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Analytical solution for determination the control parameter in the inverse parabolic equation using HAM

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Abstract

In this article, the homotopy analysis method (HAM) for obtaining the analytical solution of the inverse parabolic problem and computing the unknown time-dependent parameter is introduced. The series solution is developed and the recurrence relations are given explicitly. Special attention is given to satisfy the convergence of the proposed method. A comparison of HAM with the variational iteration method is made. In the HAM, we use the auxiliary parameter $\hbar$ to control with a simple way in the convergence region of the solution series. Applying this method with several examples is presented to show the accuracy, simplicity and efficiency of the proposed approach.

Keywords: Analytical solution; Inverse parabolic equation; Homotopy analysis method

MSC 2010 No.: 65N25, 41H04

1. Introduction

The HAM is developed in 1992 by Liao in Liao (1992) and Liao (2004). This method has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors (Abbasbandy (2007), Hayat et al. (2004), Inc (2007), and Jafari and Seifi (2009)), and references therein. We aim in this work to effectively employ the HAM to establish the analytical solution of the inverse parabolic problem and compute the unknown time-dependent parameter (Moebbi and Dehghan (2010), and Tatari and Dehghan (2007)). By using the present method, numerical results can be obtained with using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for large values of x and t. Unlike other numerical methods are given low degree of accuracy for large values of x and t. Therefore, the HAM handles linear and nonlinear problems without any assumption and restriction (Khader (2013), Sweilam and Khader (2011)).

Recently, it has caught much attention that many physical phenomena can be described in terms of parabolic partial differential equations with a source control parameter. These models arise, for example, in the study of heat conduction processes, thermo-elasticity, chemical diffusion and control theory (Cannon et al. (1994) and Cannon et al. (1992)).

In recent papers growing attention has been given, to analysis and implement of accurate methods for the numerical solution of parabolic inverse problems, i.e., the determination of an unknown function $c(t)$ in the parabolic partial differential equations. In this paper, we implement the HAM to obtain the solution of the proposed problem. Test problems have been considered to ensure that the HAM is accurate and efficient compared with the previous ones. Also, a comparison of HAM with the variational iteration method (VIM) is made (He (1999)) where the VIM is a special case of the presented method HAM (Gorder (2015)).

The paper has been organized as follows. In Section 2, the mathematical formulation is given. In Section 3, the basic idea of homotopy analysis method is described. In Section 4, applying HAM for inverse parabolic problem and computing an unknown time-dependent parameter is introduced and study the convergence of the exact solution. Discussion and conclusions are presented in Section 5.

2. Mathematical formulation

The inverse parabolic equation in a bounded domain takes the following form,

$$\psi_t(\bar{x},t) = \Delta\psi(\bar{x},t) + c(t)\,\psi(\bar{x},t) + s(\bar{x},t), \quad 0 \le t \le T, \quad \bar{x} \in \Re^n, \tag{1}$$

with initial condition

$$\psi(\bar{x},0) = f(\bar{x}), \qquad \bar{x} \in \Re^n. \tag{2}$$

We add an additional condition in the following form

$$\psi(\bar{x}_0,t) = E(t), \quad 0 \le t \le T, \quad \bar{x} \in \Re^n, \tag{3}$$

where Δ is the Laplace operator, $\Re^n$ is the spatial domain of the problem, $n = 1, 2, 3$, $\bar{x} = (x_1, x_2, ..., x_n)$, and s and E are given functions while ψ and c are unknown functions (Tatari and Dehghan (2007)).

Equation (1) can be interpreted as the heat transfer process with a source parameter present where the temperature at a point $\bar{x}_0$ in the spatial domain at time t is given by Equation (3) (Dehghan (2002)).

Before applying the proposed procedure to Equation (1), we use a pair of transformations as follows (Dehghan (2002) and Dehghan (2003)):

$$u(\bar{x}, t) = \psi(\bar{x}, t)\exp\Big(-\int_0^t c(\tau)d\tau\Big), \tag{4}$$

$$r(t) = \exp\Big(-\int_0^t c(\tau)d\tau\Big). \tag{5}$$

Then, Equation (1) transforms to a new partial differential equation which we call reformed equation for (1),

$$u_t(\bar{x}, t) = \Delta u(\bar{x}, t) + r(t)s(\bar{x}, t), \quad 0 \le t \le T, \quad \bar{x} \in \Re^n, \tag{6}$$

subject to the initial condition

$$u(\bar{x}, 0) = f(\bar{x}), \quad \bar{x} \in \Re^n, \tag{7}$$

and the boundary condition

$$u(\bar{x}_0, t) = r(t)E(t), \quad 0 \le t \le T. \tag{8}$$

Assume $E(t) \neq 0$, the later is equivalent to

$$r(t) = \frac{u(\bar{x}_0, t)}{E(t)}, \tag{9}$$

with this transformation, $c(t)$ is disappeared and its role is represented implicitly by $r(t)$. So, we can overcome the difficulties in handling with $c(t)$ and obtain the following equation (Dehghan (2001))

$$u_t(\bar{x}, t) = \Delta u(\bar{x}, t) + \frac{u(\bar{x}_0, t)}{E(t)}s(\bar{x}, t), \quad 0 \le t \le T, \quad \bar{x} \in \Re^n. \tag{10}$$

3. Basic idea of HAM

To illustrate the basic idea of HAM (Liao (1992) and Liao (2004)), we consider the following differential equation

$$N[u(\ell,t)] = 0, \tag{11}$$

where N is a linear operator for this problem, ℓ and t denote independent variables, $u(\ell,t)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way.

3.1. Zeroth-order deformation equation

In Liao (1992), Liao constructed the so-called zeroth-order deformation equation

$$(1-q)\pounds[\phi(\ell,t;q) - u_0(\ell,t)] = q\hbar N[\phi(\ell,t;q)], \tag{12}$$

where $\pounds$ is an auxiliary linear operator, $u_0(\ell,t)$ is an initial guess, $\hbar \neq 0$ is an auxiliary parameter and $q \in [0,1]$ is the embedding parameter. Obviously, when $q = 0$ and $q = 1$, it holds respectively

$$\phi(\ell,t;0) = u_0(\ell,t), \qquad \phi(\ell,t;1) = u(\ell,t). \tag{13}$$

Thus, as q increases from 0 to 1, the solution $\phi(\ell,t;q)$ varies from $u_0(\ell,t)$ to $u(\ell,t)$. Expanding $\phi(\ell,t;q)$ in Taylor series with respect to the embedding parameter q, one has

$$\phi(\ell,t;q) = u_0(\ell,t) + \sum_{m=1}^{\infty} u_m(\ell,t)q^m, \tag{14}$$

where

$$u_m(\ell,t) = \frac{1}{m!}\frac{\partial^m \phi(\ell,t;q)}{\partial q^m}\Big|_{q=0}. \tag{15}$$

Assume that the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are selected such that the series (14) is convergent at $q = 1$, Then, at $q = 1$, and using (13), the series (14) becomes

$$u(\ell,t) = u_0(\ell,t) + \sum_{m=1}^{\infty} u_m(\ell,t). \tag{16}$$

3.2. The m^{th} order deformation equation

Define the vector

$$\vec{u}_m(\ell,t) = [u_0(\ell,t), u_1(\ell,t), ..., u_m(\ell,t)]. \tag{17}$$

Differentiation of Equation (12) m times with respect to the embedding parameter q, then, setting $q = 0$ and dividing them by $m!$, finally using (15), we have the so-called m^{th}-order deformation equations

$$\pounds[u_m(\ell,t) - \delta_m u_{m-1}(\ell,t)] = \hbar\,\Re_m(\vec{u}_{m-1}), \tag{18}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\ell,t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{19}$$

and

$$\delta_m = \begin{cases} 0, & m \le 1; \\ 1, & m > 1. \end{cases} \tag{20}$$

Applying $\pounds^{-1}$ on both sides of Equation (18), we get

$$u_m(\ell,t) = \delta_m u_{m-1}(\ell,t) + \hbar\,\pounds^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{21}$$

In this way, it is easily to obtain u_m for $m \ge 1$, at N^{th} order, we have

$$u(\ell,t) \cong \sum_{m=0}^{N} u_m(\ell,t). \tag{22}$$

When $N \to \infty$, we get an accurate approximation of the original Equation (11). For the convergence of the proposed method, we refer the reader to Liao (2003). If Equation (11) admits unique solution, then, this method will produce this unique solution. If Equation (11) does not possess a unique solution, HAM will give a solution among many other (possible) solutions. For more details about the convergence, we state the following theorem.

Theorem 3.1. (Yin-Ping and Zhi-Bin (2008))

As long as the series (16) is convergent, where $u_m(\ell,t)$ is governed by the higher-order deformation equation (18) under the definitions (19) and (20), it must be a solution of the original Equation (11).

Proof:

As the series (16) is convergent, it holds that

$$\lim_{m\to\infty} u_m(\ell,t) = 0.$$

Using (18) and (20), we have

$$\sum_{m=1}^{\infty} \pounds[u_m(\ell,t) - \delta_m u_{m-1}(\ell,t)] = \hbar \sum_{m=1}^{\infty} \Re_m(\vec{u}_{m-1}).$$

Due to the linearity of the derivative, it follows

$$\sum_{m=1}^{\infty} \pounds[u_m(\ell,t) - \delta_m u_{m-1}(\ell,t)] = \pounds\left[\sum_{m=1}^{\infty}[u_m(\ell,t) - \delta_m u_{m-1}(\ell,t)]\right]$$

$$= \pounds\left[\lim_{m\to\infty} u_m(\ell,t)\right] = \pounds[0] = 0.$$

Therefore,

$$\hbar \sum_{m=1}^{\infty} \Re_m(\vec{u}_{m-1}(\ell,t)) = 0.$$

Since $\hbar \neq 0$, we have according to the definition (18) that

$$\sum_{m=1}^{\infty} \Re_m(\vec{u}_{m-1}(\ell,t)) = \sum_{m=1}^{\infty}\left[\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}} N\left[\sum_{n=1}^{\infty} u_n(\ell,t) q^n\right]\right]_{q=0}\right] = 0.$$

We note that there is no derivative with respect to q in the nonlinear operator N.

So, $N[\sum_{n=0}^{\infty} u_n(\ell,t)q^n]$ can be looked upon as a polynomial on q. By using the binomial expansion theorem, we obtain $N[\sum_{n=0}^{\infty} u_n(\ell,t)] = 0$, such as for $N = u(\ell,t)u_\ell^2$. Letting $u(\ell,t) = u_0(\ell,t) + u_1(\ell,t)q + u_2(\ell,t)q^2$, it can be easily verified that

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left[\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}} N\left[\sum_{n=1}^{2} u_n(\ell,t) q^n\right]\right]_{q=0}\right] \\
&\quad = [u_0(\ell,t) + u_1(\ell,t) + u_2(\ell,t)].[(u_0(\ell,t) + u_1(\ell,t) + u_2(\ell,t))_\ell]^2 \\
&\quad = N[u_0(\ell,t) + u_1(\ell,t) + u_2(\ell,t)].
\end{aligned}$$

This ends the proof. ■

4. Applications the proposed method

We apply the HAM to problem (He (1999)) to illustrate the strength and the efficiency of the method. We will make a comparison with the VIM (He (1999)).

One dimensional inverse parabolic problem (Example 1)

Consider problem (10) in the case $n = 1,\ T = 1$, in the domain $[0, 1]$, (Shakeri and Dehghan (2007), Tatari and Dehghan (2007)), where

$$f(x) = \cos(\pi x) + \sin(\pi x), \qquad E(t) = \sqrt{2}\, e^{-t^2},$$

$$s(x,t) = (\pi^2 - (t+1)^2)e^{-t^2}\,(\cos(\pi x) + \sin(\pi x)),$$

with $x_0 = 0.25$. The exact solution of this problem is

$$u(x,t) = e^{-t^2}\,(\cos(\pi x) + \sin(\pi x)),$$

and $c(t) = 1+t^2$. Also, $\psi(x,t) = e^{-\frac{t^3}{3}-t^2-t}\,(\cos(\pi x)+\sin(\pi x))$, is the exact solution of the reformed problem.

Now, to implement HAM, we choose the linear operator

$$\pounds[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t}, \tag{23}$$

with the property $\pounds[c_1] = 0$ where c_1 is a constant. We now define a linear operator as

$$N[\phi(x,t;q)] = \phi_t(x,t;q) - \Delta\phi(x,t;q) - \frac{\phi(x_0,t;q)}{E(t)} s(x,t). \tag{24}$$

Using above definition, we construct the zeroth-order deformation equation

$$(1-q)\pounds[\phi(x,t;q) - u_0(x,t)] = q\hbar\, N[\phi(x,t;q)]. \tag{25}$$

For $q = 0$ and $q = 1$, we can write

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t). \tag{26}$$

Thus, we obtain the m^{th}-order deformation equations

$$\pounds[u_m(x,t) - \delta_m u_{m-1}(x,t)] = \hbar\,\Re_m(\vec{u}_{m-1}),$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial \phi_{m-1}(x,t;q)}{\partial t} - \Delta\phi_{m-1}(x,t;q) - \frac{\phi_{m-1}(x_0,t;q)}{E(t)} s(x,t). \tag{27}$$

Now the solution of the m^{th} order deformation equations for $m \geq 1$ becomes

$$u_m(x,t) = \delta_m u_{m-1}(x,t) + \hbar\,\pounds^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{28}$$

This in turn gives the first few components of the approximate solution.

We start with initial approximation $u_0(x,t) = \cos(\pi x) + \sin(\pi x)$. Since $\tilde{u} = u_0 + u_1 + u_2 + ...$ From the above equations (28), we obtain u_m's as follows

$$u_0(x,t) = \cos(\pi x) + \sin(\pi x),$$

$$u_1(x,t) = \hbar\,(t + t^2 + \frac{t^3}{3})\,(\cos(\pi x) + \sin(\pi x)),$$

$$u_2(x,t) = \hbar^2 \frac{(t + t^2 + \frac{t^3}{3})^2}{2!}(\cos(\pi x) + \sin(\pi x)),$$

$$u_3(x,t) = \hbar^3 \frac{(t + t^2 + \frac{t^3}{3})^3}{3!}(\cos(\pi x) + \sin(\pi x)),$$

other components of the approximate solution can be obtained in the same manner.

Using the initial approximation u_0 and from the above equations, we can identify u_m's for $m = 1, 2, ...$ and therefore $\tilde{u} = u_0 + u_1 + u_2 + ...$ is obtained. From (4), (5) and (9), we can obtain $\tilde{r}, \tilde{u}$ and $\tilde{c}$ as approximations for r , u and c, respectively, as follows,

$$\tilde{r} = \frac{\tilde{u}(\bar{x}_0,t)}{E(t)}, \qquad \tilde{u} = \frac{\psi(\bar{x},t)}{\tilde{r}(t)}, \qquad \tilde{c} = -\frac{\tilde{r}\,'(\bar{x}_0,t)}{\tilde{r}(t)}. \tag{29}$$

For the case $\hbar = -1$, these components can be reduced to the following form,

$$\begin{aligned} u_1(x,t) &= (-\frac{t^3}{3} - t^2 - t)\,(\cos(\pi x) + \sin(\pi x)),\\ u_2(x,t) &= \frac{(\frac{t^3}{3} + t^2 + t)^2}{2!}\,(\cos(\pi x) + \sin(\pi x)),\\ u_3(x,t) &= \frac{(-\frac{t^3}{3} - t^2 - t)^3}{3!}\,(\cos(\pi x) + \sin(\pi x)). \end{aligned}$$

Other components of the approximate solution can be obtained in the same manner. Generally we have

$$u_n(x,t) = \frac{(-\frac{t^3}{3} - t^2 - t)^n}{n!}\,(\cos(\pi x) + \sin(\pi x)),$$

and thus,

$$\tilde{u}(x,t) = u_0 + u_1 + u_2 + ... = \sum_{n=0}^{\infty} \frac{(-\frac{t^3}{3} - t^2 - t)^n}{n!}\,(\cos(\pi x) + \sin(\pi x))$$

$$= e^{-\frac{t^3}{3} - t^2 - t}(\cos(\pi x) + \sin(\pi x)).$$

This is the exact solution of the reformed equation. The solution of the main problem is obtained in the following form

$$\tilde{r} = \frac{e^{-\frac{t^3}{3}-t^2-t}(\cos(0.25\pi)+\sin(0.25\pi))}{\sqrt{2}\,e^{-t^2}} = e^{-\frac{t^3}{3}-t}, \tag{30}$$

$$\tilde{u} = \frac{e^{-\frac{t^3}{3}-t^2-t}(\cos(\pi x)+\sin(\pi x))}{e^{-\frac{t^3}{3}-t}} = e^{-t^2}(\cos(\pi x)+\sin(\pi x)), \tag{31}$$

$$\tilde{c} = -\frac{(-t^2-1)e^{-\frac{t^3}{3}-t}}{e^{-\frac{t^3}{3}-t}} = 1+t^2. \tag{32}$$

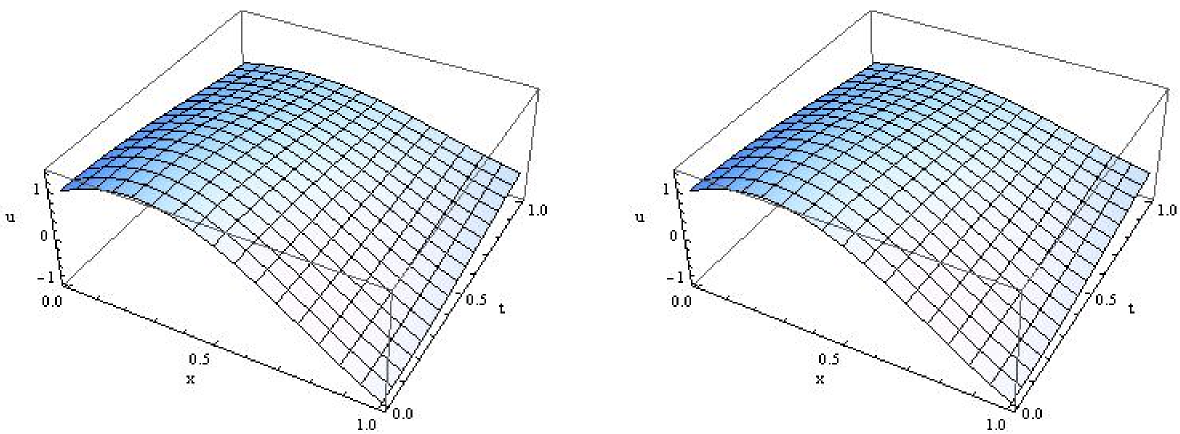

Figure 1. The approximate solution (Left) at $\hbar = -0.5$ and the exact solution (Right).

Figure 1, presents the behavior of the approximate solution with $\hbar = -0.5$ and the exact solution in the interval $t \in [0,1]$. From this figure, we can conclude that the approximate solution by using the proposed method is in excellent agreement with the exact solution.

Two dimensional inverse parabolic problem (Example 2)

Consider problem (10) in the case $n = 2$, $T = 1$, in the domain $[0,1]^2$, (Shakeri and Dehghan (2007), Tatari and Dehghan (2007)), where

$$f(x,y) = \sin\Big(\frac{\pi}{4}(x+2\,y)\Big),$$

$$s(x,y,t) = \Big(\frac{5\,\pi^2}{16} - 5\,t\Big)\,e^t \sin\Big(\frac{\pi}{4}(x+2\,y)\Big),$$

$$E(t) = \sin(0.2\pi)\,e^t,$$

with $(x_0, y_0) = (0.4, 0.2)$. The exact solution of this problem is

$$u(x, y, t) = e^t \sin\Big(\frac{\pi}{4}(x + 2\,y)\Big),$$

and $c(t) = 1 + 5\,t$. Also, $\psi(x, y, t) = e^{-\frac{5\,t^2}{2}} \sin\Big(\frac{\pi}{4}(x + 2\,y)\Big)$, is the exact solution of the reformed problem.

Now, to implement HAM, we choose the linear operator

$$\pounds[\phi(x, y, t; q)] = \frac{\partial \phi(x, y, t; q)}{\partial t}, \tag{33}$$

with the property, $\pounds[c_1] = 0$, where c_1 is a constant. We now define a linear operator as

$$N[\phi(x, y, t; q)] = \phi_t(x, y, t; q) - \Delta\phi(x, y, t; q) - \frac{\phi(x_0, y_0, t; q)}{E(t)} s(x, y, t). \tag{34}$$

Using above definition, we construct the zeroth-order deformation equation

$$(1 - q)\pounds[\phi(x, y, t; q) - u_0(x, y, t)] = q\hbar\, N[\phi(x, y, t; q)]. \tag{35}$$

For $q = 0$ and $q = 1$, we can write

$$\phi(x, y, t; 0) = u_0(x, y, t), \qquad \phi(x, y, t; 1) = u(x, y, t). \tag{36}$$

Thus, we obtain the m^{th} order deformation equations

$$\pounds[u_m(x, y, t) - \delta_m u_{m-1}(x, y, t)] = \hbar\, \Re_m(\vec{u}_{m-1}),$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial \phi_{m-1}(x, y, t; q)}{\partial t} - \Delta\phi_{m-1}(x, y, t; q) - \frac{\phi_{m-1}(x_0, y_0, t; q)}{E(t)}\, s(x, y, t). \tag{37}$$

Now the solution of the m^{th} order deformation equations for $m \geq 1$ becomes

$$u_m(x, y, t) = \delta_m u_{m-1}(x, y, t) + \hbar\, \pounds^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{38}$$

This in turn gives the first few components of the approximate solution.

We start with initial approximation $u_0(x, y, t) = \sin\Big(\frac{\pi}{4}(x + 2\,y)\Big)$. Since $\tilde{u} = u_0 + u_1 + u_2 + ...$ from the above equations (38), we can obtain u_m's as follows

$$u_0(x, y, t) = \sin\Big(\frac{\pi}{4}(x + 2\,y)\Big),$$

$$u_1(x, y, t) = \hbar\,(\frac{5\,t^2}{2}) \sin\Big(\frac{\pi}{4}(x + 2\,y)\Big),$$

$$u_2(x, y, t) = \hbar\,(\frac{5\,t^2}{2}) \sin\Big(\frac{\pi}{4}(x + 2\,y)\Big) + \hbar^2\Big(\frac{5\,t^2}{2} + \frac{(\frac{5\,t^2}{2})^2}{2!}\Big) \sin\Big(\frac{\pi}{4}(x + 2\,y)\Big),$$

$$u_3(x,y,t)=\hbar^2\Big(\frac{5\,t^2}{2}+\frac{(\frac{5\,t^2}{2})^2}{2!}\Big)\sin\Big(\frac{\pi}{4}(x+2\,y)\Big)+\hbar^3\Big(\frac{5\,t^2}{2}+\frac{(\frac{5\,t^2}{2})^2}{2!}+\frac{(\frac{5\,t^2}{2})^3}{3!}\Big)\sin\Big(\frac{\pi}{4}(x+2\,y)\Big).$$

Other components of the approximate solution can be obtained in the same manner. For the case $\hbar=-1$, these components can be reduced to the following form,

$$u_1(x,y,t)=\Big(-\frac{5\,t^2}{2}\Big)\sin\Big(\frac{\pi}{4}(x+2\,y)\Big),$$

$$u_2(x,y,t)=\frac{(-\frac{5\,t^2}{2})^2}{2!}\sin\Big(\frac{\pi}{4}(x+2\,y)\Big),$$

$$u_3(x,y,t)=\frac{(-\frac{5\,t^2}{2})^3}{3!}\sin\Big(\frac{\pi}{4}(x+2\,y)\Big).$$

Other components of the approximate solution can be obtained in the same manner. Generally we have

$$u_n(x,y,t)=\frac{(-\frac{5\,t^2}{2})^n}{n!}\sin\Big(\frac{\pi}{4}(x+2\,y)\Big),$$

and thus,

$$\tilde{u}(x,y,t)=u_0+u_1+u_2+...=\sum_{n=0}^{\infty}\frac{(-\frac{5\,t^2}{2})^n}{n!}\sin\Big(\frac{\pi}{4}(x+2\,y)\Big)=e^{-\frac{5\,t^2}{2}}\sin\Big(\frac{\pi}{4}(x+2\,y)\Big).$$

This is the exact solution of the reformed equation. The solution of the main problem is obtained in the following form

$$\tilde{r}=\frac{e^{-\frac{5\,t^2}{2}}\sin(\frac{\pi}{4}(0.4+0.4))}{e^t\sin(0.2\pi)}=e^{-\frac{5\,t^2}{2}-t}, \tag{39}$$

$$\tilde{u}=\frac{e^{-\frac{5\,t^2}{2}}\sin(\frac{\pi}{4}(x+2\,y))}{e^{-\frac{5\,t^2}{2}-t}}=e^t\sin\Big(\frac{\pi}{4}(x+2\,y)\Big), \tag{40}$$

$$\tilde{c}=-\frac{(-5\,t-1)e^{-\frac{5\,t^2}{2}-t}}{e^{-\frac{5\,t^2}{2}-t}}=1+5\,t. \tag{41}$$

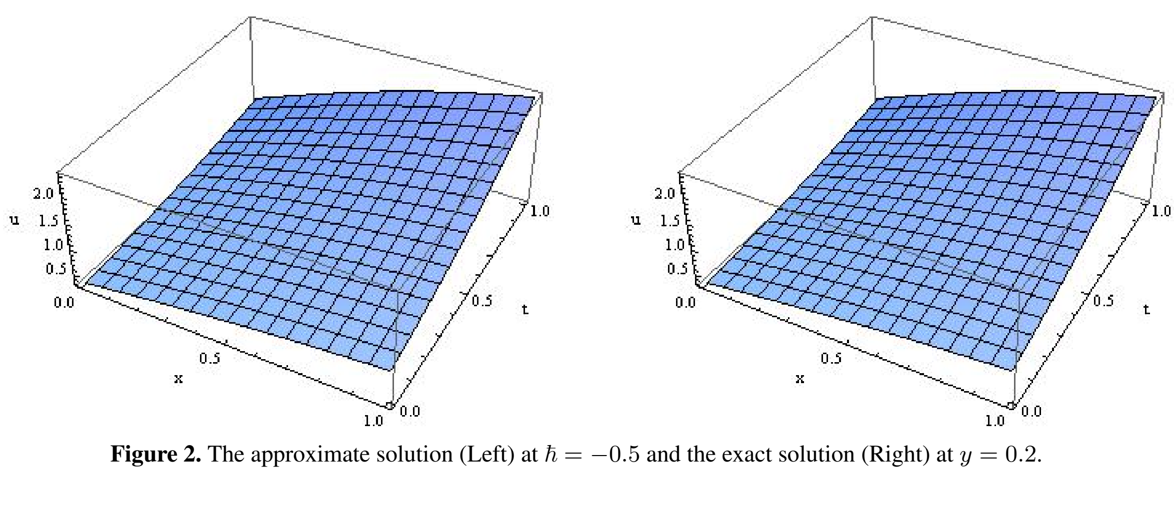

Figure 2. The approximate solution (Left) at $\hbar=-0.5$ and the exact solution (Right) at $y=0.2$.

Figure 2 presents the behavior of the approximate solution with $\hbar = -0.5$ and the exact solution at $y = 0.2$ in the interval $t \in [0, 1]$. From this figure, we can conclude that the solution by using the proposed method and the exact solution are in excellent agreement.

Three-dimensional inverse parabolic problem (Example 3)

Consider problem (10) in the case $n = 3, T = 1$, in the domain $[0, 1]^3$, (Shakeri and Dehghan (2007), Tatari and Dehghan (2007)), where

$$f(x,y,z) = \sin\Big(\frac{\pi}{4}(x + 2\,y + 3\,z)\Big),$$

$$s(x,y,z,t) = \Big(\frac{7\,\pi^2}{16} - 10\,t\Big)\, e^t \sin\Big(\frac{\pi}{4}(x + 2\,y + 3\,z)\Big),$$

$$E(t) = e^t,$$

with $(x_0, y_0, z_0) = (0.6, 0.4, 0.2)$. The exact solution of this problem is

$$u(x,y,z,t) = e^t \sin\Big(\frac{\pi}{4}(x + 2\,y + 3\,z)\Big),$$

and $c(t) = 1 + 10\,t$. Also, $\psi(x,y,z,t) = e^{-5\,t^2} \sin\Big(\frac{\pi}{4}(x + 2\,y + 3\,z)\Big)$ is the exact solution of the reformed problem.

Now, to implement HAM, we choose the linear operator

$$\pounds[\phi(x,y,z,t;q)] = \frac{\partial \phi(x,y,z,t;q)}{\partial t}, \tag{42}$$

with the property $\pounds[c_1] = 0$ where c_1 is a constant. We now define a linear operator as

$$N[\phi(x,y,z,t;q)] = \phi_t(x,y,z,t;q) - \Delta\phi(x,y,z,t;q) - \frac{\phi(x_0,y_0,z_0,t;q)}{E(t)}\, s(x,y,z,t). \tag{43}$$

Using above definition, we construct the zeroth-order deformation equation

$$(1-q)\pounds[\phi(x,y,z,t;q) - u_0(x,y,z,t)] = q\hbar\, N[\phi(x,y,z,t;q)]. \tag{44}$$

For $q = 0$ and $q = 1$, we can write

$$\phi(x,y,z,t;0) = u_0(x,y,z,t), \qquad \phi(x,y,z,t;1) = u(x,y,z,t). \tag{45}$$

Thus, we obtain the m^{th} order deformation equations

$$\pounds[u_m(x,y,z,t) - \delta_m u_{m-1}(x,y,z,t)] = \hbar\, \Re_m(\vec{u}_{m-1}),$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial \phi_{m-1}(x,y,z,t;q)}{\partial t} - \Delta\phi_{m-1}(x,y,z,t;q) - \frac{\phi_{m-1}(x_0,y_0,z_0,t;q)}{E(t)} s(x,y,z,t). \quad (46)$$

Now the solution of m^{th} order deformation equations for $m \geq 1$ becomes

$$u_m(x,y,z,t) = \delta_m u_{m-1}(x,y,z,t) + \hbar\, \pounds^{-1}[\Re_m(\vec{u}_{m-1})]. \quad (47)$$

This in turn gives the first few components of the approximate solution

We start with initial approximation $u_0(x,y,z,t) = \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big)$. Since $\tilde{u} = u_0 + u_1 + u_2 + ...$ from the above equations, we can obtain u_m's as follows,

$$u_0(x,y,z,t) = \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big),$$

$$u_1(x,y,z,t) = \hbar\,(5\,t^2)\, \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big),$$

$$u_2(x,y,z,t) = \hbar\,(5\,t^2) \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big) + \hbar^2\Big(5\,t^2 + \frac{(5\,t^2)^2}{2!}\Big)\, \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big),$$

$$u_3(x,y,z,t) = \hbar^2\Big(5\,t^2 + \frac{(5\,t^2)^2}{2!}\Big)\, \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big) + \hbar^3\Big(5\,t^2 + \frac{(5\,t^2)^2}{2!} + \frac{(5\,t^2)^3}{3!}\Big)\, \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big).$$

Other components of the approximate solution can be obtained in the same manner.

For the case that $\hbar = -1$, these components can be reduced to the following form,

$$u_1(x,y,z,t) = (-5\,t^2) \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big),$$

$$u_2(x,y,z,t) = \frac{(-5\,t^2)^2}{2!} \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big),$$

$$u_3(x,y,z,t) = \frac{(-5\,t^2)^3}{3!} \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big).$$

Generally we have

$$u_n(x,y,z,t) = \frac{(-5\,t^2)^n}{n!} \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big),$$

and thus,

$$\tilde{u}(x,y,z,t) = u_0 + u_1 + u_2 + ... = \sum_{n=0}^{\infty} \frac{(-5\,t^2)^n}{n!} \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big) = e^{-5\,t^2} \sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big).$$

This is the exact solution of the reformed equation. The solution of the main problem is obtained in the following form

$$\tilde{r} = \frac{e^{-5\,t^2}\sin\Big(\frac{\pi}{4}(0.6+0.8+0.6)\Big)}{e^t} = e^{-5\,t^2-t}, \tag{48}$$

$$\tilde{u} = \frac{e^{-5\,t^2}\sin\Big(\frac{\pi}{4}(x+2\,y+3\,z)\Big)}{e^{-5\,t^2-t}} = e^t\sin(\frac{\pi}{4}(x+2\,y+3\,z)\Big), \tag{49}$$

$$\tilde{c} = -\frac{(-10\,t-1)e^{-5\,t^2-t}}{e^{-5\,t^2-t}} = 1+10\,t. \tag{50}$$

From Equations (48)-(50), we can see that HAM solution converges to the exact solution. Also, for different values of the parameter $\hbar$ we note that there is a complete agreement between computed results by present method and the exact solution. Table 1 gives a comparison between the error of the proposed method (HAM) and VIM (He (1999)) at $x = 0.6,\ y = 0.4$. From these results we can see that the presented approach is more efficient than the other different methods, regarding HAM which takes four components only of the solution.

It is noted that our approximate solutions converge at $\hbar = -0.5$ (see Tables 1 and 2). The explicit analytic expression given by Equation (47) contains the auxiliary parameter $\hbar$, which gives the convergence region and rate of approximation for the HAM. However, the errors can be further be reduced by calculating higher order approximations. This proves that the HAM is a very useful analytic method to get accurate analytic solutions to linear and strongly nonlinear problems (Abbasbandy (2007), Inc (2007), Liao (2004)).

Table 1: Comparison between the error of HAM and VIM at $\hbar = -1$.

t	0.0	0.2	0.4	0.6	0.8	1.0
VIM	4.4409 e-15	2.9636 e-9	4.2147 e-9	4.1563 e-9	2.8473 e-9	2.2205 e-15
HAM	4.4408 e-15	2.9636 e-9	4.2146 e-9	4.1563 e-9	2.8472 e-9	2.2205 e-15

Table 2: The error of solution using HAM at different values of time at $\hbar = -0.5$.

t	0.0	0.2	0.4	0.6	0.8	1.0
HAM	2.6645 e-15	4.2109 e-4	6.5477 e-4	6.1783 e-4	3.5913 e-4	2.2205 e-15

Liao (1992) showed that whatever a solution series converges it will be one of the solutions of the considered problem. Liao (1992) and Liao (2004) presented the controlled auxiliary parameter $\hbar$ to control in the rate of convergence of the approximate solutions obtained by the HAM. We obtain the VIM solution of the inverse parabolic problem and compute the unknown time-dependent parameter when $\hbar = -1$. Also, it is noted that our approximate solutions converge at $\hbar - 0.5$ (see Tables 1 and 2). The present exact solution is calculated at the above mentioned values of $\hbar$. The explicit analytic expression contains the auxiliary parameter $\hbar$, which gives the convergence region and rate of approximation for the HAM. If we take $\hbar = -1$ in the series solution (47) then, we get the VIM. Also the homotopy perturbation method (HPM) is a special case of HAM at $\hbar = -1$. But VIM and HPM solutions are valid only for the value of $\hbar = -1$. This proves that the HAM is a very useful analytic method to get accurate analytic solutions to linear problems (Liao (1992) and Liao (2004)).

5. The discussion and conclusion

In this article, we used HAM for obtaining the numerical solutions of the inverse parabolic problem and computing the unknown time-dependent parameter. A clear conclusions can be drawn from the numerical results that HAM provides highly accurate numerical solutions without spatial discretizations for the nonlinear PDEs. From Figures 1 and 2 and Tables 1 and 2, we can conclude that the numerical solution using HAM is given in more accuracy. The proposed technique was tested on some examples and gave the satisfactory results. This method avoids linearization and physically unrealistic assumptions. The capability, effectiveness and convenience of this method were revealed by obtaining the analytical solutions of the model and comparing them with ADM. The main advantage of the HAM is that this method provides the solution of the problem without calculating Adomian's polynomials as in Adomian decomposition method (Dehghan (2004)).

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