Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 05, Issue 2 (December 2010), pp. 488 – 494
(Previously, Vol. 05, Issue 10, pp. 1585 – 1591)

Applications and Applied Mathematics:
An International Journal
(AAM)

Optimal Correction of Infeasible System in Linear Equality via Genetic Algorithm

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Received: March 21, 2010; Accepted: November 17, 2010

Abstract

This work is focused on the optimal correction of infeasible system of linear equality. In this paper, for correcting this system, we will make the changes just in the coefficient matrix by using l_2 norm and show that solving this problem is equivalent to solving a fractional quadratic problem. To solve this problem, we use the genetic algorithm. Some examples are provided to illustrate the efficiency and validity of the proposed method.

Keywords: Fractional problem; genetic algorithm; infeasible system; optimal correction

MSC 2000: 15A39, 90C25

1. Introduction

One of the frequently encountered issues in applied science is how to deal with infeasible systems [Censor et al. (2008)]. Here, we consider an infeasible system of linear equality with following form:

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$. In other words, there is no $x \in R^n$ for which (1) is feasible.

We could argue numerous reasons for the infeasibility of the system (1), including errors in data, errors in modeling, and many other reasons [Amaral et al. (2008)]. Because the remodeling of a problem, finding its errors, and generally removing its obstacles to feasibility might require a considerable amount of time and expense-and might result in yet another infeasible system-we are reluctant to do so. We, therefore, focus on optimal correction of the given system.

In this work, for correcting this system, we will make the least changes just in the coefficient matrix and obtain a fractional objective function of two quadratic functions as follows:

$$\min_x \frac{\|(Ax-b)\|^2}{\|x\|^2}. \tag{2}$$

The genetic algorithm [Mitchell (1997)] applied for solving (2) and results shows that this method is efficient with high accuracy.

This paper is organized as follows: In Section 2, genetic algorithm is reviewed. Optimal correction of infeasible linear equality system is demonstrated in Section 3. In Section 4, some numerical examples are tested. Concluding remarks are given in Section 5.

We now describe our notation. In this paper all vectors will be column vectors and we denote the n-dimensional real space by $\mathbb{R}^n$. The notation $A \in R^{m \times n}$ will signify a real $m \times n$ matrix. We mean by A^T, $\|.\|$ and $\|.\|_\infty$, the transpose of matrix A and Euclidean norm and infinity norm respectively.

2. The Genetic Algorithm

GAS are searching techniques using the mechanics of natural selection and natural genetics for efficient global searches [Mitchell (1997)]. In comparison to the conventional searching algorithms, GAS has the following characteristics: (a) GAS works directly with the discrete points coded by finite length strings (chromosomes), not the real parameters themselves; (b) GAS considers a group of points (called a population size) in the search space in every iteration, not a single point; (c) GAS uses fitness function information instead of derivatives or other auxiliary knowledge; and (d) GAS uses probabilistic transition rules instead of deterministic rules. Generally, a simple GA consists of the three basic genetic operators: (a) Reproduction; (b) Crossover; and (c) Mutation. They are described as follows.

2.1. Reproduction

Reproduction is a process for deciding how many copies of individual strings should be produced in the mating pool according to their fitness value. The reproduction operation allows strings with higher fitness value to have larger number of copies, and the strings with lower fitness values have a relatively smaller number of copies or even none at all. This is an artificial version of natural selection (strings with higher fitness values will have more chances to survive).

2.2. Crossover

Crossover is a recombined operator for two high-fitness strings (parents) to produce two offsprings by matching their desirable qualities through a random process. In this paper, the uniform crossover method is adopted. The procedure is to select a pair of strings from the mating pool at random, then, a mark is selected at random. Finally, two new strings are generated by swapping all characters correspond to the position of the mark where the bit is “1”. Although the crossover is done by random selection, it is not the same as a random search through the search space. Since it is based on the reproduction process, it is an effective means of exchanging information and combining portions of high-fitness solutions.

2.3. Mutation

Mutation is a process for providing an occasional random alteration of the value at a particular string position. In the case of binary string, this simply means changing the state of a bit from 1 to 0 and vice versa. In this paper we provide a uniform mutation method. This method is to first, produce a mask and select a string randomly, and then complement the selected string value corresponds to the position of mask where the bit value is “1”. Mutation is needed because some digits at particular position in all strings may be eliminated during the reproduction and the crossover operations. So the mutation plays the role of a safeguard in GAS. It can help GAS to avoid the possibility of mistaking a local optimum for a global optimum [Marczyk (2004)].

3. Optimal Correction of Infeasible Linear Equality System

In this section, for correcting the infeasible linear equality system (1), we can make the changes just in the coefficient matrix. Therefore, we must consider the following minimization problem:

$$\min_{x} \min_{H} \|H\|^2$$

$$(A + H)x = b, \tag{3}$$

where $H \in \mathbb{R}^{m\times n}$ is a perturbation matrix. In order to simplify problem (3), we consider the following inner minimization problem:

$$\min_{H} \|H\|^2$$

$$\text{s.t} \qquad (A + H)x = b, \tag{4}$$

which is a constrained convex problem.

Theorem 1. Suppose that H^* denote the optimal solution of problem (4), then

$$H^* = -\frac{(Ax^*-b)x^{*T}}{\|x^*\|^2}, \tag{5}$$

where x^* is the optimal solution of

$$\min_x \frac{\|(Ax-b)\|^2}{\|x\|^2} \ . \tag{6}$$

Proof:

The Lagrangian function of the problem (4) is given by

$$L(H,\lambda) = \|H\|^2 - \lambda^T((A+H)x-b).$$

Since the problem (4) is convex, then the KKT necessary conditions are also sufficient and any H satisfying the KKT conditions is a global minimum [Bazaraa et al. (1993)].The KKT conditions of (4) give:

$$\frac{\partial L}{\partial H} = (2H - \lambda x^T) = 0,$$
$$\frac{\partial L}{\partial \lambda} = (A+H)x - b = 0,$$

where λ denotes the Lagrange multiplier. From the first equation we have $H = \frac{\lambda x^T}{2}$.

Finally, the second equation implies that

$$\lambda = -\frac{2(Ax-b)}{\|x\|^2},$$

and subsequently

$$H = -\frac{(Ax-b)x^T}{\|x\|^2} \ .$$

Then, we have: $\|H\|^2 = \frac{\|(Ax-b)\|^2}{\|x\|^2}$ and value of problem (4) at optimal solution is equal to: $\|H^*\|^2 = \min\limits_x \frac{\|(Ax-b)\|^2}{\|x\|^2}$, this completes the proof.

4. Computational Results

In this section we present numerical results to obtain optimal correction of infeasibility system in linear equality on various randomly generated problems. We used the genetic algorithms for solving (6). The algorithm has been tested using MATLAB 7.9.0 on a Core 2 Duo 2.53 GHz with main memory 4 GB.

Test problems are generated infeasible system (1) by using the following MATLAB code:

```
%Sgen: Generate random infeasible system

(% Input:m,n,d(density);    Output: A ∈ ℝ^(m×n) , b ∈ ℝ^m).

pl=inline('(abs(x)+x)/2');     %pl(us) function;

m=input('enter m= ');   n=input('enter n= ');  d=input('enter d= ');

m1=max(m-round(0.5*m),m-n);

A1=sprand(m1,n,d);  A1=1*(A1-0.5*spones(A1));

x=spdiags(rand(n,1),0,n,n)*10*(rand(n,1)-rand(n,1));

x=spdiags(ones(n,1)-sign(x),0,n,n)*10*(rand(n,1)-rand(n,1));

m2=m-m1;

u=randperm(m2);A2=A1(u, :);

b1=A1*x+spdiags((rand(m1,1)),0,m1,m1)*1*ones(m1,1);

b2=b1(u)+spdiags((rand(m2,1)),0,m2,m2)*10*ones(m2,1);

A=100*[A1;-A2];

b=[b1;-b2];
```

In Table 1, we present numerical experiment. In this Table, “fval” is optimal solution of (6) and x^* is the optimal point of (6) and $R = \|(A + H)x^* - b\|_\infty$.

Tables 1: Numerical experiment to obtain optimal correction of infeasible system in linear equality on various randomly generated problems (d=0.1).

Problems	m	n	fval	$\|\|x^*\|\|$	R
Test 1	10	10	5.7845	14.213	6.5e-15
Test 2	20	15	72.431	17.286	2.28e-14
Test 3	30	25	195.7117	22.753	6.53e-14
Test4	50	40	365.1035	28.416	1.52e-13
Test5	80	60	240.6992	34.015	3.3e-13
Test6	100	80	998.322	40	4.6e-13
Test7	200	150	1733.52	53.28	1.9e-12
Test8	400	250	5111.9	71.067	2.34e-12
Test9	700	450	6399.1	90.77	4.8e-12
Test10	1000	800	8451.6	110.21	7.93e-12

This results show that the genetic algorithm for correcting of infeasible system in linear equality is efficient with high accuracy.

5. Conclusion

We have successfully shown how to correct infeasible systems of linear equalities by making minimal changes just in the coefficient matrix. For this problem we need to obtain the global minimum of fractional programming. We have also presented a genetic algorithm to do this. Our computational experiments on several randomly generated problems demonstrate the superior performance of this method.

Acknowledgement

The authors would like to thank editor and referees for their helpful comments and constructive suggestions which have improved to the final preparation of the paper.

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