Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 4, Issue 1 (June 2009) pp. 218 – 236
(Previously, Vol. 4, No. 1)

Stability of an Age-structured SEIR Epidemic Model with Infectivity in Latent Period

Xue-Zhi Li and Bin Fang
Department of Mathematics
Xinyang Normal University
Henan 464000, P. R. China
xzli66@126.com

Received: April 20, 2007; Accepted: March 5, 2008

Abstract

We study an age-structured SEIR epidemic model with infectivity in the latent period. By using the theory and methods of Differential and Integral Equations, the explicit expression for the basic reproductive number R_0 is first derived. It is shown that the disease-free equilibrium is locally and globally asymptotically stable if $R_0 < 1$. It is then proved that only one endemic equilibrium exists if $R_0 > 1$ and its stability conditions are also given.

Keywords: Age-Structured; SEIR Epidemic Model; Latent Period; the Basic Reproductive Number; Endemic Equilibrium; Stability

MSC (2000) No.: 45D05, 92D05, 92D30

1. Introduction

Many diseases, such as measles, mumps, tuberculosis, HIV/AIDS, SARS, etc., have a latent period. A latent period is the lag time between exposure to a disease-causing agent and the onset of the disease the agent causes. From the point of view of mathematical modeling, this leads to SEIR or SEIRS epidemic models. These kinds of models have attracted the attention of many authors and a number of papers have been published in this area. For example, Greenhalgh (1992) considered an SEIR model that incorporates density dependence in the death rate. Cooke and Driessche (1990) introduced and analyzed the SEIRS model with two delays. Greenhalgh (1997) studied Hopf bifurcations in the SEIRS type models with density dependent contact rate and death rate. Li and Muldowney (1995) and Li et al. (2001) studied the global dynamics of the

SEIR models with a non-linear incidence rate as well as standard incidence rate. Li et al. (1999) analyzed the global dynamics of the SEIR model with vertical transmission and a bilinear incidence. Recently, Zhang and Ma (2003) analyzed the global dynamics of the SEIR model with saturating contact rate. All the models discussed above are of SEIR-type age-independent epidemic models, which are described by a system of ordinary differential equations. The importance of age structure in epidemic models has been stressed by many researchers. One may refer Busenberg (1991), Iannelli (1992, 1998), Inaba (1990), Castillo-Chavez (1989, 1998), Li (2004), etc., for the discussions concerning age-structured SIR or SIRS models, and Xue-Zhi Li (2001, 2003), etc. for discussion on age-structured SEIR or SEIRS model. We may note that all the SEIR or SEIRS models mentioned above assume that the latent individuals have no infectivity, so, a susceptible individual may become infectious only through contact with infective.

In fact, many diseases, such as tuberculosis, HIV/AIDS and SARS, etc., have a contagious latent period, a latent individual can transmit the disease to the susceptible, i.e., the individual has force of infection in both the latent and infectious periods. This fact has been noticed by some researchers and the mathematical modeling (ODE cases) and discussions have also been given, for instance, Li and Jin (2004, 2005, 2006). Whereas in this paper, we establish and study an age-structured SEIR epidemic model with infectivity in both the latent and infectious periods. By using the theory and methods of Differential and Integral Equations, we first obtain the explicit expression for the basic reproduction number R_0 under the assumption that the total population size is at stationary demographic state. Then, it is shown that the disease-free equilibrium is locally and globally asymptotically stable if $R_0 < 1$. In this case, the disease always dies out. When $R_0 > 1$, there exists unique endemic equilibrium which is asymptotically stable under certain conditions.

This paper is organized as follows: in the following section, an age-structured SEIR epidemic model with infectivity in both the infected and latent periods is introduced. The expression of reproduction number R_0 is obtained in Section 3. The local and global stabilities of the disease-free equilibrium are also studied in this section. Section 4 discusses the existence and the stability of the unique endemic equilibrium.

2. The Model

In order to formulate an age-structured SEIR epidemic model with infectivity in latent period, we need to introduce some notations. The whole population under consideration is divided into susceptible, latent, infectious and immune (or recovered) classes, where $S(a,t), E(a,t)$, $I(a,t), R(a,t)$ denote the associated density functions with these respective epidemiological age-structured classes.

Let $b(a)$ and $\mu(a)$ be the vital parameters (age-specific fertility and mortality) of the population, which we assume not to be affected by the disease, $(\alpha(a))^{-1}$ and $(\gamma(a))^{-1}$ be the average latent period and the average infectious period, respectively. We assume that $\alpha(a)$ and $\gamma(a)$ are positive and continuous. For convenience in mathematical calculations in the following sections, we assume that the infectivity of a latent individual is same as of infectivity of infectious individual. We adopt the following separable inter-cohort constitutive form for the force of infection [Castillo-Chavez and Feng (1998)]:

$$\hat{\lambda}(a,t) = k(a)\lambda(t),$$
$$\lambda(t) = \int_0^{a_+} \beta(a)\frac{E(a,t)+I(a,t)}{N(a,t)}da.$$

Here, a_+ is the maximum age that an individual of the population can attain. The age-specific infectiousness $\beta(a)$ and the age-specific contagion rate $k(a)$ satisfy the following conditions:

$$\beta(\cdot), k(\cdot) \in C[0,a_+), \text{ and } \beta(a), k(a) \geq 0 \text{ on } [0,a_+).$$

We assume that all newborns are susceptible and that an individual may become latent through contact with latent or infectious individuals, recovered individual has permanent immunity to the disease. Here we observe that the disease-induced death rate can be neglected. The dynamics of the age-structured epidemiological classes are governed by the following initial and boundary value problem:

$$\frac{\partial S(a,t)}{\partial t} + \frac{\partial S(a,t)}{\partial a} = -(\mu(a)+\hat{\lambda}(a,t))S(a,t), \tag{2.1a}$$
$$\frac{\partial E(a,t)}{\partial t} + \frac{\partial E(a,t)}{\partial a} = -(\mu(a)+\alpha(a))E(a,t)+\hat{\lambda}(a,t)S(a,t), \tag{2.1b}$$
$$\frac{\partial I(a,t)}{\partial t} + \frac{\partial I(a,t)}{\partial a} = -(\mu(a)+\gamma(a))I(a,t)+\alpha(a)E(a,t), \tag{2.1c}$$
$$\frac{\partial R(a,t)}{\partial t} + \frac{\partial R(a,t)}{\partial a} = -\mu(a)R(a,t)+\gamma(a)I(a,t), \tag{2.1d}$$
$$\hat{\lambda}(a,t) = k(a)\lambda(t), \tag{2.1e}$$
$$\lambda(t) = \int_0^{a_+} \beta(a)\frac{E(a,t)+I(a,t)}{N(a,t)}da, \tag{2.1f}$$
$$\begin{aligned} S(0,t) &= \int_0^{a_+} b(a)[S(a,t)+E(a,t)+I(a,t)+R(a,t)]da, \\ E(0,t) &= I(0,t) = R(0,t) = 0, \end{aligned} \tag{2.1g}$$
$$\begin{aligned} &S(a,0) = S_0(a), E(a,0) = E_0(a), \\ &I(a,0) = I_0(a), R(a,0) = R_0(a). \end{aligned} \tag{2.1f}$$

Summing the equations in (2.1), we obtain the following equation for the total population density $N(a,t) = S(a,t)+E(a,t)+I(a,t)+R(a,t)$:

$$\frac{\partial N(a,t)}{\partial t} + \frac{\partial N(a,t)}{\partial a} = -\mu(a)N(a,t), \tag{2.2a}$$

$$N(0,t)=\int_0^{a_+} b(a)N(a,t)da, \tag{2.2b}$$
$$N(a,0)=N_0(a). \tag{2.2c}$$

This is the standard Mckendrik-Von Forester equation. Here, we see that indeed the population dynamics is not affected by the disease. We make the following hypotheses for this problem

$$b(a)\in L^\infty([0,a_+)),\quad b(a)\geq 0 \quad \text{in} \quad [0,a_+),$$
$$\mu(a)\in L^1_{loc}[0,a_+),\quad \mu(a)\geq 0 \quad \text{in} \quad [0,a_+),$$
$$\int_0^{a_+} \mu(a)da=+\infty.$$

Furthermore, in order to deal with a steady state problem with age density given by (2.2), we assume that the net reproduction rate of the population is equal to unity and that the total population is at the steady state. This means that

$$\int_0^{a_+} b(a)e^{-\int_0^a \mu(\tau)d\tau}da=1, \tag{2.3}$$
$$N(a,t)=N_\infty(a)=b_0 e^{-\int_0^a \mu(\tau)d\tau}. \tag{2.4}$$

The condition (2.4) is valid because of (2.3). This condition also implies that, in order to deal with a significant model, we need to take initial data as follows:

$$\begin{aligned} &S_0(a)\geq 0, E_0(a)\geq 0, I_0(a)\geq 0, R_0(a)\geq 0,\\ &S_0(a)+E_0(a)+I_0(a)+R_0(a)=N_\infty(a), \end{aligned} \tag{2.5}$$

which forces the relation

$$b_0=\frac{\int_0^{a_+}[S_0(a)+E_0(a)+I_0(a)+R_0(a)]da}{\int_0^{a_+} e^{-\int_0^a \mu(\tau)d\tau}da}. \tag{2.6}$$

Now, introducing the fractions

$$s(a,t)=\frac{S(a,t)}{N_\infty(a)}, e(a,t)=\frac{E(a,t)}{N_\infty(a)}, i(a,t)=\frac{I(a,t)}{N_\infty(a)}, r(a,t)=\frac{R(a,t)}{N_\infty(a)},$$

we get the following simplified system of equation (2.1):

$$\frac{\partial s(a,t)}{\partial t}+\frac{\partial s(a,t)}{\partial a}=-k(a)\lambda(t)s(a,t), \tag{2.7a}$$

$$\frac{\partial e(a,t)}{\partial t}+\frac{\partial e(a,t)}{\partial a}=-\alpha(a)e(a,t)+k(a)\lambda(a)s(a,t), \tag{2.7b}$$

$$\frac{\partial i(a,t)}{\partial t}+\frac{\partial i(a,t)}{\partial a}=-\gamma(a)i(a,t)+\alpha(a)e(a,t), \tag{2.7c}$$

$$\frac{\partial r(a,t)}{\partial t}+\frac{\partial r(a,t)}{\partial a}=\gamma(a)i(a,t), \tag{2.7d}$$

$$\lambda(t)=\int_0^{a_+}\beta(a)[e(a,t)+i(a,t)]da, \tag{2.7e}$$

$$s(0,t)=1, e(0,t)=i(0,t)=r(0,t)=0, \tag{2.7f}$$

$$s(a,0)=s_0(a), e(a,0)=e_0(a), i(a,0)=i_0(a), r(a,0)=r_0(a), \tag{2.7g}$$

$$s(a,t)+e(a,t)+i(a,t)+r(a,t)=1. \tag{2.7h}$$

In the next section, we derive the explicit expression for R_0, a quantity that must exceed one for the disease to remain endemic (persist). In general, R_0 is called the net reproductive number, which measures the expected number of secondary infection produced by a 'typical' infected individual during its entire-death adjusted-period of infectious in a wholly susceptible population.

3. Calculation of R_0 and Stability of the Disease-Free Equilibrium

An equilibrium solution is a solution of the system (2.7) when the population state remains unchanged with time, i.e., a time-independent solution. The disease-free state is the population state when there are no infectious or latent individuals. A steady state solution $(s(a), e(a), i(a), r(a))$ of the system (2.7) must satisfy

$$\frac{ds(a)}{da}=-k(a)\Lambda s(a), \tag{3.1a}$$

$$\frac{de(a)}{da}=-\alpha(a)e(a)+k(a)\Lambda s(a), \tag{3.1b}$$

$$\frac{di(a)}{da}=-\gamma(a)i(a)+\alpha(a)e(a), \tag{3.1c}$$

$$\frac{dr(a)}{da}=\gamma(a)i(a), \tag{3.1d}$$

$$\Lambda=\int_0^{a_+}\beta(a)[e(a)+i(a)]da, \tag{3.1e}$$

$$s(0)=1, e(0)=i(0)=r(0)=0, \tag{3.1f}$$

$$s(0)+e(0)+i(0)+r(0)=1. \tag{3.1g}$$

It is easy to see that the system (3.1) always has the disease-free equilibrium, which is given by

$$s^0(a)=1, \quad e^0(a)=i^0(a)=r^0(a)=0. \tag{3.2}$$

To study the local stability of the disease-free equilibrium, we linearize the system (2.7) about (3.2). Let $\overline{s}(a,t),\overline{e}(a,t),\overline{i}(a,t),\overline{r}(a,t)$ be the perturbation in $s^0(a),e^0(a),i^0(a),r^0(a)$, respectively, i.e.,

$$s(a,t)=\overline{s}(a,t)+s^0(a), e(a,t)=\overline{e}(a,t)+e^0(a),$$
$$i(a,t)=\overline{i}(a,t)+i^0(a), r(a,t)=\overline{r}(a,t)+r^0(a).$$

These perturbations satisfy the following system of equations:

$$\frac{\partial\overline{s}(a,t)}{\partial t}+\frac{\partial\overline{s}(a,t)}{\partial a}=-k(a)\Lambda(t)s^0(a), \tag{3.3a}$$
$$\frac{\partial\overline{e}(a,t)}{\partial t}+\frac{\partial\overline{e}(a,t)}{\partial a}=-\alpha(a)\overline{e}(a,t)+k(a)\Lambda(t)s^0(a), \tag{3.3b}$$
$$\frac{\partial\overline{i}(a,t)}{\partial t}+\frac{\partial\overline{i}(a,t)}{\partial a}=-\gamma(a)\overline{i}(a,t)+\alpha(a)\overline{e}(a,t), \tag{3.3c}$$
$$\frac{\partial\overline{r}(a,t)}{\partial t}+\frac{\partial\overline{r}(a,t)}{\partial a}=\gamma(a)\overline{i}(a,t), \tag{3.3d}$$
$$\Lambda(t)=\int_0^{a_+}\beta(a)[\overline{e}(a,t)+\overline{i}(a,t)]da, \tag{3.3e}$$
$$\overline{s}(0,t)=\overline{e}(0,t)=\overline{i}(0,t)=\overline{r}(0,t)=0. \tag{3.3f}$$

We now consider exponential solutions of the system (3.3) of the form

$$\overline{s}(a,t)=\overline{s}(a)e^{\lambda t}, \qquad \overline{e}(a,t)=\overline{e}(a)e^{\lambda t},$$
$$\overline{i}(a,t)=\overline{i}(a)e^{\lambda t}, \qquad \overline{r}(a,t)=\overline{r}(a)e^{\lambda t}.$$

The functions $\overline{s}(a),\overline{e}(a),\overline{i}(a),\overline{r}(a)$ and the parameter λ satisfy the following system of equations:

$$\frac{d\overline{s}(a)}{da}=-\lambda\overline{s}(a)-k(a)\overline{\Lambda}, \tag{3.4a}$$
$$\frac{d\overline{e}(a)}{da}=-\lambda\overline{e}(a)-\alpha(a)\overline{e}(a)+k(a)\overline{\Lambda}, \tag{3.4b}$$
$$\frac{d\overline{i}(a)}{da}=-\lambda\overline{i}(a)-\gamma(a)\overline{i}(a)+\alpha(a)\overline{e}(a), \tag{3.4c}$$
$$\frac{d\overline{r}(a)}{da}=-\lambda\overline{r}(a)+\gamma(a)\overline{i}(a), \tag{3.4d}$$
$$\overline{\Lambda}=\int_0^{a_+}\beta(a)[\overline{e}(a)+\overline{i}(a)]da, \tag{3.4e}$$
$$\overline{s}(0)=\overline{e}(0)=\overline{i}(0)=\overline{r}(0)=0. \tag{3.4f}$$

Solving (3.4b) and (3.4c) we obtain

$$\bar{e}(a) = \bar{\Lambda}\int_0^a k(\tau)e^{-\int_\tau^a \alpha(\sigma)d\sigma} e^{-\lambda(a-\tau)}d\tau. \tag{3.5}$$

$$\bar{i}(a) = \int_0^a \alpha(\tau)\bar{e}(\tau)e^{-\int_\tau^a \gamma(\sigma)d\sigma} e^{-\lambda(a-\tau)}d\tau. \tag{3.6}$$

Substituting (3.5) into (3.6) and changing the order of integration we get

$$\begin{aligned}\bar{i}(a) &= \bar{\Lambda}\int_0^a \alpha(\tau)\int_0^\tau k(\eta)e^{-\int_\eta^\tau \alpha(\sigma)d\sigma} e^{-\lambda(a-\eta)}d\eta e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau \\ &= \bar{\Lambda}\int_0^a e^{-\lambda(a-\eta)}k(\eta)\int_\eta^a \alpha(\tau)e^{-\int_\eta^\tau \alpha(\sigma)d\sigma} e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau d\eta \\ &= \bar{\Lambda}\int_0^a e^{-\lambda(a-\tau)}k(\tau)\int_\tau^a \alpha(\eta)e^{-\int_\tau^\eta \alpha(\sigma)d\sigma} e^{-\int_\eta^a \gamma(\sigma)d\sigma}d\eta d\tau.\end{aligned} \tag{3.7}$$

Substituting (3.5) and (3.7) into (3.4e) it follows that

$$\bar{\Lambda} = \int_0^{a_+} \beta(a)\bar{\Lambda}\int_0^a e^{-\lambda(a-\tau)}k(\tau)[e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(\eta)e^{-\int_\tau^\eta \alpha(\sigma)d\sigma}e^{-\int_\eta^a \gamma(\sigma)d\sigma}d\eta]d\tau da. \tag{3.8}$$

By dividing both sides by $\bar{\Lambda}$ (since $\bar{\Lambda} \neq 0$) in (3.8) we get the following characteristic equation

$$1 = \int_0^{a_+} \beta(a)\int_0^a e^{-\lambda(a-\tau)}k(\tau)[e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(\eta)e^{-\int_\tau^\eta \alpha(\sigma)d\sigma}e^{-\int_\eta^a \gamma(\sigma)d\sigma}d\eta]d\tau da. \tag{3.9}$$

Let us denote the expression in the right hand side of (3.9) by $F(\lambda)$ and define the basic reproductive number as $R_0 = F(0)$, i.e.,

$$R_0 = \int_0^{a_+} \beta(a)\int_0^a k(\tau)[e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(\eta)e^{-\int_\tau^\eta \alpha(\sigma)d\sigma}e^{-\int_\eta^a \gamma(\sigma)d\sigma}d\eta]d\tau da. \tag{3.10}$$

Now, we establish the following result.

Theorem 3.1.

The disease-free equilibrium of the system (2.7) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof:

We observe that

$$F'(\lambda) < 0, \quad \lim_{\lambda \to +\infty} F(\lambda) = 0, \quad \lim_{\lambda \to -\infty} F(\lambda) = +\infty.$$

We know that equation (3.9) has a unique negative real solution λ^*, if and only if $F(0) < 1$, or $R_0 < 1$. Also, equation (3.9) has a unique positive (zero) real solution if $F(0) > 1$ ($F(0) = 1$), or $R_0 > 1$ ($R_0 = 1$). To show that λ^* is the dominant real parts of roots of $F(\lambda)$, we let $\lambda = x + iy (x, y \in IR$, where i is the imaginary unit and IR is the set of real numbers) be an arbitrary complex solution of the equation (3.9). We note that

$$1 = F(\lambda) = | F(x + yi) | \leq F(x),$$

which indicates that $Re\lambda \leq \lambda^*$, where Re denotes the real part. It follows that the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$. This completes the proof.

The basic reproduction number R_0 can be seen as a weighted value of the basic reproduction number due to latent class and the basic reproduction number due to infectious class. When $R_0 < 1$, the number of infections decreases towards zero. The basic reproductive number R_0 must exceed one for the disease to persist in the population.

The global stability of the disease-free equilibrium is demonstrated in the following theorem.

Theorem 3.2.

The disease-free equilibrium of the system (2.7) is globally asymptotically stable if $R_0 < 1$.

Proof:

To prove the global stability of the disease-free equilibrium, we have to show that $s(a,t) \to 1, e(a,t) \to 0, i(a,t) \to 0, r(a,t) \to 0$ as $t \to \infty$. Integrating equation (2.7b) and equation (2.7c) along the characteristic line we get the following:

$$e(a,t) = \begin{cases} \int_0^a k(s)\lambda(t-a+s)e^{-\int_s^a \alpha(\sigma)d\sigma} s(s,t-a+s)ds, & a \leq t, \\ e_0(a-t)e^{-\int_0^t \alpha(a-\tau)d\tau} + \int_0^t k(a-\tau)\lambda(t-\tau)e^{-\int_0^\tau \alpha(a-\sigma)d\sigma} s(a-\tau,t-\tau)d\tau, & a > t. \end{cases} \tag{3.11}$$

$$i(a,t) = \begin{cases} \int_0^a \alpha(\tau)e(\tau,t-a+\tau)e^{-\int_\tau^a \gamma(\sigma)d\sigma} d\tau, & a \leq t, \\ i_0(a-t)e^{-\int_0^t \gamma(a-\sigma)d\sigma} + \int_0^t \alpha(a-\tau)e(a-\tau,t-\tau)e^{-\int_0^\tau \gamma(a-\sigma)d\sigma} d\tau, & a > t. \end{cases} \tag{3.12}$$

For $t > a$, by substituting (3.11) into (3.12) and changing the order of integration we obtain

$$i(a,t)=\int_0^a \alpha(\tau)[\int_0^\tau k(s)\lambda(t-a+s)e^{-\int_s^\tau \alpha(\sigma)d\sigma}s(s,t-a+s)ds]e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau$$
$$=\int_0^a k(s)\lambda(t-a+s)s(s,t-a+s)\int_s^a \alpha(\tau)e^{-\int_s^\tau \alpha(\sigma)d\sigma}e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau ds. \tag{3.13}$$

Putting the values of $e(a,t)$ and $i(a,t)$ from the equations (3.11) and (3.13) respectively in the expression for $\lambda(t)$ in equation (3.4e), we obtain

$$\lambda(t)=\int_0^t \beta(a)[e(a,t)+i(a,t)]da+\int_t^{a_+} \beta(a)[e(a,t)+i(a,t)]da \tag{3.14}$$
$$=\int_0^t \beta(a)\{\int_0^a k(s)\lambda(t-a+s)s(s,t-a+s)$$
$$\cdot[e^{-\int_s^a \alpha(\sigma)d\sigma}+\int_s^a \alpha(\tau)e^{-\int_s^\tau \alpha(\sigma)d\sigma}e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau]ds\}da+\int_t^{a_+} \beta(a)[e(a,t)+i(a,t)]da.$$

Since $e(a,t)$ and $i(a,t)$ do not exceed one, and same is true for their sum, the last integral can be estimated by $\int_t^{a_+} \beta(a)da$ which decreases to zero as $t\to\infty$. It is noted that $s(a,t)\le 1$, so

$$\lambda(t)\le\int_0^t\beta(a)\{\int_0^a k(s)\lambda(t-a+s)\cdot[e^{-\int_s^a \alpha(\sigma)d\sigma}+\int_s^a \alpha(\tau)e^{-\int_s^\tau \alpha(\sigma)d\sigma}e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau]ds\}da+\int_t^{a_+} \beta(a)da.$$

Taking the limit supremum when $t\to\infty$ on the both sides of the above equation and using Fatou's Lemma we get

$$\lim_{t\to\infty}\lambda(t)\le R_0 \limsup_{t\to\infty}\lambda(t). \tag{3.15}$$

As we assume that $R_0<1$, the only way inequality (3.15) can hold is if

$$\limsup_{t\to\infty}\lambda(t)=0.$$

Using this result in (3.11) and (3.12) we see that

$$\limsup_{t\to\infty} e(a,t)=0,\quad \limsup_{t\to\infty} i(a,t)=0,$$

pointwise in a. Furthermore, from equation (2.7a), equation (2.7d) and the fact that $s(a,t), e(a,t), i(a,t)$, and $r(a,t)$ add up to one, it is clear that

$$\limsup_{t\to\infty} s(a,t)=1,\quad \limsup_{t\to\infty} r(a,t)=0.$$

Therefore, the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$. This completes the proof of Theorem 3.2.

The global stability of the disease-free equilibrium implies that, given a set of parameters satisfying $R_0 < 1$, for any initial positive number of infectious individuals, the number of latent infectious individuals and the normal infectious individuals decrease to zero. Such prediction can be of important use in planning and evaluation of disease control policy.

In the next section, we consider the existence and stability of the unique endemic equilibrium when $R_0 > 1$.

4. The Existence and Local Stability of the Endemic Equilibrium

In Section 3, we showed that the disease-free equilibrium is unstable if $R_0 > 1$. In fact, a nontrivial steady state appears at the same time, which is discussed in the following theorem.

Theorem 4.1.

There exists an endemic equilibrium of the system (2.7) if $R_0 > 1$.

Proof:

The method commonly used to find an endemic steady state for age-structured models consists of obtaining expressions for a time independent solution $(s^*(a), e^*(a), i^*(a), r^*(a))$ of the system (2.7) that satisfies

$$\frac{ds^*(a)}{da} = -k(a)\lambda^* s^*(a), \tag{4.1a}$$

$$\frac{de^*(a)}{da} = -\alpha(a)e^*(a) + k(a)\lambda^* s^*(a), \tag{4.1b}$$

$$\frac{di^*(a)}{da} = -\gamma(a)i^*(a) + \alpha(a)e^*(a), \tag{4.1c}$$

$$\frac{dr^*(a)}{da} = \gamma(a)i^*(a), \tag{4.1d}$$

$$\lambda^* = \int_0^{a_+} \beta(a)[e^*(a) + i^*(a)]da, \tag{4.1e}$$

$$s^*(0) = 1, e^*(0) = i^*(0) = r^*(0) = 0, \tag{4.1f}$$

$$s^*(a) + e^*(a) + i^*(a) + r^*(a) = 1. \tag{4.1g}$$

Solving (4.1a), (4.1b) and (4.1c) we get the following:

$$s^*(a) = e^{-\lambda^*\int_0^a k(\sigma)d\sigma}, \tag{4.2}$$

$$e^*(a) = \lambda^* \int_0^a k(\tau)s^*(\tau)e^{-\int_\tau^a \alpha(\sigma)d\sigma} d\tau, \tag{4.3}$$

$$i^*(a) = \int_0^a \alpha(\tau)e^*(\tau)e^{-\int_\tau^a \gamma(\sigma)d\sigma} d\tau. \tag{4.4}$$

Substituting the value of $s^*(a)$ from (4.2) into (4.3) we get

$$e^*(a) = \lambda^* \int_0^a k(\tau)e^{-\lambda^*\int_0^\tau k(\sigma)d\sigma} e^{-\int_\tau^a \alpha(\sigma)d\sigma} d\tau. \tag{4.5}$$

Again substituting the value of $e^*(a)$ from (4.5) into (4.4) and changing the order of integration we get

$$\begin{aligned} i^*(a) &= \int_0^a \alpha(\tau)\lambda^* \int_0^\tau k(s)e^{-\lambda^*\int_0^s k(\sigma)d\sigma} e^{-\int_s^\tau \alpha(\sigma)d\sigma} ds e^{-\int_\tau^a \gamma(\sigma)d\sigma} d\tau \\ &= \lambda^* \int_0^a k(s)e^{-\lambda^*\int_0^s k(\sigma)d\sigma} \int_s^a \alpha(\tau)e^{-\int_s^\tau \alpha(\sigma)d\sigma} e^{-\int_\tau^a \gamma(\sigma)d\sigma} d\tau ds \\ &= \lambda^* \int_0^a k(\tau)e^{-\lambda^*\int_0^\tau k(\sigma)d\sigma} \int_\tau^a \alpha(s)e^{-\int_\tau^s \alpha(\sigma)d\sigma} e^{-\int_s^a \gamma(\sigma)d\sigma} ds d\tau. \end{aligned} \tag{4.6}$$

Finally, substituting the values of $e^*(a)$ and $i^*(a)$ from the equations (4.5) and (4.6), respectively, into (4.1e), we get

$$\lambda^* = \int_0^{a_+} \beta(a)\lambda^* \int_0^a k(\tau)e^{-\lambda^*\int_0^\tau k(\sigma)d\sigma} [e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(s)e^{-\int_\tau^s \alpha(\sigma)d\sigma} e^{-\int_s^a \gamma(\sigma)d\sigma} ds] d\tau da.$$

By dividing both sides of the above equation by λ^* (since $\lambda^* \neq 0$), we obtain

$$1 = \int_0^{a_+} \beta(a)\int_0^a k(\tau)e^{-\lambda^*\int_0^\tau k(\sigma)d\sigma} [e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(s)e^{-\int_\tau^s \alpha(\sigma)d\sigma} e^{-\int_s^a \gamma(\sigma)d\sigma} ds] d\tau da. \tag{4.7}$$

Let us denote the expression in the right-hand side of (4.8) by $G(\lambda^*)$, then, $G(\lambda^*)$ is a continuous and monotone decrease function of λ^*. It is easy to see that the system (2.7) has an endemic equilibrium expressed by equation (4.2), (4.3) and (4.4) provided the equation (4.7) has a positive solution λ^*. From (3.10) it follows that $G(0) = R_0$. Hence, if $R_0 > 1$, then $G(0) > 1$. From (4.5), (4.6) and $e^*(a) + i^*(a) < 1$, it follows that

$$\lambda^* \int_0^a k(\tau)e^{-\lambda^*\int_0^\tau k(\sigma)d\sigma} [e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(s)e^{-\int_\tau^s \alpha(\sigma)d\sigma} \cdot e^{-\int_s^a \gamma(\sigma)d\sigma} ds] d\tau < 1. \tag{4.8}$$

Thus, for any $\lambda^* > 0$, from (4.7) and (4.8) we can write

$$\lambda^* G(\lambda^*) = \int_0^{a_+} \beta(a)\lambda^* \int_0^a k(\tau) e^{-\lambda^* \int_0^\tau k(\sigma)d\sigma}$$
$$\cdot [e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(s) e^{-\int_\tau^s \alpha(\sigma)d\sigma} e^{-\int_s^a \gamma(\sigma)d\sigma} ds] d\tau da \& < \int_0^{a_+} \beta(a)da.$$

Let $\beta^+ = \int_0^{a_+} \beta(a)da$. In particular, for $\lambda^* = \beta^+$, we have $G(\beta^+) < 1$, but $G(0) > 1$. Since $G(\lambda^*)$ is a continuous and monotone decrease function of λ^*, we conclude that $G(\lambda^*) = 1$ has a unique positive solution $\tilde{\lambda}^*$ in $(0, \beta^+)$. Noting the fact that $\tilde{\lambda}^* \le \beta^+$, the system (2.7) has a unique endemic equilibrium which is given by the unique solution of the system (4.1) corresponding to $\tilde{\lambda}^*$ provided $R_0 > 1$. This completes the proof.

We note that the above theorem establishes the existence of the endemic equilibrium. In the following, we try to show the local stability of this endemic equilibrium.

Let $\hat{s}(a,t)$, $\hat{e}(a,t)$, $\hat{i}(a,t)$, $\hat{r}(a,t)$ and $\hat{\lambda}(t)$ be the perturbations in $s(a,t)$, $e(a,t)$, $i(a,t)$, $r(a,t)$ and $\lambda(t)$, respectively, i.e.,

$$\begin{aligned} & s(a,t) = \hat{s}(a,t) + s^*(a), \quad e(a,t) = \hat{e}(a,t) + e^*(a), \\ & i(a,t) = \hat{i}(a,t) + i^*(a), \quad r(a,t) = \hat{r}(a,t) + r^*(a), \\ & \lambda(t) \quad = \hat{\lambda}(t) + \lambda^*. \end{aligned} \tag{4.9}$$

These perturbations satisfy the following system of equations:

$$\frac{\partial \hat{s}(a,t)}{\partial t} + \frac{\partial \hat{s}(a,t)}{\partial a} = -\lambda^* k(a)\hat{s}(a,t) - \hat{\lambda}(t)k(a)s^*(a), \tag{4.10a}$$
$$\frac{\partial \hat{e}(a,t)}{\partial t} + \frac{\partial \hat{e}(a,t)}{\partial a} = -\alpha(a)\hat{e}(a,t) + \lambda^* k(a)\hat{s}(a,t) + \hat{\lambda}(t)k(a)s^*(a), \tag{4.10b}$$
$$\frac{\partial \hat{i}(a,t)}{\partial t} + \frac{\partial \hat{i}(a,t)}{\partial a} = -\gamma(a)\hat{i}(a,t) + \alpha(a)\hat{e}(a,t), \tag{4.10c}$$
$$\frac{\partial \hat{r}(a,t)}{\partial t} + \frac{\partial \hat{r}(a,t)}{\partial a} = \gamma(a)\hat{i}(a,t), \tag{4.10d}$$
$$\hat{\lambda}(t) = \int_0^{a_+} \beta(a)[\hat{e}(a,t) + \hat{i}(a,t)]da, \tag{4.10e}$$
$$\lambda^* = \int_0^{a_+} \beta(a)[e^*(a) + i^*(a)]da. \tag{4.10f}$$

We look for the following exponential solutions of the perturbed system (4.10)

$$\hat{s}(a,t) = \tilde{s}(a)e^{\lambda t},\ \hat{e}(a,t) = \tilde{e}(a)e^{\lambda t},\ \hat{i}(a,t) = \tilde{i}(a)e^{\lambda t},\ \hat{r}(a,t) = \tilde{r}(a)e^{\lambda t}, \text{ and } \hat{\lambda}(t) = e^{\lambda t}\tilde{\lambda}.$$

The functions $\tilde{s}(a), \tilde{e}(a), \tilde{i}(a), \tilde{r}(a)$ and the parameter $\tilde{\lambda}$ satisfy the following system of equations:

$$\frac{d\tilde{s}(a)}{da} = -\lambda\tilde{s}(a) - \lambda^* k(a)\tilde{s}(a) - \tilde{\lambda}k(a)s^*(a), \tag{4.11a}$$
$$\frac{d\tilde{e}(a)}{da} = -\lambda\tilde{e}(a) - \alpha(a)\tilde{e}(a) + \lambda^* k(a)\tilde{s}(a) + \tilde{\lambda}k(a)s^*(a), \tag{4.11b}$$
$$\frac{d\tilde{i}(a)}{da} = -\lambda\tilde{i}(a) - \gamma(a)\tilde{i}(a) + \alpha(a)\tilde{e}(a), \tag{4.11c}$$
$$\frac{d\tilde{r}(a)}{da} = -\lambda\tilde{r}(a) + \gamma(a)\tilde{i}(a), \tag{5.3d}$$
$$\tilde{\lambda} = \int_0^{a_+} \beta(a)[\tilde{e}(a) + \tilde{i}(a)]da, \tag{4.11e}$$
$$\lambda^* = \int_0^{a_+} \beta(a)[e^*(a) + i^*(a)]da, \tag{4.11f}$$
$$\tilde{s}(0) = \tilde{e}(0) = \tilde{i}(0) = \tilde{r}(0) = 0. \tag{4.11g}$$

We note that the functions $\tilde{s}(a), \tilde{e}(a), \tilde{i}(a), \tilde{r}(a)$ can take both positive and negative values. Assuming $\tilde{\lambda} \neq 0$ we set

$$\bar{s}(a) = \frac{\tilde{s}(a)}{\tilde{\lambda}}, \quad \bar{e}(a) = \frac{\tilde{e}(a)}{\tilde{\lambda}}, \quad \bar{i}(a) = \frac{\tilde{i}(a)}{\tilde{\lambda}}, \quad \bar{r}(a) = \frac{\tilde{r}(a)}{\tilde{\lambda}},$$

and obtain the following system:

$$\frac{d\bar{s}(a)}{da} = -\lambda\bar{s}(a) - \lambda^* k(a)\bar{s}(a) - k(a)s^*(a), \tag{4.12a}$$
$$\frac{d\bar{e}(a)}{da} = -\lambda\bar{e}(a) - \alpha(a)\bar{e}(a) + \lambda^* k(a)\bar{s}(a) + k(a)s^*(a), \tag{4.12b}$$
$$\frac{d\bar{i}(a)}{da} = -\lambda\bar{i}(a) - \gamma(a)\bar{i}(a) + \alpha(a)\bar{e}(a), \tag{4.12c}$$
$$\frac{d\bar{r}(a)}{da} = -\lambda\bar{r}(a) + \gamma(a)\bar{i}(a), \tag{4.12d}$$
$$\lambda^* = \int_0^{a_+} \beta(a)[e^*(a) + i^*(a)]da, \tag{4.12e}$$
$$1 = \int_0^{a_+} \beta(a)[\bar{e}(a) + \bar{i}(a)]da, \tag{4.12f}$$
$$\bar{s}(0) = \bar{e}(0) = \bar{i}(0) = \bar{r}(0) = 0. \tag{4.12g}$$

Solving (4.12a), (4.12b) and (4.12c) we get the following:

$$\bar{s}(a) = -\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}e^{-\lambda^*\int_\tau^a k(\sigma)d\sigma}d\tau. \tag{4.13}$$

$$\overline{e}(a)=\int_0^a[\lambda^*k(\tau)\overline{s}(\tau)+k(\tau)s^*(\tau)]e^{-\lambda(a-\tau)}e^{-\int_\tau^a\alpha(\sigma)d\sigma}d\tau$$

$$=\lambda^*\int_0^a k(\tau)\overline{s}(\tau)e^{-\lambda(a-\tau)}e^{-\int_\tau^a\alpha(\sigma)d\sigma}d\tau+\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}e^{-\int_\tau^a\alpha(\sigma)d\sigma}d\tau. \tag{4.14}$$

$$\overline{i}(a)=\int_0^a\alpha(\tau)\overline{e}(\tau)e^{-\lambda(a-\tau)}e^{-\int_\tau^a\gamma(\sigma)d\sigma}d\tau. \tag{4.15}$$

Let us denote the expression in the right-hand side of (4.12f) by $Q(\lambda)$, i.e.

$$Q(\lambda)=\int_0^{a_+}\beta(a)[\overline{e}(a)+\overline{i}(a)]da=1. \tag{4.16}$$

We now consider Equation (4.16) in two different ways. Firstly, substituting (4.14) into (4.15) and changing the order of integration we obtain

$$\begin{aligned}\overline{i}(a)&=\lambda^*\int_0^a\alpha(\tau)[\int_0^\tau k(\eta)\overline{s}(\eta)e^{-\lambda(\tau-\eta)}e^{-\int_\eta^\tau\alpha(\sigma)d\sigma}d\eta)]e^{-\lambda(a-\tau)}e^{-\int_\tau^a\gamma(\sigma)d\sigma}d\tau\\&\quad+\int_0^a\alpha(\tau))[\int_0^\tau k(\eta)s^*(\eta)e^{-\lambda(\tau-\eta)}e^{-\int_\eta^\tau\alpha(\sigma)d\sigma}d\eta)]e^{-\lambda(a-\tau)}e^{-\int_\tau^a\gamma(\sigma)d\sigma}d\tau\\&=\lambda^*\int_0^a k(\eta)\overline{s}(\eta)e^{-\lambda(a-\eta)}\int_\eta^a\alpha(\tau)e^{-\int_\eta^\tau\alpha(\sigma)d\sigma}e^{-\int_\tau^a\gamma(\sigma)d\sigma}d\tau d\eta\\&\quad+\int_0^a k(\eta)s^*(\eta)e^{-\lambda(a-\eta)}\int_\eta^a\alpha(\tau)e^{-\int_\eta^\tau\alpha(\sigma)d\sigma}e^{-\int_\tau^a\gamma(\sigma)d\sigma}d\tau d\eta\\&=\lambda^*\int_0^a k(\tau)\overline{s}(\tau)e^{-\lambda(a-\tau)}\int_\tau^a\alpha(\eta)e^{-\int_\tau^\eta\alpha(\sigma)d\sigma}e^{-\int_\eta^a\gamma(\sigma)d\sigma}d\eta d\tau\\&\quad+\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}\int_\tau^a\alpha(\eta)e^{-\int_\tau^\eta\alpha(\sigma)d\sigma}e^{-\int_\eta^a\gamma(\sigma)d\sigma}d\eta d\tau.\end{aligned} \tag{4.17}$$

Substituting (4.14) and (4.17) into (4.16) we get an expression for $Q(\lambda)$ as follows

$$\begin{aligned}1&=\lambda^*\int_0^{a_+}\beta(a)\left\{\int_0^a k(\tau)e^{-\lambda(a-\tau)}\left[e^{-\int_\tau^a\alpha(\sigma)d\sigma}+\int_\tau^a\alpha(\eta)e^{-\int_\tau^\eta\alpha(\sigma)d\sigma}e^{-\int_\eta^a\gamma(\sigma)d\sigma}d\eta\right]\overline{s}(\tau)d\tau\right\}da\\&\quad+\int_0^{a_+}\beta(a)\left\{\int_0^a k(\tau)e^{-\lambda(a-\tau)}\left[e^{-\int_\tau^a\alpha(\sigma)d\sigma}+\int_\tau^a\alpha(\eta)e^{-\int_\tau^\eta\alpha(\sigma)d\sigma}e^{-\int_\eta^a\gamma(\sigma)d\sigma}d\eta\right]s^*(\tau)d\tau\right\}da\\&=Q(\lambda)\end{aligned} \tag{4.18}$$

Secondly, substituting (4.13) into (4.14) and changing the order of integration we have

$$\overline{e}(a)=\lambda^*\int_0^a k(\tau)\overline{s}(\tau)e^{-\lambda(a-\tau)}e^{-\int_\tau^a\alpha(\sigma)d\sigma}d\tau+\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}e^{-\int_\tau^a\alpha(\sigma)d\sigma}d\tau$$

$$= \lambda^* \int_0^a k(\tau)[-\int_0^\tau k(\eta)s^*(\eta)e^{-\lambda(\tau-\eta)}e^{-\lambda^*\int_\eta^\tau k(\sigma)d\sigma}d\eta]e^{-\lambda(a-\tau)}e^{-\int_\tau^a \alpha(\sigma)d\sigma}d\tau$$

$$+\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}e^{-\int_\tau^a \alpha(\sigma)d\sigma}d\tau$$

$$= -\lambda^* \int_0^a k(\eta)s^*(\eta)e^{-\lambda(a-\eta)}\int_\eta^a k(\tau)e^{-\lambda^*\int_\eta^\tau k(\sigma)d\sigma}e^{-\int_\tau^a \alpha(\sigma)d\sigma}d\tau d\eta$$

$$+\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}e^{-\int_\tau^a \alpha(\sigma)d\sigma}d\tau$$

$$= -\lambda^* \int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}\int_\tau^a k(\eta)e^{-\lambda^*\int_\tau^\eta k(\sigma)d\sigma}e^{-\int_\eta^a \alpha(\sigma)d\sigma}d\eta d\tau$$

$$+\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}e^{-\int_\tau^a \alpha(\sigma)d\sigma}d\tau$$

$$= \int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}[e^{-\int_\tau^a \alpha(\sigma)d\sigma} - \lambda^* \int_\tau^a k(\eta)e^{-\lambda^*\int_\tau^\eta k(\sigma)d\sigma}e^{-\int_\eta^a \alpha(\sigma)d\sigma}d\eta]d\tau. \tag{4.19}$$

Substituting (4.19) into (4.15) and changing the order of integration it follows

$$\overline{i}(a) = \int_0^a \alpha(\tau)\{\int_0^\tau k(\xi)s^*(\xi)e^{-\lambda(\tau-\xi)}[e^{-\int_\xi^\tau \alpha(\sigma)d\sigma} - \lambda^*\int_\xi^\tau k(\eta)e^{-\lambda^*\int_\xi^\eta k(\sigma)d\sigma}$$

$$\cdot e^{-\int_\eta^\tau \alpha(\sigma)d\sigma}d\eta]d\xi\}e^{-\lambda(a-\tau)}e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau$$

$$= \int_0^a \alpha(\tau)[\int_0^\tau k(\xi)s^*(\xi)e^{-\lambda(\tau-\xi)}e^{-\int_\xi^\tau \alpha(\sigma)d\sigma}d\xi]e^{-\lambda(a-\tau)}e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau$$

$$-\lambda^*\int_0^a \alpha(\tau)\{\int_0^\tau k(\xi)s^*(\xi)e^{-\lambda(\tau-\xi)}[\int_\xi^\tau k(\eta)e^{-\lambda^*\int_\xi^\eta k(\sigma)d\sigma}$$

$$\cdot e^{-\int_\eta^\tau \alpha(\sigma)d\sigma}d\eta]d\xi\}e^{-\lambda(a-\tau)}e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau$$

$$= \int_0^a k(\xi)s^*(\xi)e^{-\lambda(a-\xi)}\int_\xi^a \alpha(\tau)e^{-\int_\xi^\tau \alpha(\sigma)d\sigma}e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau d\xi - \lambda^*\int_0^a k(\xi)s^*(\xi)$$

$$\cdot e^{-\lambda(a-\xi)}\{\int_\xi^a \alpha(\tau)[\int_\xi^\tau k(\eta)e^{-\lambda^*\int_\xi^\eta k(\sigma)d\sigma}e^{-\int_\eta^\tau \alpha(\sigma)d\sigma}d\eta]e^{-\int_\tau^a \gamma(\sigma)d\sigma}d\tau\}d\xi$$

$$= \int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}\int_\tau^a \alpha(\xi)e^{-\int_\tau^\xi \alpha(\sigma)d\sigma}e^{-\int_\xi^a \gamma(\sigma)d\sigma}d\xi d\tau - \lambda^*\int_0^a k(\tau)s^*(\tau)$$

$$\cdot e^{-\lambda(a-\tau)}\{\int_\tau^a \alpha(\xi)[\int_\tau^\xi k(\eta)e^{-\lambda^*\int_\tau^\eta k(\sigma)d\sigma}e^{-\int_\eta^\xi \alpha(\sigma)d\sigma}d\eta]e^{-\int_\xi^a \gamma(\sigma)d\sigma}d\xi\}d\tau$$

$$= \int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}\{\int_\tau^a \alpha(\xi)e^{-\int_\xi^a \gamma(\sigma)d\sigma}[e^{-\int_\tau^\xi \alpha(\sigma)d\sigma} - \lambda^*\int_\tau^\xi k(\eta)$$

$$\cdot e^{-\lambda^*\int_\tau^\eta k(\sigma)d\sigma}e^{-\int_\eta^\xi \alpha(\sigma)d\sigma}d\eta]d\xi\}d\tau. \tag{4.20}$$

Substituting (4.19) and (4.20) into (4.16) we get another expression for $Q(\lambda)$ as follows

$$1=\int_0^{a_+}\beta(a)\{\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}[(e^{-\int_\tau^a \alpha(\sigma)d\sigma}-\lambda^*\int_\tau^a k(\eta)e^{-\lambda^*\int_\tau^\eta k(\sigma)d\sigma}$$
$$\cdot e^{-\int_\eta^a \alpha(\sigma)d\sigma}d\eta)+\int_\tau^a \alpha(\xi)e^{-\int_\xi^a \gamma(\sigma)d\sigma}(e^{-\int_\tau^\xi \alpha(\sigma)d\sigma}-\lambda^*\int_\tau^\xi k(\eta)e^{-\lambda^*\int_\tau^\eta k(\sigma)d\sigma}$$
$$\cdot e^{-\int_\eta^\xi \alpha(\sigma)d\sigma}d\eta)d\xi]d\tau\}da$$
$$=Q(\lambda). \tag{4.21}$$

Let

$$D(a,\tau)=g(a,\tau)+\int_\tau^a G(a,\xi)g(\xi,\tau)d\xi, \tag{4.22}$$

where

$$g(a,\tau)=e^{-\int_\tau^a \alpha(\sigma)d\sigma}-\lambda^*\int_\tau^a k(\eta)e^{-\lambda^*\int_\tau^\eta k(\sigma)d\sigma}e^{-\int_\eta^a \alpha(\sigma)d\sigma}d\eta,$$
$$G(a,\xi)=\alpha(\xi)e^{-\int_\xi^a \gamma(\sigma)d\sigma}.$$

Then, (4.21) can be rewritten as

$$Q(\lambda)=\int_0^{a_+}\beta(a)\int_0^a k(\tau)s^*(\tau)e^{-\lambda(a-\tau)}D(a,\tau)d\tau da, \tag{4.23}$$

and the following result is established

Theorem 4.2.

Let us assume

$$g(a,\tau)\ge 0, \qquad \forall 0\le\tau\le a\le a_+. \tag{4.24}$$

Then,

(a) $Q(\lambda)$ *is a decreasing function of* λ, *which tends to zero as* $\lambda\to+\infty$.

(b) $Q(0)<1$.

Proof:

(a) From equation (4.22) and assumption (4.24) it follows that the function $D(a,\tau)$ defined for $0 \le \tau \le a \le a_+$ is nonnegative and independent of λ. Equation (4.23) clearly show that $Q(\lambda) \ge 0$, which is exponentially decreasing in λ and $Q(\lambda) \to 0$ as $\lambda \to +\infty$.

(b) To show this part we use a different representation of $Q(\lambda)$. From (4.18) we have

$$Q(0) = \lambda^* \int_0^{a_+} \beta(a) \int_0^a k(\tau)[e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(\eta) e^{-\int_\tau^\eta \alpha(\sigma)d\sigma} \cdot e^{-\int_\eta^a \gamma(\sigma)d\sigma} d\eta]\overline{s}(\tau)d\tau da + \int_0^{a_+} \beta(a) \int_0^a k(\tau)[e^{-\int_\tau^a \alpha(\sigma)d\sigma} + \int_\tau^a \alpha(\eta) e^{-\int_\tau^\eta \alpha(\sigma)d\sigma} e^{-\int_\eta^a \gamma(\sigma)d\sigma} d\eta] s^*(\tau) d\tau da. \quad (4.25)$$

The equations (4.2) and (4.7) imply that the second integral above is exactly equal to one. Furthermore, from equation (4.13) it follows that $\overline{s}(a) \le 0$ under the assumption (4.24). Hence, the first integral is negative. Therefore, $Q(0) < 1$. This completes the proof.

The Theorem 4.2 and formula (4.23) imply that the equation $Q(\lambda) = 1$, which is equivalent to (4.16), has a unique real solution which is negative and that all complex solutions have real parts smaller than the unique real solution. Consequently, assumption (4.24) guarantees the stability of the endemic equilibrium. So we have the following result:

Theorem 4.3.

If $R_0 > 1$ and assumption (4.24) is satisfied, then the endemic equilibrium is locally asymptotically stable.

5. Conclusion

Here we proposed and analyzed an age-structured SEIR epidemic model with infectivity in both the latent and infectious periods. We note that, tuberculosis, HIV/AIDS and SARS, are examples of this kind of diseases. It is different from Gui-Hua Li and Zhen Jin (2004, 2005, 2006), where ODE models with infectivity in both the latent and infectious periods are considered. We determined the steady states of the model and proved globally stability results for the disease-free equilibrium and locally stability results for the endemic equilibrium under certain conditions. The question of global stability of the endemic equilibrium is still an open problem.

Acknowledgments

This work is partially supported by the NSF of China (No.10371105; No.10671166) and the NSF of Henan Province (No.0312002000). The authors would like to thank the reviewers for their comments and suggestions, and thank Dr. Mini Ghosh for her carefully reading to the manuscript and correcting the English expressions of the whole paper.

REFERENCES

Grenhalgh, D. (1992). Some results for an SEIR epidemic model with density dependence in the death rate. IMA J. Math. Appl. Med. Biol., **9**, 67-85.

Cooke, K., Driessche, and P. van den (1990). Analysis of an SEIRS epidemic model with two Delays. J. Math. Biol., **35**, 240-258.

Greenhalgh, D. (1997). Hopf bifurcation in epidemic models with a latent period and nonpermanent immunity. Math. Comput. Model., **25**, 85-93.

Li, M. Y. and Muldoweney, J. S. (1995). Global stability for SEIR modle in epidemiology. Math. Biosci., **125**, 155-167.

Li, M. Y. and Muldoweney, J. S. (2001). Global stability of a SEIR epidemic model with vertical transmission. SIAM J. Appl.Math., **62**, 58-69.

Li, M. Y., Muldoweney, J. S., Wang, L. C., Karsai, J. (1999). Global dynamics of an SEIR epidemic model with a varying total population size. Math. Biosci., **160**, 191-213.

Zhang, Juan and Ma, Zhi-en (2003). Global stability of SEIR model with saturating contact rate. Math. Biosci., **185**, 15-32.

Busenberg, S., Iannelli, M. and Thieme, H. (1991). Global behavior of an age- structured epidemic model. SIAM. J. Math. Anal., **22**(4), 1065-1080.

Iannelli, M., Milner, F. and Pugliese, A. (1992). Analytical and numerical results for the age-structured SIS epidemic model with mixed inter-intracohort transmission. SIAM. J. Math. Anal., **23**(3), 662-688.

Cha, Y., Iannelli, M. and Milner, F. (1998). Existence and uniqueness of endemic states for the age-structured SIR epidemic model. Math. Biosci., **150**, 177-190.

Castillo-Chavez, Carlos and Feng, Zhilan (1998). Global stability of age-structure model for TB and its applicaitons to optimal vaccination strategies. Math. Biosci., **151**, 135-154.

Inaba, H. (1990). Threshold and stability results for an age-structured epidemic model. J. Math. Biol., **28**, 411-434.

Li, Xue-Zhi and Gupur, Geni (2004). Global stability of an Age-structure SIRS Epidemic model with Vaccination. Discrete and Continuous Dynamical system-series B, **4**(3), 643-652.

Castillo-Chavez, Carlos, Hethcote, H., Levin, S., Liu, W. M. (1989). Epodemiological model with age structure, proportionate mixing and cross-immunity. J. Math. Biol., **17**, 233-258.

Feng, Zhilan, Huang, Wenzhang , Castillo-Chavez, Carlos (2005). Global behavior of a multi-group SIS epidemic model with age structure. Journal of Differential Equations, **218**(2), 292-324.

Li, Xue-Zhi, Gupur, Geni, Zhu, Guang-Tian (2001). Threshold and stability results for an age-structured SEIR epidemic model. Comp. Math. Appl., **42**, 883-907.

Li, Xue-Zhi, Gupur, Geni, Zhu, Guang-Tian (2003). Analysis for an age-structured SEIR epidemic model with vaccination. Inter. J. Diff. Equ. Appl., **7**(1), 47-67.

Li, Guihua and Jin, Zhen (2004). Global stability of an SEI epidemic model. Chaos, Solitions and Fractals., **21**, 925-931.

Li, Guihua and Jin, Zhen (2005). Global stability of an SEI epidemic model with general contact rate. Chaos, Solitions and Fractals., **23**, 997-1004.

Li, Guihua and Jin, Zhen (2005). Global stability of an SEIR epidemic model with infectious force in latent, infected and immune period. Chaos, Solitions and Fractals., **25**, 1177-1184.

Li, Guihua, Wang, Wendi and Jin, Zhen (2006). Global stability of an SEIR epidemic model with contact immigration. Chaos, Solitons and Fractals, **30**, 1012-1019.

Webb, G. B (1985). Theory of Nonlinear Age-Dependent Population Dynamics. New York and Basel: Marcel dekker.

Iannelli, M. (1995). Mathematical Theory of Age-structured Population Dynamics, Vol.7. Applied Mathematics Monographs, Comitato Nazionale per le Scienze Matematiche, Consiglio Nazionale delle R