\frac{G'}{G})$–expansion method (Borhanifar and Abazari (2011)), tanh method, sinh-cosh method, homotopy analysis method (HAM), variational iteration method (VIM), homotopy perturbation method (HPM) (Rajabi, et al. (2007)), Exp-function method and differential transform method (DTM).

Many problems in the fields of physics, engineering and biology are modeled by matrix differential equations (MDEs) and matrix partial differential equations (MPDEs). The differential transform method (DTM) was introduced by Borhanifar (Borhanifar and Abazari (2010a) and (2010b)) for solving linear and nonlinear problems and Zhou (1986) used it to study approximate solutions of electrical circuits.

In this paper, we extend the DTM to MPDEs, and derive analytic approximations for some important nonlinear matrix equations. The equations under consideration are matrix Fisher equation, matrix Burgers equation and matrix KdV equation. These equations are formulated as follows:

$$\begin{array}{ll} \text{Matrix Fisher Equation:} & u_t = Au_{xx} + u(I-u) + B(x,t), \\ \text{Matrix Burgers Equation:} & u_t + uu_x = Au_{xx} + B(x,t), \\ \text{Matrix KdV Equation:} & u_t = Auu_x + u_{xxx} + B(x,t), \end{array} \tag{1}$$

where $u(x,t) \in R^{n\times n}$, and $A \in R^{n\times n}$ is a constant matrix, $I \in R^{n\times n}$ is the identity matrix, and $B(x,t) \in R^{n\times n}$ is a known function matrix.

Similarities, like the appearance of u_t, u_{xx}, and u_{xxx}, in these equations motivated us to study them as a class of nonlinear MPDEs in one single work. For the physical background of the above equations one can refer to Ozdemir and Kaya (2006) and the references therein.

The layout of the paper is as follows. In Section 2, the matrix differential transform method is presented, and we show how to use this method to approximate a solution. In Section 3, some numerical results are given to clarify the method and a comparison is made with existing results. Section 4 gives a brief conclusion of this paper. Note that we have computed the numerical results by Maple programming.

2. Basic definitions

The DTM is a semi-numerical-analytic-technique that formalizes the Taylor series in a totally different manner. It was first introduced by Zhou in a study about electrical circuits.

In this paper, we extend the DTM to MPDEs. With this technique, the given MPDEs and related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear MPDEs. There is no need for linearization, perturbations or large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear MPDEs with approximations (Abazari and Kilicman (2012), Bildik and Konuralp (2006)). The basic definitions of matrix differential transform are introduced as follows.

2.1. One-dimensional matrix differential transformation

With reference to the articles (Jang et al. (2001), Khader et al. (2013)), we introduce the basic definitions of the one-dimensional matrix differential transform.

Definition 1.

If $u(t) \in \mathbb{R}^{n\times n}$ is a matrix analytical function in the domain T, then it can be differentiated continuously with respect to time t,

$$\frac{d^k u(t)}{dt^k} = \phi(t,k), \quad \forall\, t \in T, \tag{2}$$

for $t = t_i$, where $\phi(t,k) = \phi(t_i,k)$, where k belongs to the set of non-negative integers, denoted as the K domain. Therefore, Equation (2) can be written as

$$U_i(k) = \phi(t_i,k) = \left[\frac{d^k u(t)}{dt^k}\right]_{t=t_i}, \quad \forall\, k \in K, \tag{3}$$

where $U_i(k) \in \mathbb{R}^{n\times n}$ is called the spectrum of $u(t)$ at $t = t_i$, in the domain K.

Definition 2.

If $u(t) \in \mathbb{R}^{n\times n}$ can be expressed by a Taylor series about a fixed point t_i, then $u(t)$ can be represented as

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t_i)}{k!}(t-t_i)^k. \tag{4}$$

If $u_n(t)$ is the n-partial sum of a Taylor series (4), then

$$u_n(t) = \sum_{k=0}^{n} \frac{u^{(k)}(t_i)}{k!}(t-t_i)^k + R_n(t), \tag{5}$$

where $u_n(t)$ is called the nth Taylor polynomial for $u(t)$ about t_i and $R_n(t)$ is the remainder term. If $U(k)$ is defined as

$$U(k) = \frac{u^{(k)}(t_i)}{k!}, \qquad \text{where} \quad k = 0, 1, ..., \tag{6}$$

then Equation (4) reduces to

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_i)^k, \tag{7}$$

Table I: The fundamental operations of one-dimensional matrix differential transform method

Original matrix function	Matrix transformed function
$w(x) = u(x) \pm v(x)$	$W(k) = U(k) \pm V(k)$
$w(x) = c\,u(x)$	$W(k) = c\,U(k)$
$w(x) = \frac{d}{dx}u(x)$	$W(k) = (k+1)U(k+1)$
$w(x) = \frac{d^2}{dx^2}u(x)$	$W(k) = (k+1)(k+2)U(k+2)$
$w(x) = \frac{d^m}{dx^m}u(x)$	$W(k) = (k+1)...(k+m)U(k+m)$
$w(x) = u(x)v(x)$	$W(k) = U(k) \otimes V(k) = \sum_{l=0}^{k} U(l)V(l-k)$

and the n-partial sum of the Taylor series (5) reduces to

$$u_n(t) = \sum_{k=0}^{n} U(k)(t-t_i)^k + R_n(t). \tag{8}$$

The $U(k)$ defined in Equation (6) is called the matrix differential transform of matrix function $u(t)$. For simplicity assume that $t_0 = 0$; then the solution (7) reduces to

$$u(t) = \sum_{k=0}^{n} t^k U(k) + R_{n+1}(t). \tag{9}$$

From the above definitions, the concept of the one-dimensional matrix differential transform is derived from the Taylor series expansion. With Equations (6) and (7), the fundamental mathematical operations performed by the one-dimensional matrix differential transform can readily be obtained and are listed in Table I.

2.2. Two-dimensional matrix differential transformation

Consider a matrix function of two variables $w(x,t) \in \mathbb{R}^{n\times n}$. Based on the properties of the one-dimensional matrix differential transform, function $w(x,t)$ can be represented as

$$w(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i,j)x^i t^j, \tag{10}$$

where $W(i,j) \in \mathbb{R}^{n\times n}$ is called the spectrum of $w(x,t)$.

The basic definitions and operations of the two-dimensional matrix differential transform are introduced as follows.

Definition 3.

If $w(x,t) \in \mathbb{R}^{n\times n}$ is an analytical matrix function and differentiated continuously with respect to time t and x in the domain of interest, then

$$W(k,h) = \frac{1}{k!h!}\Big[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x,t)\Big]_{\substack{x=x_0\\ t=t_0}}, \tag{11}$$

where the spectrum function $W(k,h)$ is the transformed function, which is also called the T-function in brief.

In this paper, the lowercase $w(x,t)$ represents the original function while the uppercase $W(k,h)$ represents the transformed function (T-function).

Table II: The fundamental operations of two-dimensional differential transform method

Original matrix function	Matrix transformed function
$w(x,t)=u(x,t)\pm v(x,t)$	$W(k,h)=U(k,h)\pm V(k,h)$
$w(x,t)=c\,u(x,t)$	$W(k,h)=c\,U(k,h)$
$w(x,t)=\frac{\partial}{\partial x}u(x,t)$	$W(k,h)=(k+1)U(k+1,h)$
$w(x,t)=\frac{\partial}{\partial t}u(x,t)$	$W(k,h)=(h+1)U(k,h+1)$
$w(x,t)=\frac{\partial^{r+s}}{\partial x^r\partial t^s}u(x,t)$	$W(k,h)=(k+1)...(k+s)(h+1)...(h+s)U(k+r,h+s)$
$w(x,t)=u(x,t)v(x,t)$	$W(k,h)=U(k,h)\otimes V(k,h)$ $=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$

The differential inverse transform of $W(k,h)$ is defined as

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)(x-x_0)^k(t-t_0)^h. \tag{12}$$

Combining Equations (11) and (12), and assuming $x_0=t_0=0$, we obtain

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\Big[\frac{\partial^{k+h}}{\partial x^k\partial t^h}w(x,t)\Big]_{\substack{x=0\\t=0}}x^k\,t^h=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k\,t^h. \tag{13}$$

From the above definitions, the concept of the two-dimensional differential transform is derived from the two-dimensional Taylor series expansion. With Equations (11) and (12), the fundamental mathematical operations performed by two-dimensional differential transform can readily be obtained and are listed in Table II.

3. Applications and numerical results

This section is devoted to computational results. We apply the proposed method and solve some examples. These examples are chosen such that exact solutions exist.

3.1. Matrix Fisher equation

The matrix Fisher equation is of the form

$$u_t=Au_{xx}+u(I-u)+B(x,t), \tag{14}$$

where $u(x,t)\in R^{n\times n}$ and $A\in R^{n\times n}$ is a constant matrix, $I\in R^{n\times n}$ is the identity matrix and $B(x,t)\in R^{n\times n}$ is a known function matrix.

Using the matrix differential transformation method on matrix Fisher equation (14), for $k,h=0,1,...,$ we obtain

$$(h+1)U(k,h+1)=(k+1)(k+2)AU(k+2,h)+U(k,h)-U(k,h)\otimes U(k,h)+\mathbf{B}(k,h), \tag{15}$$

where $U(k,h)$ and $\mathbf{B}(k,h)$ are the matrix differential transform of $u(x,t)$ and $B(x,t)$, respectively. Note that the matrices $\mathbf{B}(k,h)$ are the matrix coefficients of x^kt^h in Taylor's expand of $B(x,t)$.

From (15), and the matrix differential transform operators of Table II, we obtain

$$U(k,h+1)=\frac{1}{(h+1)}\Big\{(k+1)(k+2)AU(k+2,h)+U(k,h)-\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)U(k-r,s)+\mathbf{B}(k,h)\Big\}. \tag{16}$$

Example 1.

Consider Equation (14) with

$$A=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix},\ B(x,t)=\begin{bmatrix}\cos^2(x)e^{(2t)} & 1-\cos(x)e^t\\ -\cos(x)e^t & e^{2(x+t)}\end{bmatrix},$$

subject to initial condition

$$u(x,0)=\begin{bmatrix}\cos(x) & -1\\ 0 & e^x\end{bmatrix}. \tag{17}$$

From initial condition (17), we obtain

$$u(x,0)=\sum_{k=0}^{\infty}U(k,0)x^k=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}x+\begin{bmatrix}-\frac{1}{2!} & 0\\ 0 & \frac{1}{2!}\end{bmatrix}x^2+\begin{bmatrix}0 & 0\\ 0 & \frac{1}{3!}\end{bmatrix}x^3+\begin{bmatrix}\frac{1}{4!} & 0\\ 0 & \frac{1}{4!}\end{bmatrix}x^4+..., \tag{18}$$

and from functions matrix $B(x,t)$, we obtain

$$\begin{aligned}B(x,t)=\sum_{k=0}^{\infty}\mathbf{B}(k,h)x^kt^h=&\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}x+\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}t+\begin{bmatrix}-1 & \frac{1}{2}\\ \frac{1}{2} & 2\end{bmatrix}x^2+\begin{bmatrix}0 & 0\\ 0 & 4\end{bmatrix}xt\\ &+\begin{bmatrix}2 & -\frac{1}{2}\\ -\frac{1}{2} & 2\end{bmatrix}t^2+\begin{bmatrix}0 & 0\\ 0 & \frac{4}{3}\end{bmatrix}x^3+\begin{bmatrix}-2 & \frac{1}{2}\\ \frac{1}{2} & 4\end{bmatrix}x^2t+\begin{bmatrix}0 & 0\\ 0 & 4\end{bmatrix}xt^2+\begin{bmatrix}\frac{4}{3} & -\frac{1}{6}\\ -\frac{1}{6} & \frac{4}{3}\end{bmatrix}t^3+...,\end{aligned} \tag{19}$$

From Equation (16), for $k,h=0,1,...$ we obtain

$$\begin{aligned}
&U(0,1)=2AU(2,0)+U(0,0)-U(0,0)^2+\mathbf{B}(0,0)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\\
&U(1,1)=6AU(3,0)+U(1,0)-2U(0,0)U(1,0)+\mathbf{B}(1,0)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\\
&U(2,1)=12AU(4,0)+U(2,0)-2U(0,0)U(2,0)-U(1,0)^2+\mathbf{B}(2,0)=\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix},\\
&U(3,1)=20AU(5,0)+U(3,0)-2U(0,0)U(3,0)-2U(1,0)U(2,0)+\mathbf{B}(3,0)=\begin{bmatrix}0 & 0\\ 0 & \frac{1}{6}\end{bmatrix},\\
&\quad\vdots\\
&U(0,2)=AU(2,1)+\tfrac{1}{2}U(0,1)-U(0,1)U(0,0)+\tfrac{1}{2}\mathbf{B}(0,1)=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix},\\
&U(1,2)=3AU(3,1)+\tfrac{1}{2}U(1,1)-U(0,1)U(1,0)-U(0,0)U(1,1)+\tfrac{1}{2}\mathbf{B}(1,1)=\begin{bmatrix}0 & 0\\ 0 & \frac{1}{2}\end{bmatrix},\\
&U(2,2)=6UA(4,1)+\tfrac{1}{2}U(2,1)-U(0,1)U(2,0)-U(0,0)U(2,1)-U(1,1)U(1,0)+\tfrac{1}{2}\mathbf{B}(2,1)\\
&\qquad=\begin{bmatrix}-\frac{1}{4} & 0\\ 0 & \frac{1}{4}\end{bmatrix},\\
&U(3,2)=10AU(5,1)+\tfrac{1}{2}U(3,1)-U(0,1)U(3,0)-U(0,0)U(3,1)-U(1,1)U(2,0)-U(1,0)U(2,1)\\
&\qquad+\tfrac{1}{2}\mathbf{B}(3,1)=\begin{bmatrix}0 & 0\\ 0 & \frac{1}{12}\end{bmatrix},\\
&\quad\vdots
\end{aligned} \tag{20}$$

In the same manner, the rest of the components can be obtained using Maple.

Substituting the quantities listed in (20) into Equation (19), when $x_0 = t_0 = 0$, the matrix approximation solution in series form is

$$u(x,t)=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}x+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}t+\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}x^2 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}xt+\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}t^2+\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{3!} \end{bmatrix}x^3+\ldots. \quad (21)$$

The closed form of the above solution is

$$u(x,t) = \begin{bmatrix} \cos(x)e^t & -1 \\ 0 & e^{(x+t)} \end{bmatrix},$$

which is exactly the same as the exact solution.

3.2. Matrix Burgers equation

The matrix Burgers equation is **of** the form

$$u_t + uu_x = Au_{xx} + B(x,t), \quad (22)$$

where $u(x,t) \in R^{n\times n}$, $A \in R^{n\times n}$ is a constant matrix, and $B(x,t) \in R^{n\times n}$ is a known function matrix.

Using the matrix differential transformation method on matrix Burgers equation (22), for $k, h = 0, 1, ...$, we obtain

$$(h+1)U(k,h+1) + U \otimes U_x|_{\substack{x=k \\ t=h}} = (k+1)(k+2)AU(k+2,h) + \mathbf{B}(k,h), \quad (23)$$

where $U(k,h)$, and $\mathbf{B}(k,h)$ are the matrix differential transform of $u(x,t)$ and $B(x,t)$, respectively.

Note that the matrices $\mathbf{B}(k,h)$ are the matrix coefficients of $x^k t^h$ in Taylor's expansion of $B(x,t)$. From (23), and matrix differential transform operators of Table II, we obtain

$$U(k,h+1) = \frac{1}{(h+1)}\Big\{ -\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s) + (k+1)(k+2)AU(k+2,h) + \mathbf{B}(k,h)\Big\}. \quad (24)$$

Example 2.

Consider Equation (22) with

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \qquad B(x,t) = \begin{bmatrix} \sin(x)\cos(x)e^{(2t)} & 1-e^{-t} \\ \sin(x)e^t & xe^t(1+e^t) \end{bmatrix},$$

subject to initial condition

$$u(x,0) = \begin{bmatrix} \sin(x) & 1 \\ 0 & x \end{bmatrix}. \quad (25)$$

From initial condition (25), we obtain

$$u(x,0)=\sum_{k=0}^{\infty}U(k,0)x^k=\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}x+\begin{bmatrix}0&0\\0&0\end{bmatrix}x^2+\begin{bmatrix}-\frac{1}{3!}&0\\0&0\end{bmatrix}x^3+\begin{bmatrix}0&0\\0&0\end{bmatrix}x^4+..., \tag{26}$$

and from functions matrix $B(x,t)$, we obtain

$$\begin{aligned}B(x,t)=\sum_{k=0}^{\infty}\mathbf{B}(k,h)x^kt^h=&\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\1&2\end{bmatrix}x+\begin{bmatrix}0&1\\0&0\end{bmatrix}t+\begin{bmatrix}0&0\\0&0\end{bmatrix}x^2+\begin{bmatrix}2&0\\1&3\end{bmatrix}xt\\&+\begin{bmatrix}0&-\frac{1}{2}\\0&0\end{bmatrix}t^2+\begin{bmatrix}-\frac{2}{3}&0\\-\frac{1}{6}&0\end{bmatrix}x^3+\begin{bmatrix}0&0\\0&0\end{bmatrix}x^2t+\begin{bmatrix}2&0\\\frac{1}{2}&\frac{5}{2}\end{bmatrix}xt^2+\dots.\end{aligned} \tag{27}$$

From Equation (24), for $k,h=0,1,...$ we obtain

$$\begin{aligned}
&U(0,1)=\mathbf{B}(0,0)+2AU(2,0)-U(0,0)U(1,0)=\begin{bmatrix}0&-1\\0&0\end{bmatrix},\\
&U(1,1)=\mathbf{B}(1,0)+6AU(3,0)-2U(0,0)U(2,0)-U(1,0)^2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\\
&U(2,1)=\mathbf{B}(2,0)+12AU(4,0)-3U(0,0)U(3,0)-3U(1,0)U(2,0)=\begin{bmatrix}0&0\\0&0\end{bmatrix},\\
&U(3,1)=\mathbf{B}(3,0)+20AU(5,0)-4U(0,0)U(4,0)-4U(1,0)U(3,0)-2U(2,0)^2=\begin{bmatrix}-\frac{1}{6}&0\\0&0\end{bmatrix},\\
&\quad\vdots\\
&U(0,2)=\tfrac{1}{2}\mathbf{B}(0,1)+AU(2,1)-\tfrac{1}{2}U(0,1)U(1,0)-\tfrac{1}{2}U(0,0)U(1,1)=\begin{bmatrix}0&\frac{1}{2}\\0&0\end{bmatrix},\\
&U(1,2)=\tfrac{1}{2}\mathbf{B}(1,1)+3AU(3,1)-U(0,1)U(2,0)-U(0,0)U(2,1)-U(1,1)U(1,0)=\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix},\\
&U(2,2)=\tfrac{1}{2}\mathbf{B}(2,1)+6AU(4,1)-\tfrac{3}{2}U(0,1)U(3,0)-\tfrac{3}{2}U(0,0)U(3,1)-\tfrac{3}{2}U(1,1)U(2,0)-\tfrac{3}{2}U(1,0)U(2,1)\\
&\qquad=\begin{bmatrix}0&0\\0&0\end{bmatrix},\\
&U(3,2)=\tfrac{1}{2}\mathbf{B}(3,1)+10AU(5,1)-2U(0,1)U(4,0)-2U(0,0)U(4,1)-2U(1,1)U(3,0)-2U(1,0)U(3,1)\\
&\qquad-2U(2,1)U(2,0)=\begin{bmatrix}-\frac{1}{12}&0\\0&0\end{bmatrix},\\
&\quad\vdots
\end{aligned} \tag{28}$$

In the same manner, the rest of the components can be obtained using Maple.

Substituting the quantities listed in (28) into Equation (27), when $x_0 = t_0 = 0$, the matrix approximation solution in series form is

$$u(x,t)=\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\1&2\end{bmatrix}x+\begin{bmatrix}0&-1\\0&0\end{bmatrix}t+\begin{bmatrix}0&0\\0&0\end{bmatrix}x^2+\begin{bmatrix}1&0\\0&1\end{bmatrix}xt+\begin{bmatrix}0&\frac{1}{2}\\0&0\end{bmatrix}t^2+\begin{bmatrix}-\frac{1}{6}&0\\0&0\end{bmatrix}x^3+\dots. \tag{29}$$

The closed form of the above solution is

$$u(x,t)=\begin{bmatrix}\sin(x)e^t & e^{-t}\\ 0 & xe^t\end{bmatrix},$$

which is exactly the same as the exact solution.

3.3. Matrix KdV equation

The matrix KdV equation is of the form (Athorne and Fordy (1987), Özer (1998))

$$u_t = Auu_x + u_{xxx} + B(x,t), \tag{30}$$

where $u(x,t) \in R^{n\times n}$, and $A \in R^{n\times n}$ is a constant matrix and $B(x,t) \in R^{n\times n}$ is known function matrix.

Similar to the previous example using the DTM on matrix KdV equation (30), for $k, h = 0, 1, ...,$ we obtain

$$(h+1)U(k,h+1) = AU \otimes U_x|_{\substack{x=k\\t=h}} + (k+1)(k+2)(k+3)U(k+3,h) + \mathbf{B}(k,h), \tag{31}$$

where $U(k,h)$, and $\mathbf{B}(k,h)$ are the matrix differential transforms of $u(x,t)$ and $B(x,t)$, respectively, and the matrices $\mathbf{B}(k,h)$ are the matrix coefficients of $x^k t^h$ in a Taylor expansion of $B(x,t)$.

From (31), and matrix differential transform operators of Table II, we obtain

$$\begin{aligned} U(k,h+1) = \frac{1}{(h+1)}\Big\{ & A\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s) \\ & +(k+1)(k+2)(k+3)U(k+3,h) + \mathbf{B}(k,h)\Big\}. \end{aligned} \tag{32}$$

Example 3. Consider Equation (30) with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \; B(x,t) = \begin{bmatrix} -e^{2(x+t)} & e^{2(x-t)} - e^{(x-t)} \\ -e^{2(x+t)} & -3e^{(x-t)} \end{bmatrix},$$

subject to initial condition

$$u(x,0) = \begin{bmatrix} e^x & 1 \\ 0 & e^x \end{bmatrix}. \tag{33}$$

From initial condition (33), we obtain

$$u(x,0) = \sum_{k=0}^{\infty} U(k,0)x^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}x + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}x^2 + \begin{bmatrix} \frac{1}{3!} & 0 \\ 0 & \frac{1}{3!} \end{bmatrix}x^3 + \begin{bmatrix} \frac{1}{4!} & 0 \\ 0 & \frac{1}{4!} \end{bmatrix}x^4 + ..., \tag{34}$$

and from functions matrix $B(x,t)$, we obtain

$$\begin{aligned} B(x,t) = \sum_{k=0}^{\infty}\mathbf{B}(k,h)x^k t^h = & \begin{bmatrix} -1 & 0 \\ -1 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -2 & -3 \end{bmatrix}x + \begin{bmatrix} -2 & -2 \\ 0 & -2 \end{bmatrix}t + \begin{bmatrix} -2 & \frac{3}{2} \\ -2 & -\frac{3}{2} \end{bmatrix}x^2 + \begin{bmatrix} -4 & -2 \\ 0 & -4 \end{bmatrix}xt \\ & + \begin{bmatrix} -2 & \frac{3}{2} \\ -2 & -\frac{3}{2} \end{bmatrix}t^2 + \begin{bmatrix} -\frac{4}{3} & \frac{7}{6} \\ -\frac{4}{3} & -\frac{1}{2} \end{bmatrix}x^3 + \begin{bmatrix} -4 & -\frac{7}{2} \\ -4 & \frac{3}{2} \end{bmatrix}x^2 t + \begin{bmatrix} -4 & \frac{7}{2} \\ -4 & -\frac{3}{2} \end{bmatrix}xt^2 + \dots. \end{aligned} \tag{35}$$

From Equation (32), for $k, h = 0, 1, ...$ we obtain

$$\begin{aligned}
U(0,1) &= B(0,0) + 6U(3,0) + AU(0,0)U(1,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\\
U(1,1) &= B(1,0) + 24U(4,0) + 2AU(0,0)U(2,0) + AU(1,0)^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\\
U(2,1) &= B(2,0) + 60U(5,0) + 3AU(0,0)U(3,0) + 3AU(1,0)U(2,0) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix},\\
U(3,1) &= B(3,0) + 120U(6,0) + 4AU(0,0)U(4,0) + 4AU(1,0)U(3,0) + 2AU(2,0)^2 = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{bmatrix},\\
&\vdots\\
U(0,2) &= \tfrac{1}{2}B(0,1) + 3U(3,1) + \tfrac{1}{2}AU(0,1)U(1,0) + \tfrac{1}{2}AU(0,0)U(1,1) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix},\\
U(1,2) &= \tfrac{1}{2}B(1,1) + 12U(4,1) + AU(0,1)U(2,0) + AU(0,0)U(2,1) + AU(1,1)U(1,0) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix},\\
U(2,2) &= \tfrac{1}{2}B(2,1) + 30U(5,1) + \tfrac{3}{2}AU(0,1)U(3,0) + \tfrac{3}{2}AU(0,0)U(3,1) + \tfrac{3}{2}AU(1,1)U(2,0)\\
&\quad + \tfrac{3}{2}AU(1,0)U(2,1) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix},\\
U(3,2) &= \tfrac{1}{2}B(3,1) + 60U(6,1) + 2AU(0,1)U(4,0) + 2AU(0,0)U(4,1) + 2AU(1,1)U(3,0)\\
&\quad + 2AU(1,0)U(3,1) + 2AU(2,1)U(2,0) = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{12} \end{bmatrix}.\\
&\vdots
\end{aligned} \tag{36}$$

In the same manner, the rest of the components can be obtained using Maple. Substituting the quantities listed in (36) into Equation (35), when $x_0 = t_0 = 0$, the matrix approximation solution in series form is

$$\begin{aligned}
u(x,t) = &\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} x^2 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} xt\\
&+ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} t^2 + \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} x^3 + \ldots .
\end{aligned} \tag{37}$$

The closed form of the above solution is

$$u(x,t) = \begin{bmatrix} e^{(x+t)} & 1 \\ 0 & e^{(x-t)} \end{bmatrix},$$

which is exactly the same as the exact solution.

4. Conclusions

In this paper, we have shown that the MDTM can be used successfully to solve a system of nonlinear partial differential equations. This method is simple and easy to use and solves any MPDE without any need for discretizing the variables. The results of given test examples show that the MDTM results agree with the Taylor series solution of the exact solution. MDTM is powerful compared to the other approximation methods such as ADM, VIM, HPM, and HAM. It it not affected by computation round-off errors. Also, the proposed method is useful for finding an accurate approximation of the exact solution. The symbolic calculation software package Maple is used in the derivations. The method gives rapidly converging series solutions. The accuracy of the obtained solution can be improved by taking more terms in the solution. In many cases, the series solutions obtained with MDTM can be written in exact closed form.

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