Bianchi Type I Magnetized Cosmological Model in Bimetric Theory of Gravitation

M. S. Borkar

Post Graduate Department of Mathematics
Mahatma Jotiba Fule Educational Campus, Amravati Road
R. T. M. Nagpur University, Nagpur – 440033 (India)
borkar.mukund@rediffmail.com; borkar.mukund@yahoo.com

S. S. Charjan

Department of Mathematics,
Late K. Z. S. Science College
Bramhani (kalmeshwar) – 441501
Dis. Nagpur India
sscharjan@rediffmail.com

Abstract

In this paper, we attempt to construct a Bianchi Type I magnetized cosmological model in Rosen's bimetric theory of gravitation by using the techniques of Letelier and Stachel under the condition that $A = (BC)^n$, where $n > 0$ in Rosen's field equations. The physical and geometrical significance of the model are discussed. It is important to note the added dimension of this paper include the introduction of a magnetic field on the model. Prior to this, all it has been is a vacuum model in bimetric gravitation.

Keywords: Bimetric theory; cosmic string; magnetic field; Bianchi Type I

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1. Introduction

In the early phase of the universe, there is no definite evidence that the present day universe (FRW universe) was of the same type. Therefore it is important to study different Bianchi Type models in the context of the early phase of the universe. In this regards we plan to study various Bianchi Type models in bimetric theory of gravitation. Earlier, Borkar and Charjan (2009, 2010) have developed the models like Bianchi Type I string dust cosmological model with magnetic field in bimetric relativity (2009) , The charged perfect fluid distribution in bimetric theory of relativity (2010) , LRS Bianchi Type I string dust magnetized cosmological models in bimetric theory of relativity (2010) and Bianchi Type I bulk viscous fluid string dust cosmological model with magnetic field in bimetric relativity (2010). Several new theories of gravitation have been formulated which are considered to be alternatives to Einstein's theory of gravitation. The most important among them is Rosen's (1977) bimetric theory of gravitation.

The Rosen's bimetric theory is the theory of gravitation based on two metrics. One is the fundamental metric tensor g_{ij} describes the gravitational potential and the second metric γ_{ij} refers to the flat space–time and describes the inertial forces associated with the acceleration of the frame of reference. The metric tensor g_{ij} determine the Riemannian geometry of the curved space time which plays the same role as given in Einstein's general relativity and it interacts with matter.

The background metric γ_{ij} refers to the geometry of the empty universe (no matter but gravitation is there) and describe the inertial forces. The metric tensor γ_{ij} has no direct physical significance but appears in the field equations. Therefore it interacts with g_{ij} but not directly with matter. One can regard γ_{ij} as giving the geometry that would exists if there were no matter.

In the absence of matter one would have $g_{ij} = \gamma_{ij}$. Moreover, the bimetric theory also satisfied the covariance and equivalence principles; the formation of general relativity. The theory agrees with the present observational facts pertaining to general relativity [For details one may refer Karade (1980), Katore and Rane (2006) and Rosen (1974, 1977)] . Thus at every point of space–time, there are two metrics

$$ds^2 = g_{ij} dx^i dx^j$$

(1)

$$d\eta^2 = \gamma_{ij} dx^i dx^j$$

(2)

The field equations of Rosen's (1974) bimetric theory of gravitation are

$$N_i^j - \frac{1}{2} N \, \delta_i^j = -8\,\pi k T_i^{\,j},$$

(3)

where

$$N_i^{\ j} = \frac{1}{2}\gamma^{pr}\left(g^{sj}g_{si|p}\right)_{|r}, \ N = N_i^{\ i}, \ k = \sqrt{\frac{g}{\gamma}}$$

together with $g = \det(g_{ij})$ and $\gamma = \det(\gamma_{ij})$. Here, the vertical bar $(\,|\,)$ stands for γ–covariant differentiation and $T_i^{\ j}$ is the energy–momentum tensor of matter fields.

Several aspects of bimetric theory of gravitation have been studied by Rosen (1974, 1977), Karade (1980), Israelit (1981), Katore and Rane (2006), Khadekar and Tade (2007). In particular, Reddy and Venkateswara Rao (1998) have obtained some Bianchi Type cosmological models in bimetric theory of gravitation. The purpose of Rosen's bimetric theory is to get rid of the singularities that occur in general relativity that was appearing in the big–bang of the cosmological model; this has rekindled a lot of interest and study of in cosmological models related to Rosen's bimetric theory of gravitation.

In bimetric theory, according to Rosen (1974) the background metric tensor γ_{ij} should not be taken as describing an empty universe but one in which the cosmological principles hold. Hence Rosen proposed that the metric γ_{ij} be taken as the metric tensor of exactly this kind of universe. In accordance with this principle, the large scale structure of the universe is ideally assumed to the same aspect from everywhere in space and at all times. This principle is not quite valid on a small scale structure due to irregularities in the matter distribution and is equally invalid on large scale structure due to evolution of the matter. We however assume the idial case of binding cosmological principle. It does not apply to g_{ij} and the matter in the universe but to the metric γ_{ij}. Hence γ_{ij} describes a space–time of constant curvature.

In the context of general relativity cosmic strings do not occur in Bianchi Type models (see K. D. Krori et al. (1994)). In it some Bianchi Type cosmological models – two in four and one in higher dimensions– are studied by Krori et al (1994). They show that the cosmic strings do not occur in Bianchi Type V cosmology. Bali and Dave (2003), Bali and Upadhaya (2003), Bali and Singh (2005), Bali and Pareek (2007) have all investigated Bianchi Type IX, I and V string cosmological models under different physical conditions in general relativity. Raj Bali and Anjali (2006) have investigated Bianchi Type I magnetized string cosmological model in general relativity by introducing the condition $A = (BC)^n$, where $n > 0$ in Einstein field equations, whereas Raj Bali and Umesh Kumar Pareek (2007) have deduced Bianchi Type I string dust cosmological model with magnetic field in general relativity by imposing the condition $A = N\,(BC)^{-n}$, where $n > 0$ and N is proportionality constant, in Einstein field equations. Further M. S. Borkar and S. S. Charjan (2009) have examined Bianchi Type I string dust

cosmological model with magnetic field in bimetric relativity under the condition $A = N \, (BC)^{-n}$, where $n > 0$ and N is proportionality constant in Rosen's field equations.

Researchers like Bali et al. (2003,2006), Pradhan et al. (2007), Wang (2004, (2006), Rothore et al. (2008), Pradhan (2009), Borkar and Charjan (2009,2010) developed the models in the field of bulk viscous fluid solutions and Bianchi Type string models which are the most useful as well as in bimetric theory of gravitation. In order to achieve our magnetized model in bimetric theory of gravitation, we have freely emphasized the technique and symbols of Bali et al. and Borkar et al. (2010).

2. Solutions of Rosen's Field Equations

We consider Bianchi Type I metric in the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 , \tag{4}$$

where A, B and C are functions of t – alone. Here, $B = C$ otherwise, we get LRS Bianchi Type I model.

The flat metric corresponding to metric (4) is

$$d\eta^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{5}$$

The energy momentum tensor $T_i^{\ j}$ for the string dust with magnetic field is taken as

$$T_i^{\ j} = (\in + p) \, \upsilon_i \upsilon^j + p \, g_i^{\ j} - \lambda \, x_i \, x^j + E_i^{\ j} \tag{6}$$

with

$$\upsilon^i \upsilon_i = -x_i \, x^i = -1 \tag{7}$$

$$\upsilon^i x_i = 0. \tag{8}$$

In this model $\in$ and λ denote the rest energy density and the tension density of the system of strings respectively, p is the pressure, υ^i is the flow vector and x_i the direction of strings.

The electromagnetic field E_{ij} is given by Lichnerowicz (1967)

$$E_{ij} = \vec{\mu} \left[|h|^2 \left(\upsilon_i \, \upsilon_j + \frac{1}{2} g_{ij} \right) - h_i \, h_j \right] \tag{9}$$

The four velocity vector υ_i is given by

$$g_{ij}\,\upsilon^i\upsilon^j = -1 \tag{10}$$

and $\bar{\mu}$ is the magnetic permeability and h_i is the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}}\in_{ijkl} F^{kl}\upsilon^j\ , \tag{11}$$

where F_{kl} is the electromagnetic field tensor and $\in_{ijkl}$ is the Levi Civita tensor density.

Assume the comoving coordinates and hence we have $\upsilon^1 = \upsilon^2 = \upsilon^3 = 0,\ \upsilon^4 = 1.$

Further we assume that the incident magnetic field is taken along x–axis, so that

$$h_1 \neq 0, \quad h_2 = h_3 = h_4 = 0.$$

The first set of Maxwell's equation

$$F_{[ij,k]} = 0 \tag{12}$$

yields

$$F_{23} = \text{constant} = \text{H (say)}.$$

Due to the assumption of infinite electrical conductivity we have

$$F_{14} = F_{24} = F_{34} = 0.$$

The only non–vanishing component of F_{ij} is F_{23}.

So that

$$h_1 = \frac{AH}{\bar{\mu}\,BC} \tag{13}$$

and

$$|h|^2 = \frac{H^2}{\bar{\mu}^2 B^2 C^2}. \tag{14}$$

From equation (9) we obtain

$$-E_1^1 = E_2^2 = E_3^3 = -E_4^4 = \frac{H^2}{2\,\overline{\mu}B^2C^2}$$

(15)

Equation (6) of energy momentum tensor yield

$$T_1^1 = \left(p - \lambda - \frac{H^2}{2\,\overline{\mu}B^2C^2} \right), T_2^2 = T_3^3 = \left(p + \frac{H^2}{2\,\overline{\mu}B^2C^2} \right), T_4^4 = \left(-\in - \frac{H^2}{2\,\overline{\mu}B^2C^2} \right).$$

(16)

The Rosen's field equations (3) for the metric (4) and (5) with the help of (16) gives

$$\frac{-A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{A_4^2}{A^2} - \frac{B_4^2}{B^2} - \frac{C_4^2}{C^2} = 16\pi ABC\left(-p + \lambda + \frac{H^2}{2\,\overline{\mu}B^2C^2} \right),$$

(17)

$$\frac{A_{44}}{A} - \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{A_4^2}{A^2} + \frac{B_4^2}{B^2} - \frac{C_4^2}{C^2} = 16\pi ABC\left(-p - \frac{H^2}{2\,\overline{\mu}B^2C^2} \right),$$

(18)

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{A_4^2}{A^2} - \frac{B_4^2}{B^2} + \frac{C_4^2}{C^2} = 16\pi ABC\left(-p - \frac{H^2}{2\,\overline{\mu}B^2C^2} \right),$$

(19)

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{A_4^2}{A^2} - \frac{B_4^2}{B^2} - \frac{C_4^2}{C^2} = 16\pi ABC\left(\in + \frac{H^2}{2\,\overline{\mu}B^2C^2} \right),$$

(20)

where $A_4 = \dfrac{dA}{dt}, B_4 = \dfrac{dB}{dt}, C_4 = \dfrac{dC}{dt},$ *etc.*

From equations (18) and (19), we obtain

$$\frac{C_{44}}{C} - \frac{B_{44}}{B} = \frac{C_4^2}{C^2} - \frac{B_4^2}{B^2}.$$

(21)

Equations (17) and (18) leads to

$$2\frac{B_{44}}{B} - 2\frac{A_{44}}{A} + 2\frac{A_4^2}{A^2} - 2\frac{B_4^2}{B^2} = 16\pi ABC\left(\lambda + \frac{2K}{B^2C^2} \right),$$

(22)

where

$$K = \frac{H^2}{2\,\overline{\overline{\mu}}}.$$

Equations (20) and (22) , after using string dust condition $\left(\in = \lambda\right)$ [Zel'dovich (1980)] , lead to

$$\frac{B_{44}}{B} - 3\frac{A_{44}}{A} - \frac{C_{44}}{C} + 3\frac{A_4^2}{A^2} - \frac{B_4^2}{B^2} + \frac{C_4^2}{C^2} = 16\pi A\left(\frac{K}{BC}\right).$$ (23)

Equations (17) to (20) are four equations, in six unknowns A, B, C, λ, $\in$ and p and, therefore, to deduce a determinate solution, we assume two conditions. First is that the component σ_1^1 of shear tensor σ_i^j is proportional to the expansion (θ) which leads to

$$A = (BC)^n , \qquad \text{where } n > 0$$ (24)

and second is Zel'dovich condition

$$\in = \lambda.$$

Using the first condition [equation (24)], in equation (23), we write

$$(1 - 3n)\frac{B_{44}}{B} - (3n+1)\frac{C_{44}}{C} + (3n-1)\frac{B_4^2}{B^2} + (3n+1)\frac{C_4^2}{C^2} = 16\pi K(BC)^{n-1}.$$ (25)

Now the equation (21), rewrite as

$$\frac{\left(CB_4 - BC_4\right)_4}{\left(CB_4 - BC_4\right)} = \frac{B_4}{B} + \frac{C_4}{C},$$ (26)

which on integrating , we get

$$C^2\left(\frac{B}{C}\right)_4 = LBC,$$ (27)

where L is the constant of integration.

Assume $BC = \mu$, $\frac{B}{C} = v$, then $B^2 = \mu v$, $C^2 = \frac{\mu}{v}$. In view of these relations, equation (27) becomes

$$\frac{v_4}{v} = L.$$ (28)

Now equation (25) after using (24) and assumptions $BC = \mu \; and \; \frac{B}{C} = v$ leads to

$$-3n\frac{\mu_{44}}{\mu} + 3n\left(\frac{\mu_4^2}{\mu^2}\right) - \left(\frac{v_4}{v^2}\right)^2 + \frac{v_{44}}{v} = 16\pi\mu^{(n-1)} K.$$ (29)

The expressions (28) and (29) yield

$$2\mu_{44} - 2\frac{\mu_4^2}{\mu} = \frac{-32\pi\,\mu^n K}{3n}, \tag{30}$$

which reduces to

$$\frac{d}{d\mu}\left[f^2\right] - 2\frac{f^2}{\mu} = \frac{-32\pi\,\mu^n K}{3n}, \tag{31}$$

where $\mu_4 = f(\mu)$.

The differential equation (31) has solution

$$f^2 = P\mu^2 - \frac{32\pi\,K\mu^{n+1}}{3n(n-1)}, \tag{32}$$

where P is the constant of integration.

From equation (28) we write

$$\log v = \int \frac{L\,d\mu}{\sqrt{P\mu^2 - \dfrac{32\pi\,K\mu^{n+1}}{3n(n-1)}}} + \log b, \tag{33}$$

where b is the constant of integration.

Now, using $\mu_4 = f(\mu)$ and expression (32), the metric (4) will be

$$ds^2 = -\frac{d\mu^2}{\left[P\mu^2 - \dfrac{32\pi\,K\mu^{n+1}}{3n(n-1)}\right]} + \mu^{2n}dx^2 + \mu v\,dy^2 + \frac{\mu}{v}dz^2, \tag{34}$$

where v is determined by equation (33).

After suitable transformation of co–ordinates

$$\mu = T,\ x = X,\ y = Y,\ z = Z \ \text{i.e.,}$$

$$d\mu = dT,\quad dx = dX,\ dy = dY,\ dz = dZ.$$

Above metric can be re–written as

$$ds^2 = -\frac{dT^2}{\left[PT^2 - \dfrac{32\pi\,KT^{n+1}}{3n\,(n-1)}\right]} + T^{2n}dX^2 + T\,v\,dY^2 + \frac{T}{v}dZ^2.$$

(35)

This is the Bianchi Type I magnetized cosmological model in bimetric theory of gravitation.

3. Observations

The energy density $\in$, the string tension density λ, the pressure p for the model (35) are

$$\in (= \lambda) = \frac{-(4n+1)}{3n}\,\frac{K}{T^2}\,, \qquad n > 0$$

(36)

$$p = \frac{-2}{3}\,\frac{K}{T^2}.$$

(37)

The strong energy conditions of Hawking and Ellis (1973):

$(\in - p) > 0$ and $(\in + p) > 0$ must be satisfied by our model (35).

For this model (35), these conditions are specifically

$$(-2n-1\,) > 0 \qquad or \qquad n < -\tfrac{1}{2}$$

(38)

and

$$(-6n-1) > 0 \ or \qquad n < -\tfrac{1}{6},$$

which contradict the assumption that $n > 0$.

The expansion θ given by $\left(\dfrac{A_4}{A} + \dfrac{B_4}{B} + \dfrac{C_4}{C}\right)$ becomes

$$\theta = \frac{(n+1)f}{T}, \text{for the model (35),}$$

which [after using (32)] reduces to

$$\theta = (n+1)\left[P - \frac{32\pi\,K\,T^{n-1}}{3n(n-1)}\right]^{\frac{1}{2}}.$$

(39)

The components of shear tensor $\left(\sigma_i^j\right)$ are given by

$$\sigma_1^1 = \frac{1}{3}\left[\frac{2A_4}{A} - \frac{B_4}{B} - \frac{C_4}{C}\right]$$

or

$$\sigma_1^1 = \frac{(2n-1)}{3}\left[P - \frac{32\pi K T^{n-1}}{3n(n-1)}\right]^{\frac{1}{2}} \tag{40}$$

$$\sigma_2^2 = -\frac{1}{2}\ \sigma_1^1 + \frac{L}{2} \tag{41}$$

$$\sigma_3^3 = -\frac{1}{2}\ \sigma_1^1 - \frac{L}{2} \tag{42}$$

$$\sigma_4^4 = 0. \tag{43}$$

In the absence of magnetic field, i.e., for $K = 0$, we have

$$\in\ (= \lambda) = 0, \quad\text{and}\quad p = 0 \tag{44}$$

$$\theta = (n+1)\ P^{\frac{1}{2}} \tag{45}$$

$$\sigma_1^1 = \frac{(2n-1)}{3}\ P^{\frac{1}{2}}$$

$$\sigma_2^2 = \frac{(2n-1)}{6}\ P^{\frac{1}{2}} + \frac{L}{2}$$

$$\sigma_3^3 = \frac{(2n-1)}{6}\ P^{\frac{1}{2}} - \frac{L}{2} \tag{46}$$

$$\sigma_4^4 = 0.$$

4.   Geometrical and Physical Significance

From equations (36) and (37), it is seen that the pressure p the rest energy density $\in$ and the string tension density λ in the model (35), are all negative showing that such a Bianchi Type I magnetized cosmological model (35) does not exist in bimetric theory of gravitation, under the condition $A = (BC)^n$, where $n > 0$ in Rosen's field equations.

The strong energy conditions $(\in - p) > 0$ and $(\in + p) > 0$ stated by Hawking and Ellis (1973) leads to

$$(-2n - 1) > 0 \qquad or \qquad n < -\tfrac{1}{2}$$

and

$$(-6n - 1) > 0 \ or \qquad n < -\tfrac{1}{6},$$

from which, it is observed that n is negative for which the strong energy conditions of Hawking and Ellis (1973) satisfies. But this violet the assuming condition $n > 0$ in $A = (BC)^n$ and therefore it is concludes that the strong energy conditions of Hawking and Ellis (1973) does not satisfied, for $n > 0$.

From the results (39) – (42), it is concludes that the expansion θ and shear σ of the model (35) are imaginary, when $P < 0$ which support the non–existence of the model in the presence of magnetic field.

In the absence of magnetic field, *i.e.,* for $K = 0$, we have $\in = p = 0$ and therefore we get vacuum model and there is no expansion and no shear in the model for constants P and L.

Hence our models (35) strongly suggest a vacuum universe without expansion and shearing and no magnetic field, since forcing it to be in magnetic field jeopardized its very existence.

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