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Solution for System Of Fractional Partial Differential Equations

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Abstract

The purpose of this article is to discuss solutions of different initial value problems (IVPs) for system of fractional differential equations. These equations appear in physical processes such as transportation and anomalous diffusion. The iteration method is successfully developed and series solution of IVPs at hand are obtained which converges to a function known as solution function of the IVPs. Graphical representation of solution of some IVPs are given using Mathematical software "MATLAB".

Keywords: Riemann-Liouville fractional integral; Caputo fractional derivative; System of fractional transport and Burger's equations; Mittag-Leffler functions; Iteration method

MSC 2010 No.: 35C10, 35F40, 35F55

1. Introduction

Many problems in mathematical, physical, chemical, biological sciences and technologies are governed by differential equations. In recent years, fractional differential equations have attracted many researchers due to their applications in the field of visco elasticity, feed back amplifiers, electrical circuits, electroanalytical chemistry, fractional multipoles, etc. (see Debnath (2003), Metzler and Klafter (2000), Daftardar-Gejji and Bhalekar (2006)).

Various methods such as Green's function method (Seneider and Wyss (1989)), Finite sine transform method by Agrawal (2002), method of images and Fourier transform (Metzler and Klafter (2000)), separation of variables (Daftardar-Gejji and Bhalekar (2006)), Adomian decomposition method (Adomian (1994), Adomian (1998), Babolian, Vahidi and Shoja (2014), Dhaigude, Jadhav-Kanade and Mahmood (2014), Jawad (2015), Khodabakhshi, Vaezpour and Baleanu (2014), Kucuk, Yidiger and Celik (2014), Pratiban and Balachandran (2012), Saha and Ray (2014)), and an iterative method (Bhalekar, Daftardar-Gejji (2008), Daftardar-Gejji and Bhalekar (2008), Daftardar-Gejji and Jafari (2006), Dhaigude and Dhaigude (2012), Dhaigude and Nikam (2012), Dhaigude, Kanade and Dhaigude (2016), Kanade and Dhaigude (2016), Kokak and Yeildirim (2011), ur Rahman, Yaseen and Kamran (2016)) are successfully applied to obtain solution of varies problems.

We organize the paper as follows: In Section 2 useful definitions of fractional calculus are given. In Section 3, we develop the iterative method for system of equations. Section 4 consists of solutions of initial value problems for system of fractional partial differential equations. In Section 5 the solutions of initial value problems for system of fractional transport equations as well as fractional Burger's equations are obtained as an application of above iterative method. Concluding remarks are in the last section.

2. Preliminaries

We define the Caputo partial fractional derivative. It follows the Riemann-Liouville fractional integral (see Podlubny (1999), Khilbas, Shivastava and Trujillo (2006)).

Definition 2.1.

The (left sided) Riemann-Liouville fractional integral of order μ, $\mu > 0$ of a function $u(x,t) \in C_\alpha$, $\alpha \geq -1$ is denoted by $I_t^\mu u(x,t)$ and is defined as

$$I_t^\mu u(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} u(x,\tau) d\tau,\ t > 0.$$

Definition 2.2.

The (left sided) Caputo partial fractional derivative of a function $u(x,t) \in C_l^m$, with respect to "t" is denoted by $D_t^\mu u(x,t)$ and is defined as

$$D_t^\mu u(x,t) = \begin{cases} \frac{\partial^m}{\partial t^m} u(x,t), & \mu = m,\ m \in N, \\ I_t^{m-\mu} \frac{\partial^m}{\partial t^m} u(x,t), & m-1 < \mu < m, \end{cases}$$

where $I_t^\mu u(x,t)$ is Riemann-Liouville fractional integral of order μ, $\mu > 0$.

Note that,

$$I_t^\mu D_t^\mu u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k} u(x,0) \frac{t^k}{k!}, \quad m-1 < \mu \leq m, \; m \in N,$$

and

$$I_t^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\nu+\mu}.$$

Mittag-Leffler Function (Podlubny (1999)).

In 1902, Mittag-Leffler introduced the one parameter function commonly known as Mittag-Leffler function which is denoted by $E_\alpha(z)$ and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (\alpha > 0). \tag{1}$$

■

Example 2.4.

If we put $\alpha = 1$ then equation (1) becomes,

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Example 2.5.

If we put $\alpha = 2$ then equation (1) becomes,

$$E_2(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k!)} = \cosh z.$$

3. Analysis of Iterative Method

Now we extend the iterative method for system of equations developed in Daftardar-Gejji and Jafari (2006). Consider the system of functional equations,

$$u(x,t) = f(x,t) + L(u(x,t)) + N(u(x,t), v(x,t)), \tag{2}$$

$$v(x,t) = g(x,t) + M(v(x,t)) + P(u(x,t), v(x,t)), \tag{3}$$

where $f(x,t)$ and $g(x,t)$ are known continuous functions. Note that L, M are linear and N, P are linear in lower order or nonlinear operators in lower order respectively. Consider series solutions of the equations (2) and (3) as

$$u(x,t) = \sum u_i(x,t) = u_0 + u_1 + u_2 + ..., \tag{4}$$

$$v(x,t) = \sum v_i(x,t) = v_0 + v_1 + v_2 + \dots . \tag{5}$$

Since L and M are linear operators, we have,

$$L\bigg(\sum_{i=0}^{\infty} u_i(x,t)\bigg) = L(u_0(x,t)) + L(u_1(x,t)) + ..., \tag{6}$$

$$M\bigg(\sum_{i=0}^{\infty} v_i(x,t)\bigg) = M(v_0(x,t)) + M(v_1(x,t)) + \dots . \tag{7}$$

Decompose nonlinear operators N and P as in Gejji and Jafari (2006),

$$N\left(\sum_{i=0}^{\infty} u_i, \sum_{i=0}^{\infty} v_i\right) = N(u_0, v_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j, \sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} u_j, \sum_{j=0}^{i-1} v_j\right)\right\}, \tag{8}$$

$$P\left(\sum_{i=0}^{\infty} v_i, \sum_{i=0}^{\infty} u_i\right) = P(v_0, u_0) + \sum_{i=1}^{\infty}\left\{P\left(\sum_{j=0}^{i} v_j, \sum_{j=0}^{i} u_j\right) - P\left(\sum_{j=0}^{i-1} v_j, \sum_{j=0}^{i-1} u_j\right)\right\}. \tag{9}$$

From Equations (4), (6) and (8), then Equation (2) is equivalent to

$$\begin{aligned}\sum_{i=0}^{\infty} u_i = f(x,t) + \sum_{i=0}^{\infty} L(u_i) + N(u_0, v_0) +& \\ \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j, \sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} u_j, \sum_{j=0}^{i-1} v_j\right)\right\}&,\end{aligned} \tag{10}$$

and from Equations (5), (7) and (9), then Equation (3) is equivalent to

$$\begin{aligned}\sum_{i=0}^{\infty} v_i = g(x,t) + \sum_{i=0}^{\infty} M(v_i) + P(v_0, u_0) +& \\ \sum_{i=1}^{\infty}\left\{P\left(\sum_{j=0}^{i} v_j, \sum_{j=0}^{i} u_j\right) - P\left(\sum_{j=0}^{i-1} v_j, \sum_{j=0}^{i-1} u_j\right)\right\}&,\end{aligned} \tag{11}$$

$$\begin{aligned}u_0 + u_1 + .. = f + L(u_0) + L(u_1) + L(u_2) + ... + N(u_0, v_0) + [N(u_0 + u_1, v_0 + v_1)\\ - N(u_0, v_0)] + [N(u_0 + u_1 + u_2, v_0 + v_1 + v_2) - N(u_0 + u_1, v_0 + v_1)]\\ + [N(u_0 + u_1 + u_2 + u_3, v_0 + v_1 + v_2 + v_3) -\\ N(u_0 + u_1 + u_2, v_0 + v_1 + v_2)] + ...,\end{aligned}$$

and

$$\begin{aligned}v_0 + v_1 + .. = g + M(v_0) + M(v_1) + M(v_2) + ... + P(v_0, u_0) + [P(v_0 + v_1, u_0 + u_1)\\ - P(v_0 + u_0)] + [P(v_0 + v_1 + v_2, u_0 + u_1 + u_2) - P(v_0 + v_1, u_0 + u_1)]\\ + [P(v_0 + v_1 + v_2 + v_3, u_0 + u_1 + u_2 + u_3) -\\ P(v_0 + v_1 + v_2, u_0 + u_1 + u_2)] + \dots .\end{aligned}$$

Now we define the iterations as follows:

$$\begin{aligned}
u_0 &= f,\\
v_0 &= g,\\
u_1 &= L(u_0) + N(u_0, v_0),\\
v_1 &= M(v_0) + P(v_0, u_0),\\
u_2 &= L(u_1) + [N(u_0 + u_1, v_0 + v_1) - N(u_0, v_0)],\\
v_2 &= M(v_1) + [P(v_0 + v_1, u_0 + u_1) - P(v_0, u_0)],\\
u_{m+1} &= L(u_m) + [N(u_0 + ... + u_m, v_0 + ...+, v_m) - N(u_0 + ... + u_{m-1},\\
&\qquad v_0 + ...+, v_{m-1})], m = 0, 1, 2, ...,\\
v_{m+1} &= M(v_m) + [P(v_0 + ... + v_m, u_0 + ... + u_m) - P(v_0 + ... + v_{m-1},\\
&\qquad u_0 + ... + u_{m-1})], m = 0, 1, 2, ...
\end{aligned}$$

4. Fractional Initial Value Problem

Consider the IVP for general system of fractional partial differential equations

$$\begin{aligned}
D_t^\alpha u(x,t) = \sum_{j=1}^{n} a_j D_{x_j}^{\delta_j} u(x,t) + \sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t) +\\
\sum_{j=1}^{n} c_j D_{x_j}^{\gamma_j} u(x,t) + N(u(x,t), v(x,t));
\end{aligned} \tag{12}$$

$$\begin{aligned}
D_t^\alpha v(x,t) = \sum_{j=1}^{n} a_j^* D_{x_j}^{\delta_j} v(x,t) + \sum_{j=1}^{n} b_j^* D_{x_j}^{\beta_j} v(x,t) +\\
\sum_{j=1}^{n} c_j^* D_{x_j}^{\gamma_j} v(x,t) + P(u(x,t), v(x,t));
\end{aligned} \tag{13}$$

$m-1 < \alpha \le m, 3 < \delta_j \le 4, 1 < \beta_j \le 2, 0 < \gamma_j \le 1, m \in N$,
where $x = (x_1, x_2, ...x_n) \in R^n, a_j, b_j, c_j$ and a_j^*, b_j^*, c_j^* are real constants, $0 \le t \le T$, with initial conditions

$$\frac{\partial^k u(x,0)}{\partial t^k} = h_k(x), \;\; 0 \le k \le m-1, \tag{14}$$

$$\frac{\partial^k v(x,0)}{\partial t^k} = h_k^*(x), \;\; 0 \le k \le m-1. \tag{15}$$

Applying inverse operator I_t^α to both Equations (12) and (13) on both sides we obtain,

$$\begin{aligned}
I_t^\alpha D_t^\alpha u(x,t) = I_t^\alpha \Big(\sum_{j=1}^{n} a_j D_{x_j}^{\delta_j} u(x,t) + \sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t)\\
+ \sum_{j=1}^{n} c_j D_{x_j}^{\gamma_j} u(x,t) + N(u(x,t), v(x,t)) \Big),
\end{aligned}$$

and

$$I_t^{\alpha} D_t^{\alpha} v(x,t) = I_t^{\alpha}\Bigg(\sum_{j=1}^{n} a_j^* D_{x_j}^{\delta_j} v(x,t) + \sum_{j=1}^{n} b_j^* D_{x_j}^{\beta_j} v(x,t) + \sum_{j=1}^{n} c_j^* D_{x_j}^{\gamma_j} v(x,t) + P(u(x,t), v(x,t))\Bigg).$$

Using initial conditions (14) and (15), we get,

$$u(x,t) = \sum_{k=0}^{m-1} h_k(x)\frac{t^k}{k!} + I_t^{\alpha}\Bigg(\sum_{j=1}^{n} a_j D_{x_j}^{\delta_j} u(x,t) + \sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t) + \sum_{j=1}^{n} c_j D_{x_j}^{\gamma_j} u(x,t) + N(u(x,t), v(x,t))\Bigg), \tag{16}$$

$$v(x,t) = \sum_{k=0}^{m-1} h_k^*(x)\frac{t^k}{k!} + I_t^{\alpha}\Bigg(\sum_{j=1}^{n} a_j^* D_{x_j}^{\delta_j} v(x,t) + \sum_{j=1}^{n} b_j^* D_{x_j}^{\beta_j} v(x,t) + \sum_{j=1}^{n} c_j^* D_{x_j}^{\gamma_j} v(x,t) + P(u(x,t), v(x,t))\Bigg). \tag{17}$$

Equations (16) and (17) have the forms of Equations (2) and (3) with

$$f(x,t) = \sum_{k=0}^{m-1} h_k(x)\frac{t^k}{k!}, g(x,t) = \sum_{k=0}^{m-1} h_k^*(x)\frac{t^k}{k!},$$

$$L(u(x,t)) = I_t^{\alpha}\Bigg(\sum_{j=1}^{n} a_j D_{x_j}^{\delta_j} u(x,t) + \sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t) + \sum_{j=1}^{n} c_j D_{x_j}^{\gamma_j} u(x,t)\Bigg),$$

$$M(v(x,t)) = I_t^{\alpha}\Bigg(\sum_{j=1}^{n} a_j^* D_{x_j}^{\delta_j} v(x,t) + \sum_{j=1}^{n} b_j^* D_{x_j}^{\beta_j} v(x,t) + \sum_{j=1}^{n} c_j^* D_{x_j}^{\gamma_j} v(x,t)\Bigg),$$

$$N(u(x,t), v(x,t)) = I_t^{\alpha}\Bigg(N(u(x,t), v(x,t))\Bigg),$$

$$P(u(x,t), v(x,t)) = I_t^{\alpha}\Bigg(P(u(x,t), v(x,t))\Bigg),$$

and can be solved by the iteration method developed in Section 2.

5. Applications

In this section, we discuss some illustrative examples for linear and nonlinear system of fractional transport equations and fractional Burger's equations, respectively.

5.1. System of Time Fractional Transport Equation

These equations appear in the mathematical description of many phenomena in classical and statistical mechanics. Now we consider an example of special system of time fractional transport equations with suitable initial conditions.

Example 5.1.

Consider the linear system of fractional transport equations,

$$D_t^{\alpha} u + D_x u + u - v = 0, \tag{18}$$

$$D_t^{\alpha} v + D_x v - u + v = 0, \quad 0 < \alpha \leq 1, \tag{19}$$

with initial conditions

$$u(x,0) = \sinh x, \tag{20}$$

$$v(x,0) = \cosh x. \tag{21}$$

Solution:

The system of equations can be written in operator form

$$D_t^{\alpha} u = L(u) + N(u,v),$$
$$D_t^{\alpha} u = M(v) + P(v,u),$$

where $L = -\frac{\partial}{\partial x}$, $N(u,v) = -u + v$, $M = -\frac{\partial}{\partial x}$ and $P(u,v) = -v + u$. Here the operators and L, M are linear operators as well as the operators N, P are linear in lower order. We look for the series solution

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{22}$$

$$v(x,t) = \sum_{i=0}^{\infty} v_i(x,t). \tag{23}$$

Applying iterative method developed in Section 3, we get,

$$\begin{aligned} u(x,t) =& u(x,0) + \sum_{i=0}^{\infty} L(u_i) + N(u_0, v_0) + \\ & \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} u_j, \sum_{j=0}^{i} v_j \right) - N\left(\sum_{j=0}^{i-1} u_j, \sum_{j=0}^{i-1} v_j \right) \right\}, \\ v(x,t) =& v(x,0) + \sum_{i=0}^{\infty} M(v_i) + P(v_0, u_0) + \\ & \sum_{i=1}^{\infty} \left\{ P\left(\sum_{j=0}^{i} v_j, \sum_{j=0}^{i} u_j \right) - P\left(\sum_{j=0}^{i-1} v_j, \sum_{j=0}^{i-1} u_j \right) \right\}. \end{aligned}$$

Comparing both sides we get,

$$\begin{aligned}
u_0 =& u(x,0) = \sinh x, \\
v_0 =& v(x,0) = \cosh x, \\
u_1 =& I_t^\alpha\Big[L(u_0)\Big] + I_t^\alpha\Big[N(u_0,v_0)\Big] = (-\sinh x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\
v_1 =& I_t^\alpha\Big[M(v_0)\Big] + I_t^\alpha\Big[P(v_0,u_0)\Big] = (-\cosh x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\
u_2 =& I_t^\alpha\Big[L(u_1)\Big] + I_t^\alpha\Big[N(u_0+u_1,v_0+v_1) - N(u_0,v_0)\Big] = (\sinh x)\frac{(t^\alpha)^2}{\Gamma(2\alpha+1)}, \\
v_2 =& I_t^\alpha\Big[M(v_1)\Big] + I_t^\alpha\Big[P(v_0+v_1,u_0+u_1) - P(v_0,u_0)\Big] = (\cosh x)\frac{(t^\alpha)^2}{\Gamma(2\alpha+1)}
\end{aligned}$$

and so on in general, we get,

$$\begin{aligned}
u_i =& I_t^\alpha\Big[L(u_{i-1})\Big] + I_t^\alpha\Big[N(u_0+...+u_{i-1},v_0+...+v_{i-1}) - \\
& \qquad N(u_0+...+u_{i-2},v_0+...+v_{i-2})\Big] \\
=& (-1)^i(\sinh x)^i\frac{(t^\alpha)^i}{\Gamma(i\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
v_i =& I_t^\alpha\Big[M(v_{i-1})\Big] + I_t^\alpha\Big[P(v_0+...+v_{i-1},u_0+...+u_{i-1}) - \\
& \qquad P(u_0+...+u_{i-2},u_0+...+u_{i-2})\Big] \\
=& (-1)^i(\cosh x)\frac{(t^\alpha)^i}{\Gamma(i\alpha+1)}.
\end{aligned}$$

Substituting $u_0, u_1, u_2, u_3, ...$ and $v_0, v_1, v_2, ...$ in (22) and (23), we get the solution of the system (18)- (21)

$$u(x,t) = \sinh x\left[\sum_{i=0}^{\infty}(-1)^i\frac{(t^\alpha)^i}{\Gamma(i\alpha+1)}\right] = \sinh x E_\alpha(-t^\alpha),$$

$$v(x,t) = \cosh x\left[\sum_{i=0}^{\infty}(-1)^i\frac{(t^\alpha)^i}{\Gamma(i\alpha+1)}\right] = \cosh x E_\alpha(-t^\alpha). \qquad \blacksquare$$

5.2. System of Time Fractional Burger's Equations

Fractional diffusion equations play an important role in describing anomalous diffusion. The modeling of the dynamics of anomalous process by means of fractional differential equations has provided good results in the field of science and engineering Khilbas, Shivastava and Trujillo (2006),

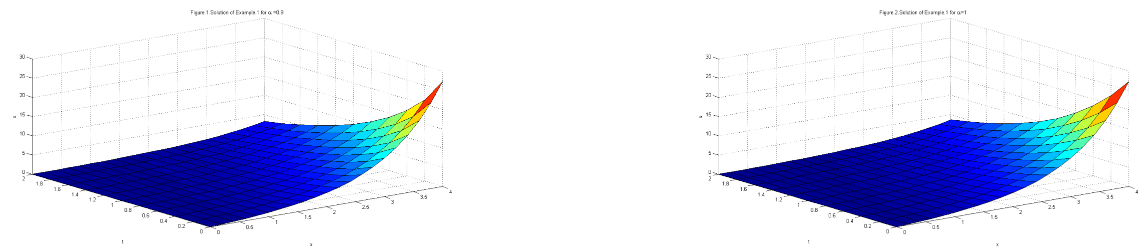

Figure 1. The graphical representation of the solution of Example 5.1 for $u(x,t)$, $\alpha = 0.9$ and $\alpha = 1$.

Figure 2. The graphical representation of the solution of Example 5.1 for $v(x,t)$, $\alpha = 0.9$ and $\alpha = 1$.

Zaslavsky (2002) and references therein. Some of them are motion of tracer particles in turbulent flow Richardson (1926), chaotic dynamics Shlesinger (1993).

Fractional diffusion equations also account for typical anomalous features which may be observed in many systems such as the case of dispersive transport in amorphous semiconductors Metzler and Klafter (2000) as well as diffusion of free carriers in multiple trapping Bisquert (2003).

Now we study some special systems of anomalous diffusion in the next examples with suitable initial conditions.

Example 5.2.

Consider the IVP for nonlinear system of fractional Burger's equations,

$$D_t^{\alpha} u - D_x^2 u + u D_x u - uv = 0, \tag{24}$$

$$D_t^{\alpha} v - D_x^2 v + v D_x v + uv = 0, \quad 0 < \alpha \leq 1, \tag{25}$$

with initial condition

$$u(x,0) = \sin x, \tag{26}$$

$$v(x,0) = \cos x. \tag{27}$$

Solution:

The IVP (24)- (25) can be written in operator form

$$D_t^{\alpha} u = L(u) + N(u,v),$$

$$D_t^{\alpha} v = M(v) + P(v,u),$$

where $L = \frac{\partial^2}{\partial x^2}$, $N(u,v) = -u\frac{\partial u}{\partial x} + uv$, $M = \frac{\partial^2}{\partial x^2}$ and $P(u,v) = -v\frac{\partial v}{\partial x} - uv$.

We look for the series solution

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{28}$$
$$v(x,t) = \sum_{i=0}^{\infty} v_i(x,t). \tag{29}$$

Applying the iterative method developed in Section 3, we get,

$$\begin{aligned}
u_0 &= u(x,0) = \sin x,\\
v_0 &= v(x,0) = \cos x,\\
u_1 &= I_t^{\alpha}\Big[L(u_0)\Big] + I_t^{\alpha}\Big[N(u_0,v_0)\Big] = -\sin x\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
v_1 &= I_t^{\alpha}\Big[M(v_0)\Big] + I_t^{\alpha}\Big[P(v_0,u_0)\Big] = -\cos x\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
u_2 &= I_t^{\alpha}\Big[L(u_1)\Big] + I_t^{\alpha}\Big[N(u_0+u_1,v_0+v_1) - N(u_0,v_0)\Big] = \sin x\frac{(t^{\alpha})^2}{\Gamma(2\alpha+1)},\\
v_2 &= I_t^{\alpha}\Big[M(v_1)\Big] + I_t^{\alpha}\Big[P(v_0+v_1,u_0+u_1) - P(v_0,u_0)\Big] = \cos x\frac{(t^{\alpha})^2}{\Gamma(2\alpha+1)}.
\end{aligned}$$

And so on in general, we get,

$$\begin{aligned}
u_i &= I_t^{\alpha}\Big[L(u_{i-1})\Big] + I_t^{\alpha}\Big[N(u_0+...+u_{i-1},v_0+...v_{i-1}) - \\
&\qquad\qquad N(u_0+...+u_{i-2},v_0+...v_{i-2})\Big]\\
&= (-1)^i(\sin x)^i\frac{(t^{\alpha})^i}{\Gamma(i\alpha+1)},\\
v_i &= I_t^{\alpha}\Big[M(v_{i-1})\Big] + I_t^{\alpha}\Big[P(v_0+...+v_{i-1},u_0+...v_{i-1}) - \\
&\qquad\qquad P(v_0+...+v_{i-2},u_0+...u_{i-2})\Big]\\
&= (-1)^i(\cos x)\frac{(t^{\alpha})^i}{\Gamma(i\alpha+1)}.
\end{aligned}$$

Substituting $u_0, u_1, u_2, ...$ and $v_0, v_1, v_2, ...$ in (28) and (29), we get the solution of IVP (24) and (25) with initial conditions (26) and (27) is

$$\begin{aligned}
u(x,t) &= \sin x\left[\sum_{i=0}^{\infty}(-1)^i\frac{(t^{\alpha})^i}{\Gamma(i\alpha+1)}\right] = \sin x E_{\alpha}(-t^{\alpha}),\\
v(x,t) &= \cos x\left[\sum_{i=0}^{\infty}(-1)^i\frac{(t^{\alpha})^i}{\Gamma(i\alpha+1)}\right] = \cos x E_{\alpha}(-t^{\alpha}).
\end{aligned}$$

■

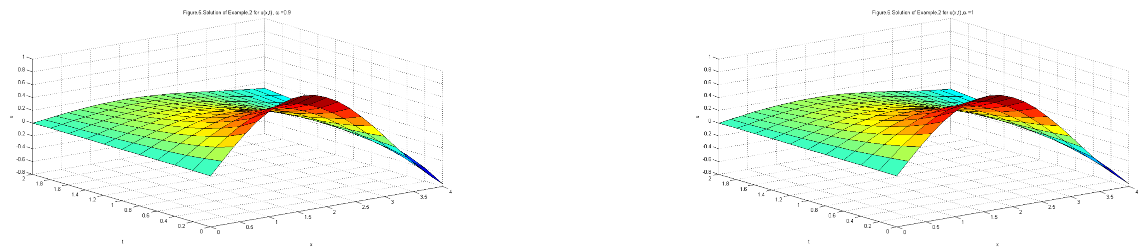

Figure 3. The graphical representation of the solution of Example 5.2 for $u(x,t)$, $\alpha = 0.9$ and $\alpha = 1$.

Figure 4. The graphical representation of the solution of Example 5.2 for $v(x,t)$, $\alpha = 0.9$ and $\alpha = 1$.

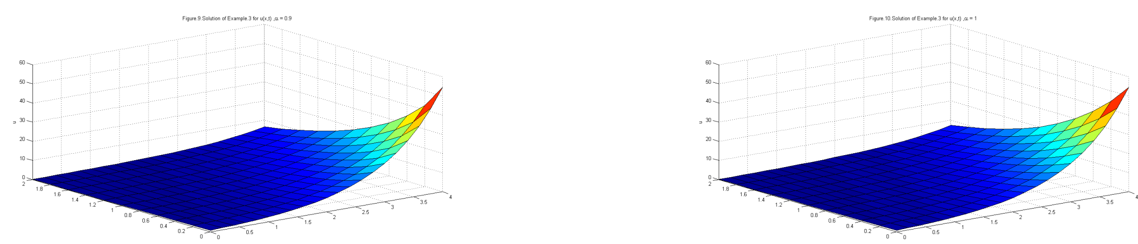

Figure 5. The graphical representation of the solution of Example 5.3 for $u(x,t)$, $\alpha = 0.9$ and $\alpha = 1$.

Example 5.3.

Consider the IVP for nonlinear system of fractional Burger's equations,

$$D_t^\alpha u + D_x^2 u + 2uD_x u + D_x(uv) = 0, \tag{30}$$

$$D_t^\alpha v + D_x^2 v + 2vD_x v + D_x(uv) = 0, 0 < \alpha \leq 1, \tag{31}$$

with initial condition

$$u(x,0) = e^x, \tag{32}$$

$$v(x,0) = -e^x. \tag{33}$$

Solution:

The above system can be written in operator form

$$D_t^\alpha u = L(u) + N(u,v),$$
$$D_t^\alpha v = M(v) + P(u,v),$$

where $L = -\frac{\partial^2}{\partial x^2}$, $N(u,v) = -2u\frac{\partial u}{\partial x} - \frac{\partial(uv)}{\partial x}$, $M = -\frac{\partial^2}{\partial x^2}$ and $P(u,v) = -2v\frac{\partial v}{\partial x} - \frac{\partial(uv)}{\partial x}$. We look for

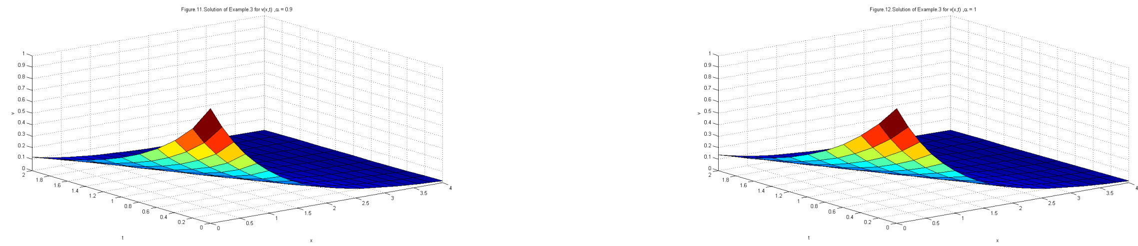

Figure 6. The graphical representation of the solution of Example 5.3 for $v(x,t)$, $\alpha = 0.9$ and $\alpha = 1$.

the series solution

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{34}$$

$$v(x,t) = \sum_{j=0}^{\infty} v_j(x,t). \tag{35}$$

Applying iterative method developed in Section 3, we get,

$$\begin{aligned}
u_0 &= u(x,0) = e^x, \\
v_0 &= v(x,0) = -e^x, \\
u_1 &= I_t^\alpha\Big[L(u_0)\Big] + I_t^\alpha\Big[N(u_0,v_0)\Big] = -e^x \frac{t^\alpha}{\Gamma(\alpha+1)}, \\
v_1 &= I_t^\alpha\Big[M(v_0)\Big] + I_t^\alpha\Big[P(v_0,u_0)\Big] = e^x \frac{t^\alpha}{\Gamma(\alpha+1)}, \\
u_2 &= I_t^\alpha\Big[L(u_1)\Big] + I_t^\alpha\Big[N(u_0+u_1, v_0+v_1) - N(u_0,v_0)\Big] = e^x \frac{(t^\alpha)^2}{\Gamma(2\alpha+1)}, \\
v_2 &= I_t^\alpha\Big[M(v_1)\Big] + I_t^\alpha\Big[P(v_0+v_1, u_0+u_1) - P(v_0,u_0)\Big] = -e^x \frac{(t^\alpha)^2}{\Gamma(2\alpha+1)}.
\end{aligned}$$

And so on in general, we get,

$$\begin{aligned}
u_i &= I_t^\alpha\Big[L(u_{i-1})\Big] + I_t^\alpha\Big[N(u_0+\dots+u_{i-1}, v_0+\dots+v_{i-1}) - \\
&\qquad\qquad N(u_0+\dots+u_{i-2}, v_0+\dots+v_{i-2})\Big] \\
&= (-1)^i (e^x)^i \frac{(t^\alpha)^i}{\Gamma(i\alpha+1)}, \\
v_i &= I_t^\alpha\Big[M(v_{i-1})\Big] + I_t^\alpha\Big[P(v_0+\dots+v_{i-1}, u_0+\dots+u_{i-1}) - \\
&\qquad\qquad P(v_0+\dots+v_{i-2}, u_0+\dots+u_{i-2})\Big] \\
&= (-1)^i (-e^x) \frac{(t^\alpha)^i}{\Gamma(j\alpha+1)}.
\end{aligned}$$

Substituting $u_0, u_1, u_2, \dots$ and $v_0, v_1, v_2, \dots$ in (34) and (35) we get the solution of the IVP (30) – (31)

with initial conditions (32), (33) is

$$\begin{aligned} u(x,t) =& e^x \left[\sum_{i=0}^{\infty} (-1)^i \frac{(t^\alpha)^i}{\Gamma(i\alpha+1)} \right] = e^x E_\alpha(-t^\alpha), \\ v(x,t) =& -e^x \left[\sum_{i=0}^{\infty} (-1)^i \frac{(t^\alpha)^i}{\Gamma(i\alpha+1)} \right] = (-e^x) E_\alpha(-t^\alpha). \end{aligned}$$ ■

6. Conclusion

In this paper we have developed an iterative method for system of fractional partial differential equations with suitable initial conditions. This method is applied to some IVPs and obtained exact and approximate analytical solutions. Graphical representations of solutions of some IVPs are given using MATLAB. We have shown that the method is capable of reducing volume of computational work as compared to other classical methods, and maintain the high level of accuracy of numerical results. Also it can be seen that Iterative method has a clear advantage over other methods for solving nonlinear problems. We conclude that the iterative method can be considered as a nice refinement in existing numerical techniques and have wide applications.

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