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Iterative Solution of Fractional Diffusion Equation Modelling Anomalous Diffusion

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Abstract

In this article, we study the fractional diffusion equation with spatial Riesz fractional derivative. The continuation of the solution of this fractional equation to the solution of the corresponding integer order equation is proved. The series solution is obtained based on properties of Riesz fractional derivative operator and utilizing the optimal homotopy analysis method (OHAM). Numerical simulations are presented to validate the method and to show the effect of changing the fractional derivative parameter on the solution behavior.

Keywords: Fractional diffusion equation; Riesz derivative; Caputo derivative; Optimal homotopy analysis method; Residual error

MSC 2010 No.: 14F35, 26A33, 76R50

1. Introduction

Fractional derivatives have found numerous applications in different fields of applied science. One process, in which fractional derivatives have been successfully applied, is called anomalous diffusion. This type of diffusion is characterized by the nonlinear dependence of the mean square displacement $x(t)$ of a diffusing particle over time t: $x^2(t) \propto k_\alpha t^\alpha$ and it can be interpreted as the Lévy stable densities. On the other hand, in the case of classical diffusion, linear dependence $x^2(t) \propto kt$ occurs and it follows Gaussian statistics and Fick's second law for running processes at time t. A detailed discussion of anomalous diffusion is presented in (Metzler and Klafter, 2000).

Anomalous diffusion is described by fractional partial differential equations (FPDEs) in which classical derivatives are replaced by derivatives of fractional order. Studies have been devoted for a type of anomalous diffusion modeled by the fractional diffusion equation with spatial Riesz and Riesz-Feller fractional derivatives (see (Gorenflo et al., 2002), (Gorenflo and Vivoli, 2003), (Tarasov and Zaslavsky, 2006), (Ciesielski and Leszczynski, 2006), (Zhang and Liu, 2007), (Lin et al., 2009), (Yang et al., 2010), (Elsaid, 2010) and (Elsaid, 2011)).

Yet very few articles dealt with applying iterative techniques to Riesz FPDEs. This is due to the difficulty in repeated application of Riesz fractional derivative to solution components. This work is based on properties that show repetitive behavior for complex exponential function, hence sine and cosine functions, when Riesz fractional derivative is applied to them (see (Elsaid, 2010) and (Elsaid, 2011)). By representing a function by its Fourier series or Fourier integral, an iterative scheme is deduced for this type of FPDEs.

In this work, the motivation is to establish the continuation of the solution of the fractional-order diffusion equation with spatial derivative in Riesz sense to the exact solution of the corresponding integer-order equation as the order of the fractional derivative approaches its integer limit. This objective is carried out theoretically then via approximate series solution obtained iteratively by applying the OHAM. We consider the space-fractional diffusion equation of the form

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \ k(u)\ R_x^\alpha u(x,t) + p(u), \ \ -\infty < x < \infty, \ t > 0, \\ u(x,0) = f(x), \end{cases} \tag{1}$$

where R_x^α denotes the Riesz fractional derivative (in space) of order α. The parameter α is restricted to the conditions $0 < \alpha < 2$ and $\alpha \neq 1$. The two functions k and p are continuous functions in u.

This paper is organized as follows. In section two, basic definitions of fractional derivative operators involved are presented. Proof of solution continuation is presented in section three. The OHAM is illustrated in section four. The results of numerical experiments are presented in section five. Section six contains the conclusion of this work.

2. Fractional derivatives and integrals

Definition.

A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^m if $f^m \in C_\mu$, $m \in \mathbb{N}$.

Definition.

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(x) \in C_\mu$, $\mu \geq -1$ is defined as

$$\begin{cases} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \ \alpha > 0, \ x > 0, \\ J^0 f(x) = f(x). \end{cases} \tag{2}$$

Definition.

The fractional derivative in Riemann-Liouville sense of $f(x), m \in \mathbb{N}$, $x > 0$ is defined as

$$\mathbf{D}_x^\alpha f(t) = \frac{d^m}{dx^m} J^{m-\beta} f(x), \ m-1 < \beta < m. \tag{3}$$

Definition.

The fractional derivative in Caputo sense of $f(x) \in C_{-1}^m$, $m \in \mathbb{N}$, $x > 0$ is defined as

$$^C D_x^\beta f(x) = \begin{cases} J^{m-\beta} \frac{d^m}{dx^m} f(x), \ m-1 < \beta < m, \\ \frac{d^m}{dx^m} f(x), \qquad \beta = m. \end{cases} \tag{4}$$

Definition.

The Riesz partial fractional derivative R_x^α is defined as (Gorenflo et al., 2002)

$$R_x^\alpha u(x) = -\frac{1}{2\cos(\alpha\pi/2)} [D_+^\alpha u(x) + D_-^\alpha u(x)], \qquad 0 < \alpha < 2, \ \alpha \neq 1, \tag{5}$$

where $D_\pm^\alpha u(x)$ are the Weyl fractional derivatives

$$D_\pm^\alpha u(x) = \begin{cases} \pm \frac{d}{dx} W_\pm^{1-\alpha} u(x), 0 < \alpha < 1, \\ \frac{d^2}{dx^2} W_\pm^{2-\alpha} u(x), 1 < \alpha < 2, \end{cases} \tag{6}$$

and $W_\pm^\beta$ denote the Weyl fractional integrals of order $\beta > 0$, given by

$$\begin{aligned} W_+^\beta u(x) &= \frac{1}{\Gamma(\beta)} \int\limits_{-\infty}^x (x-z)^{\beta-1} u(z) dz, \\ W_-^\beta u(x) &= \frac{1}{\Gamma(\beta)} \int\limits_x^\infty (z-x)^{\beta-1} u(z) dz. \end{aligned} \tag{7}$$

When $\alpha = 0$ the Weyl fractional derivative degenerates into the identity operator

$$D_\pm^0 u(x) = u(x). \tag{8}$$

For continuity we have

$$D_\pm^1 u(x) = \pm \frac{d}{dx} u(x), \ D_\pm^2 u(x) = \frac{d^2}{dx^2} u(x). \tag{9}$$

Evidently, in case $\alpha = 2$, we define

$$R_x^\alpha u(x) = \frac{d^2}{dx^2} u(x). \tag{10}$$

For the case $\alpha = 1$ we have

$$R_x^1 u(x) = \frac{d}{dx} Hu(x) \tag{11}$$

$$= \frac{d}{dx}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(z)}{z-x} dz, \tag{12}$$

where H is the Hilbert transform and the integral is understood in the Cauchy principal value sense.

3. Continuation of the solution

In this section, we prove the continuation of the solution to fractional-order diffusion equation with Riesz spatial derivative to the solution of the corresponding integer-order equation. We begin by defining Riesz fractional derivative in Caputo sense on unbounded domain.

Definition.

The Riesz fractional derivative in Caputo sense is defined by

$$^{C}R_t^{\alpha} f(t) = -\frac{[^{C}D_{+}^{\alpha} f(t) +^{C} D_{-}^{\alpha} f(t)]}{2\cos(\alpha\pi/2)},\ 0 < \alpha < 2,\ \alpha \neq 1, \tag{13}$$

where $^{C}D_{+}^{\alpha}$ and $^{C}D_{-}^{\alpha}$ are Weyl fractional derivatives defined in Caputo sense for $\alpha \in (0, 2)$, $\alpha \neq 1$ as

$$^{C}D_{\pm}^{\alpha} f(t) = \begin{cases} \pm W_{\pm}^{1-\alpha} \frac{d}{dt} f(t),\ 0 < \alpha < 1, \\ W_{\pm}^{2-\alpha} \frac{d^2}{dt^2} f(t),\ 1 < \alpha < 2, \end{cases} \tag{14}$$

and defined for $\alpha = 0$, 1, and 2 as

$$\begin{cases} ^{C}D_{\pm}^{0} f(t) = f(t), \\ ^{C}D_{\pm}^{1} f(t) = \pm\frac{d}{dt} f(t), \\ ^{C}D_{\pm}^{2} f(t) = \frac{d^2}{dt^2} f(t). \end{cases} \tag{15}$$

In the following lemma, we establish the equivalence between the classical definition of Riesz fractional derivative and the definition we proposed in Caputo sense.

Lemma 1.

Let f belong to the class of "good functions" ((Miller and Ross, 1993)). Then for $\alpha \in (0, 2)$, $\alpha \neq 1$,

$$^{C}R_x^{\alpha} f(x) \equiv R_x^{\alpha} f(x).$$

Proof:

Consider the case $\alpha \in (0, 1)$, Riesz fractional derivative is defined by

$$R_x^{\alpha} f(x) = \frac{-C(\alpha)}{2}\frac{d}{dx}[\int_{-\infty}^{x} \frac{f(z)}{(x-z)^{\alpha}} dz - \int_{x}^{\infty} \frac{f(y)}{(y-x)^{\alpha}} dy]. \tag{16}$$

where $C(\alpha) = \frac{1}{\cos(\alpha\pi/2)\Gamma(1-\alpha)}$. Substituting $x-z=\lambda$ and $y-x=\mu$ in first and second integrals, respectively,

$$R_x^\alpha f(x) = \frac{C(\alpha)}{2}\frac{d}{dx}[\int_\infty^0 \frac{f(x-\lambda)}{\lambda^\alpha}d\lambda + \int_0^\infty \frac{f(x+\mu)}{\mu^\alpha}d\mu], \tag{17}$$

which can be written as

$$\begin{aligned} cR_x^\alpha f(x) &= \frac{C(\alpha)}{2}\frac{d}{dx}\int_0^\infty \frac{f(x+\tau)-f(x-\tau)}{\tau^\alpha}d\tau \\ &= \frac{C(\alpha)}{2}\int_0^\infty \frac{\partial}{\partial x}\left[\frac{f(x+\tau)-f(x-\tau)}{\tau^\alpha}\right]d\tau. \end{aligned} \tag{18}$$

For ${}^C R_x^\alpha$, we have

$${}^C R_x^\alpha f(x) = \frac{-C(\alpha)}{2}[\int_{-\infty}^x \frac{f'(z)}{(x-z)^\alpha}dz - \int_x^\infty \frac{f'(y)}{(y-x)^\alpha}dy], \tag{19}$$

which by the same substitution yields

$${}^C R_x^\alpha f(x) = \frac{C(\alpha)}{2}\int_0^\infty \left[\frac{f^{'}(x+\tau)-f^{'}(x-\tau)}{\tau^\alpha}\right]d\tau, \tag{20}$$

which is the same as (18). The Case $\alpha \in (1,2)$ can be proved in a similar manner. □

As the equivalence between the Riemann and Caputo definitions of Riesz fractional derivative is deduced, in the following theorem, the continuation of the solution obtained is proved.

Theorem 1.

If f(x) is a function in $L^1(-\infty,\infty)$, then the exact solution u_α of the space fractional diffusion equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = & R_x^\alpha u(x,t), \quad -\infty < x < \infty, \ t>0, \\ & u(x,0) = f(x), \end{cases} \tag{21}$$

is given by

$$u_\alpha(x,t) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \exp(-\omega^\alpha t)f(v)\cos(\omega(x-v))dvd\omega. \tag{22}$$

Theorem 2.

Let $\alpha \in (1,2)$, $f(x)$ be a function in $L^1(-\infty,\infty)$, and u_α displayed in (22) be the solution of the space-fractional problem (21), then

$$\lim_{\alpha\to 2} u_\alpha(x,t) = u(x,t),$$

where $u(x,t)$ is the exact solution of the integer-order diffusion equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), \ -\infty < x < \infty, \ t > 0, \\ \qquad u(x,0) = f(x). \end{cases} \tag{23}$$

Proof:

Consider the set of functions

$$\begin{aligned} \varphi_n(\omega) &= \frac{1}{\pi}\exp(-\omega^{2-\frac{1}{n+1}}t)\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v))dv, \\ \omega &\in (0,\infty), \ n = 1,2,.... \end{aligned} \tag{24}$$

Then,

$$|\varphi_n(\omega)| \leq \frac{1}{\pi}\left|\exp(-\omega^{2-\frac{1}{n+1}}t)\right|\int_{-\infty}^{\infty}|f(v)|\,dv,$$

and as $f(x) \in L^1(-\infty,\infty)$, there exists a constant $M > 0$ such that

$$|\varphi_n(\omega)| \leq \frac{M}{\pi}\exp(-\omega^{2-\frac{1}{n+1}}t). \tag{25}$$

Then, for a fixed time $t > 0$ and $n = 1,2,...$

$$|\varphi_n(\omega)| \leq g(\omega), \ \omega \in (0,\infty), \tag{26}$$

where

$$g(\omega) = \frac{M}{\pi}\exp(-\omega^{3/2}).$$

Since $g(\omega)$ belongs to $L^1(0,\infty)$ and setting $\alpha = 2 - \frac{1}{n+1}$, by Lebesgue dominated convergence theorem we have

$$\begin{aligned} c\lim_{\alpha\to 2} u_\alpha(x,t) &= \lim_{n\to\infty}\int_0^{\infty}\varphi_n(\omega)d\omega \\ &= \int_0^{\infty}\lim_{n\to\infty}\varphi_n(\omega)d\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\exp(-\omega^2 t)\ f(v)\cos[\omega(x-v)]dvd\omega \\ &= u(x,t), \end{aligned} \tag{27}$$

which is the exact solution of the integer-order diffusion equation (23). □

4. Optimal homotopy analysis method (OHAM)

We begin by illustrating the classical homotopy analysis method (HAM). Consider the following nonlinear equation

$$N[u(x,t)] = 0, \tag{28}$$

where N is a nonlinear operator, $u(x,t)$ is the unknown function and x and t denote spatial and temporal independent variables, respectively. By generalizing the traditional homotopy method Liao constructs the so-called zero-order deformation equation (Liao, 2003)

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = p\hbar H(x,t)N[\phi(x,t;p)], \tag{29}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, $H(x,t)$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t;p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$, we have $\phi(x,t;0) = u_0(x,t)$, $\phi(x,t;1) = u(x,t)$, respectively. Thus, as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. By expanding $\phi(x,t;p)$ in a Taylor series with respect to p, we have

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m, \tag{30}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m} \mid_{p=0} . \tag{31}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then, as proved in (Liao, 2003), series (30) converges at $p = 1$ and one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{32}$$

which must be one of solutions of the original nonlinear equation, as proved in (Liao, 2003). Using definition (31), the governing equation of the HAM can be deduced from the zero-order deformation equation (29) as follows. Define the vector

$$\overrightarrow{u}_n = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}. \tag{33}$$

From equation (29), the so-called m th-order deformation equation is given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)\Re_m[\overrightarrow{u}_{m-1}(x,t)], \tag{34}$$

where

$$\Re_m[\overrightarrow{u}_{m-1}] = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}} \mid_{p=0}, \tag{35}$$

and

$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1. \end{cases} \tag{36}$$

Applying the inverse operator L^{-1} to both sides of (34), $u_m(x,t)$ can be easily solved for by symbolic computations software. The HAM has been successfully applied to solve various classes of equations and applied problems (see (Wu et al., 2005), (Song and Zhang, 2007), (Bouremel, 2007), (Song and Zhang, 2009), (Cang et al., 2009), (Sedighi et al., 2012) and (Martin, 2013)).

In the classical HAM, choosing the value of parameter $\hbar$ depends on inspecting the graph of the quantity of interest; the solution or one of its derivatives. Yet, when $H(x,t)$ is fixed, it is obvious that $u_m(x,t)$ contains only one control parameter $\hbar$. Thus, by constructing a formula for the residual error, in (Liao, 2010) Liao proposed that the OHAM solution be obtained by choosing the value for parameter $\hbar$ that minimizes the error. His technique is deduced from the optimal homotopy asymptotic method presented in the work of Marinca et al., (see (Marinca et al., 2008), (Marinca and Herişanu, 2008), and (Marinca et al., 2009)). Here, the averaged residual error defined for ordinary differential equations in (Liao, 2010) is generalized to the case of two variable partial differential equations in the following form

$$E_m(\hbar) = \frac{1}{MK}\sum_{i=0}^{M}\sum_{j=0}^{K}\left[N\sum_{n=0}^{m} u_n\left(\frac{i}{M},\frac{j}{K}\right)\right]^2, \tag{37}$$

which is a nonlinear algebraic equation of one unknown, the convergence-control parameter $\hbar$. Thus the optimal value of $\hbar$ is determined by the minimum of the averaged residual error E_m to ensure the fast convergence of the homotopy series.

To apply the OHAM recursive technique to the problem, a repeated evaluation of Riesz fractional derivative to solution components is needed. This obstacle is overcome by using property of Riesz fractional derivative in the following lemma.

Lemma 2.

Let $\alpha \in (0,2)$, $\alpha \neq 1$. Then,

$$R_x^\alpha(e^{i\omega x}) = -\omega^\alpha e^{i(\omega x)}, \tag{38}$$

or in a trigonometric form

$$R_x^\alpha \sin(\omega x) = -\omega^\alpha \sin(\omega x), \tag{39}$$

$$R_x^\alpha \cos(\omega x) = -\omega^\alpha \cos(\omega x). \tag{40}$$

Proof:

See (Elsaid, 2011) and (Elsaid, 2010). □

5. Numerical simulation

In this section, we consider linear and nonlinear problems to illustrate the efficiency of the method of solution to this type of problems and to illustrate the continuation of the solution we proved in Section 3.

Example 1.

Consider problem (1) with $k(u) = 1,\ g(u) = a\ u$ and $f(x) = A\ \sin(\pi x/b)$

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = R_x^\alpha u(x,t) + a\ u, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = A\ \sin(\pi x/b), & \end{cases} \tag{41}$$

where $a,\ A,$ and b are real constants.

Here m th-order deformation equation, with $H(x,t) = 1$, for this linear problem is given by

$$\frac{\partial}{\partial t}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar\left(\frac{\partial}{\partial t}(u_{m-1}) - R_x^\alpha(u_{m-1}) - u_{m-1}\right). \tag{42}$$

Then the inverse integral operator is applied to both sides with $u_0 = u(x,0) = A\ \sin(\pi x/b)$ to obtain the series solution terms. The first three terms are given by

$$\begin{aligned} cu_0 &= A\ \sin\left(\frac{\pi x}{b}\right) \\ u_1 &= Ah\left(-a + \left(\frac{\pi}{b}\right)^\alpha\right)t\ \sin\left(\frac{\pi x}{b}\right) \\ u_2 &= \frac{1}{2}Ah\left(-a + \left(\frac{\pi}{b}\right)^\alpha\right)\left(2 + h\left(2 - at + \left(\frac{\pi}{b}\right)^\alpha t\right)\right)\ t\ \sin\left(\frac{\pi x}{b}\right) \\ u_3 &= \frac{1}{6}Ah\left(-a + \left(\frac{\pi}{b}\right)^\alpha\right)[6 + h\left(12 - 6at + 6\left(\frac{\pi}{b}\right)^\alpha\right)t \\ &\quad + h\left(6 + \left(-a + \left(\frac{\pi}{b}\right)^\alpha\right)\left(6 - at + \left(\frac{\pi}{b}\right)^\alpha t\right)t\right)]t\ \sin\left(\frac{\pi x}{b}\right). \end{aligned}$$

The series solution is $u = u_0 + u_1 + u_2 + u_3 + ...$ Figure (1) shows the effect of the fractional order derivative α on the behavior of the solution which indicates that the amplitude of the solution is attenuated as α increases. The plots represents the sum of the first six terms (u_0 to u_5) in the OHAM series when $a = 1.0,\ A = 0.1\,,\ b = 1.0,\ t = 0.2,\ 0 < x < 1$ and the fractional parameter $\alpha = 1.7,\ 1.8, 1.9$ and 2. At these different values of $\alpha = 1.7,\ 1.8, 1.9$ and 2,the optimal convergence control parameter $\hbar$ is calculated as be $\hbar = -0.908224,\ -0.896584, -0.883822$ and -0.86996, and the residual error is $E = 5.5 * 10^{-9},\ 2.7 * 10^{-8},\ 1.1 * 10^{-7}$ and $4.7 * 10^{-7}$, respectively.

Example 2.

Consider the nonlinear space-fractional diffusion problem (1) with $k(u) = u,\ g(u) = 4\ u^2 - 4u$ and $f(x) = \sin(2x)$

$$u_t = u\ R_x^\alpha u + 4u^2 - 4u,\ \ -\infty < x < \infty,\ t > 0, \tag{43}$$

subject to the initial condition

$$u(x,0) = \sin(2x).$$

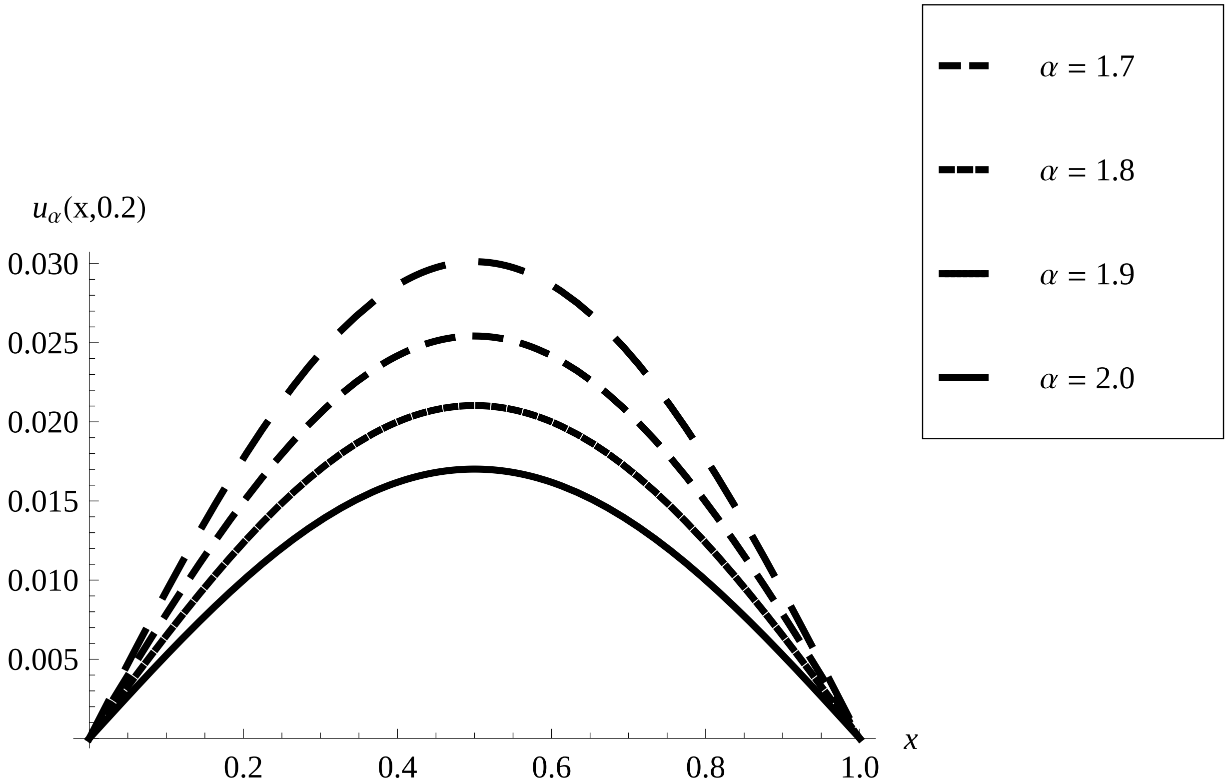

Figure 1: The behavior of the solution of (41) when $a = 1.0, A = 0.1, b = 1.0, t = 0.2$ and the fractional parameter $\alpha = 1.7, 1.8, 1.9$ and 2.

Here we choose the auxiliary linear operator as

$$L[\phi] = \frac{\partial}{\partial t}(\phi), \tag{44}$$

and operator N is chosen as

$$N[\phi] = \phi_t - \phi R_x^\alpha(\phi) - 4\phi^2 + 4\phi. \tag{45}$$

Then, mth-order deformation equation for this problem is given by

$$\frac{\partial}{\partial t}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)\Re_m[\overrightarrow{u}_{m-1}(x,t)], \tag{46}$$

where $\Re_m[\overrightarrow{u}_{m-1}(x,t)]$ is given by

$$\Re_m[\overrightarrow{u}_{m-1}(x,t)] = \frac{\partial}{\partial t}(u_{m-1}) - \sum_{k=0}^{m-1} u_k R_x^\alpha(u_{m-1-k}) - 4\sum_{k=0}^{m-1} u_k u_{m-1-k} + 4u_{m-1}. \tag{47}$$

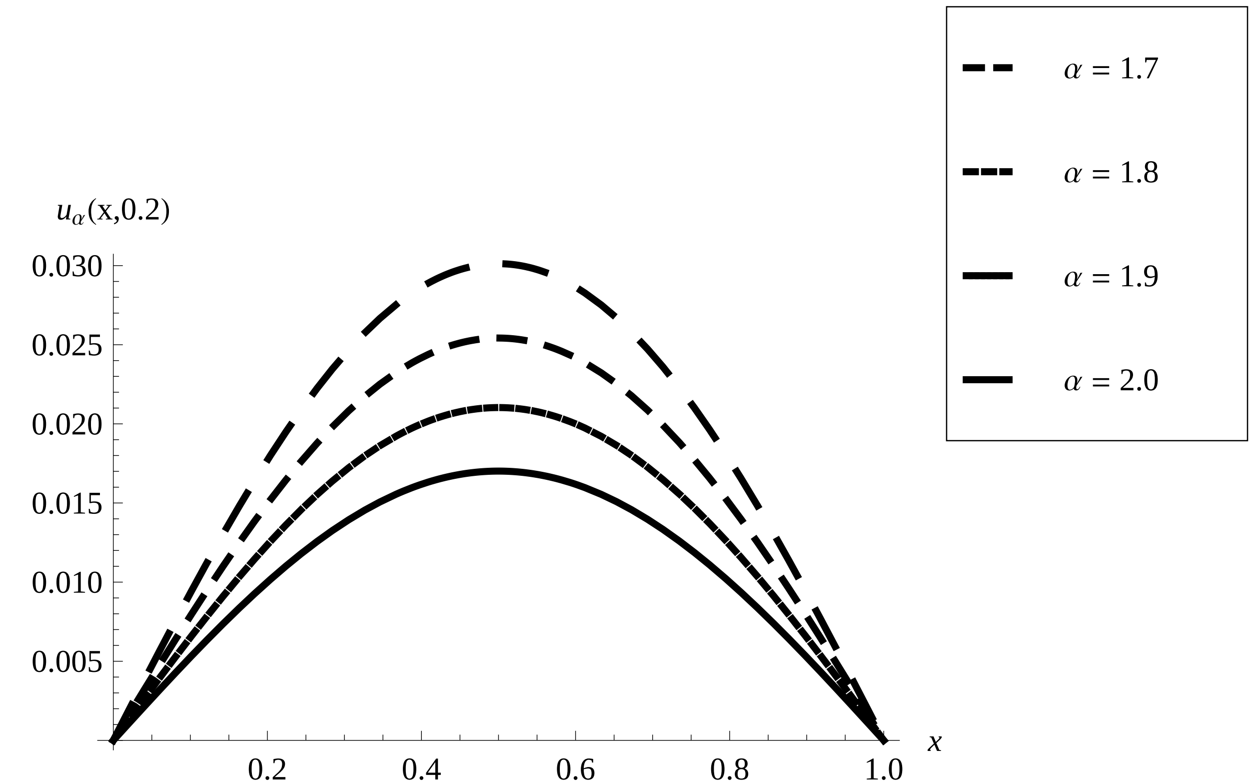

Figure 2: The series solution of Example 2 at different values of a compared with the exact solution of the corresponding integer order problem.

We choose $H(x,t) = 1$ and $u_0 = \sin(2x)$. By applying the inverse integral operator, we obtain

$$
\begin{aligned}
cu_0 &= \sin(2x) \\
u_1 &= \frac{1}{2}\hbar t\ \left(-4 + 2^{\alpha} + 4\cos(4x) - 2^{\alpha}\cos(4x) + 8\sin(2x)\right) \\
u_2 &= -\frac{1}{4}\hbar t sin(2x)[-16 - 16\hbar - 64\hbar t - 4^{\alpha}\hbar t + 32\hbar t cos(4x) + 4^{\alpha}\left(1 + 2^{\alpha}\right)\hbar t cos(4x) \\
&\quad -(3)\ 2^{2+\alpha}\hbar t cos(4x) - 4^{1+\alpha}\hbar t cos(4x) + (3)\ 2^{2+\alpha}\hbar t + 16 sin(2x) + 16\hbar sin(2x] \\
&\quad +96\hbar t sin(2x) - 2^{2+\alpha} sin(2x) - 2^{2+\alpha}\hbar sin(2x) - 3\ 2^{3+\alpha}\hbar t sin(2x)] \\
&\quad \vdots
\end{aligned}
$$

and the solution is thus obtained as

$$u = u_0 + u_1 + u_2 + u_3 + \ldots.$$

The solution behavior as the Riesz parameter α changes is shown in Figure (2) at a fixed time $t = 0.2$. As α increases, the amplitude of the sinusoidal behavior in solution decreases. As α tends to 2, the series solution approximately coincides with the exact solution of the corresponding integer order equation ($u = e^{-4t}\sin(2x)$, represented by the solid line in the graph). The optimal convergence parameter $\hbar$ in each case is obtained by minimizing the residual error displayed in (37) for $K = 10, M = 5$ in the interval $0 \le x \le 2$ and $0 \le t \le 0.25$. The series displayed in plots is the partial sum of the first five terms; $n = 5$ (summing u_0 to u_4). At different values of the fractional parameter $\alpha = 1.7,\ 1.8$ and 1.9, the optimal convergence control parameter is found to be $\hbar = -0.6720,\ -0.6431$ and -0.6300, and the residual error is $E = 0.00059, 0.00111$ and 0.00150, respectively.

6. Conclusion

A definition of the fractional-order Riesz derivative in the Caputo sense is proposed and its equivalence with the classical definition is proved. Then, we proved the continuation of the solution of the fractional order anomalous diffusion equation with Riesz spatial derivative to the corresponding integer order problem. The iterative series solution for the fractional equation is obtained using the OHAM. The advantage of using this technique is the ability to estimate an approximation to the residual error. The results obtained illustrate graphically the continuation of the solution we proved theoretically.

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