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Rate of Convergence for Generalized Szász–Mirakyan operators in exponential weighted space

[1]Sevilay Kırcı Serenbay & [2]Özge Dalmanoğlu

Department of Mathematics Education
Başkent University
Ankara, Turkey
[1]sevilaykirci@gmail.com, [2]ozgedalmanoglu@gmail.com

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Abstract

In the present paper, generalized Szász–Mirakyan operators in exponential weighted space of functions of one variable are introduced. Using a method given by Rempulska and Walczak, some theorems on the degree of approximation are investigated. Furthermore, a numerical example with an illustrative graphic is given to show comparison for the error estimates of the operators.

Keywords: Szász type operators; exponential weighted space; polynomial weighted space; degree of approximation; error estimation; modulus of continuity; modulus of smoothness

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1. Introduction

Let f be a function defined on $[0, \infty)$. The Szász-Mirakyan operators S_n applied to f are given by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}. \tag{1}$$

Becker et. al. investigated the approximation properties of the operators S_n in the exponential weight space and proved main approximation theorems for these operators. In 2002, İspir and Atakut modified the operator S_n as follows:

$$S_n^*(f;x) = \sum_{k=0}^{\infty} p_k(a_n x) f\left(\frac{k}{b_n}\right) \quad , \quad x \in \mathbb{R}_0, \tag{2}$$

where

$$p_k(a_n x) = e^{-a_n x} \frac{(a_n x)^k}{k!},$$

$\mathbb{R}_0 = [0, \infty)$, $\{a_n\}$ and $\{b_n\}$ are increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{1}{b_n} = 0, \; \frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right).$$

It is easily seen that $S_n^*(f;x)$ is a positive linear operator. When $a_n = b_n = n$ in (2), we obtain the original Szász-Mirakyan operators given in (1). İspir and Atakut (2002) and Walczak (2002) studied some approximation properties of these operators in polynomial weighted spaces of continuous and unbounded functions defined on positive semi-axis. İspir and Atakut also obtained the order of approximation and investigated the bivariate case of the operators.

Herzog (2003) investigated Szász-Mirakyan type operators defined by modified Bessel functions. Besides this paper, Herzog (2015) also presented the approximation properties of these operators for functions from exponential weight spaces. The bivariate version of the same operators are also introduced by the same author.

Before proceeding further, let us first introduce the exponential weighted spaces, which in this paper we denote by C_r, for $r > 0$ a fixed number. Let $C(\mathbb{R}_0)$ be the set of all real-valued functions continuous on $\mathbb{R}_0 = [0, \infty)$. The exponential weighted space is defined as

$$C_r = \{f \in C(\mathbb{R}_0) : \nu_r f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\}, \tag{3}$$

where ν_r is the exponential weight function defined by

$$\nu_r(x) := e^{-rx}, \; x \in R_0 := [0, \infty). \tag{4}$$

The norm in this space is given by the formula

$$\|f\|_r \equiv \|f(.)\|_r := \sup_{x \in R_0} \nu_r(x) |f(x)|. \tag{5}$$

Rempulska and Walczak examined the approximation properties of the following modified Szász-Mirakyan operators,

$$S_n\left(f;r;x\right)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n+r}\right),\quad r>0,\ x\in R_0, \tag{6}$$

for function $f\in B_r$, where $B_r, r>0$, denotes the space of all real-valued functions f defined on R_0 for which $\nu_r f$ is bounded function on R_0 and the norm is given by (5).

Note that, in this paper we consider the modulus of continuity of $f\in C_r$,

$$\omega_1(f;C_r;t):=\sup_{0\le h\le t}\left\|\Delta_h f\left(.\right)\right\|_r$$

and modulus of smoothness of $f\in C_r$,

$$\omega_2(f;C_r;t)=\sup_{0\le h\le t}\left\|\Delta_h^2 f\left(.\right)\right\|_r,$$

for $t\ge 0$, where

$$\Delta_h f(x):=f(x+h)-f(x),\qquad \Delta_h^2 f(x):=f(x)-2f(x+h)+f(x+2h),$$

for $x,h\in\mathbb{R}_0$. (See De Vore and Lorentz (1993)). Moreover, for fixed $m\in\mathbb{N}$, the function space C_r^m is defined as

$$C_r^m=\left\{f\in C_r:f^{(k)}\in C_r,k=1,2,...,m\right\}.$$

Inspired by the method used by İspir and Atakut (2002) as well as Rempulska and Walczak (2001), we generalize the operators $S_n\left(f;r;x\right)$ given in (6) with the help of the sequences (a_n) and (b_n). For functions $f\in B_r$ and $n\in\mathbb{N}$, we define the following generalized Szász-Mirakyan operators

$$S_n^*\left(f;r,x\right):=\begin{cases}\sum_{k=0}^{\infty}p_k(a_nx)f\left(\dfrac{k}{r+b_n}\right), & x>0\\ f\left(0\right), & x=0,\end{cases} \tag{7}$$

where

$$p_k(a_nx)=e^{-a_nx}\frac{(a_nx)^k}{k!}, \tag{8}$$

$R_0=[0,\infty)$, $\{a_n\}$ and $\{b_n\}$ are given increasing and unbounded sequences of positive numbers satisying the conditions

$$(a_n) \leq (b_n), \quad \lim_{n\to\infty} \frac{1}{r+b_n} = 0, \quad \frac{a_n}{r+b_n} = 1 + O\left(\frac{1}{r+b_n}\right), \tag{9}$$

for all $n \in \mathbb{N}$. We shall prove some approximation theorems in the exponential weighted space of functions by these generalized operators. We will give some auxiliary Lemmas in Section 2. Section 3 is devoted to our main results. We shall prove our approximation theorems in this section.

2. Auxiliary Results and Lemmas

In this section we shall mention basic properties of the operators $S_n^*\left(f;r,x\right)$.We also give some auxiliary Lemmas that we need when proving our main results.

Lemma 2.1.

For each $n \in \mathbb{N}, x \in \mathbb{R}_0$ and $r > 0$, we have

$$\begin{aligned} S_n^*\left(1;r,x\right) &= 1, \\ S_n^*\left(t;r,x\right) &= \frac{a_n x}{r+b_n}, \\ S_n^*\left(t^2;r,x\right) &= \frac{a_n x + \left(a_n x\right)^2}{\left(r+b_n\right)^2}, \\ S_n^*\left(\left(t-x\right);r,x\right) &= x\left(\frac{a_n}{r+b_n} - 1\right), \\ S_n^*\left(\left(t-x\right)^2;r,x\right) &= \frac{x^2\left(a_n - \left(r+b_n\right)\right)^2 + a_n x}{\left(r+b_n\right)^2}. \end{aligned} \tag{10}$$

Lemma 2.2.

For each $n \in \mathbb{N}, x \in \mathbb{R}_0$ and $r > 0$, we have

$$\begin{aligned} S_n^*\left(e^{rt};r;x\right) &= e^{c_n x}, \\ S_n^*\left(te^{rt};r;x\right) &= \frac{a_n x}{r+b_n} e^{\frac{r}{r+b_n}} e^{c_n x}, \\ S_n^*\left(t^2 e^{rt};r;x\right) &= \frac{a_n x}{\left(r+b_n\right)^2} e^{\frac{r}{r+b_n}} \left\{a_n x e^{\frac{r}{r+b_n}} + 1\right\} e^{c_n x}, \end{aligned} \tag{11}$$

where

$$c_n := a_n\left(e^{\frac{r}{r+b_n}} - 1\right). \tag{12}$$

By elementary calculations we get the equalities given above. So we omit the proofs.

With the help of the identities in (10), (11) and (12), we shall prove two main lemmas.

Lemma 2.3.

Let $r>0$ be a fixed number. For all $n\in\mathbb{N}$ and $f\in B_r$, we have

$$\left\|S_n^*\left(\frac{1}{\nu_r};r,.\right)\right\|_r\leq 1. \tag{13}$$

Proof:

From the series expansion of the exponential function one can find

$$a_n\left(e^{\frac{r}{r+b_n}}-1\right)<\frac{a_n}{b_n}r, \tag{14}$$

from which we have $c_n-r<0$ by (9). Since

$$\nu_r(x)S_n^*\left(\frac{1}{\nu_r(t)};r,x\right)=e^{(c_n-r)x}, \tag{15}$$

by (5) and the above inequality, we get the desired result. ■

Theorem 2.4.

Let $r>0$ be a fixed number. For all $n\in\mathbb{N}$ and $f\in B_r$, we have

$$\left\|S_n^*\left(f;r,.\right)\right\|_r\leq\left\|f\right\|_r. \tag{16}$$

Proof:

From (13) and from the definiton of the norm on the space B_r, we get

$$\begin{aligned}\left\|S_n^*\left(f;r,.\right)\right\|_r&=\sup_{x\in R_0}\nu_r(x)\left|S_n^*\left(f(t);r,x\right)\right|\leq\sup_{x\in R_0}\nu_r(x)S_n^*\left(\left|f(t)\right|;r,x\right)\\&=\sup_{x\in R_0}\nu_r(x)S_n^*\left(\nu_r(t)\left|f(t)\right|\frac{1}{\nu_r(t)};r,x\right)\\&\leq\left\|f\right\|_r\sup_{x\in R_0}\nu_r(x)S_n^*\left(\frac{1}{\nu_r(t)};r,x\right)\\&\leq\left\|f\right\|_r.\end{aligned}$$

■

Inequality (16) shows that $S_n^*\left(f;r,x\right)$ is a positive linear operator from the space B_r into C_r.

Lemma 2.5.

For every fixed $r > 0$, for all $x \in R_0$ and $n \in N$ we have

$$\nu_r(x) S_n^*\left(\frac{(t-x)^2}{\nu_r(t)}; r; x\right) \le 2e^{2r/(r+b_n)}\left[\left(\frac{a_n}{r+b_n}-1\right)^2+\left(\frac{r}{r+b_n}\right)^2\right]x^2 + \frac{a_n}{(r+b_n)^2}e^{r/(r+b_n)}x. \tag{17}$$

Proof:

Since $S_n^*\left((t-x)^2 e^{rt}; r; x\right) = S_n^*\left(t^2 e^{rt}; r; x\right) - 2xS_n^*\left(te^{rt}; r; x\right) + x^2 S_n^*\left(e^{rt}; r; x\right)$, with the help of the equalities in (11), we have

$$\begin{aligned} S_n^*\left((t-x)^2 e^{rt}; r; x\right) &= \frac{a_n x}{(r+b_n)^2} e^{\frac{r}{r+b_n}}\left\{a_n x e^{\frac{r}{r+b_n}} + 1\right\} e^{c_n x} - 2x\frac{a_n x}{r+b_n} e^{\frac{r}{r+b_n}} e^{c_n x} + x^2 e^{c_n x} \\ &= \left\{x^2\left(\frac{a_n}{r+b_n}e^{r/(r+b_n)}-1\right)^2 + \frac{a_n x}{(r+b_n)^2}e^{r/(r+b_n)}\right\} e^{c_n x}, \end{aligned} \tag{18}$$

where c_n is given in (12). Using the inequalities $(a+b)^2 \le 2(a^2+b^2)$ and $e^t - 1 \le te^t$ for $t \ge 0$, we get

$$\begin{aligned} \left(\frac{a_n}{r+b_n}e^{r/(r+b_n)}-1\right)^2 &\le 2\left\{\left(\frac{a_n}{r+b_n}-1\right)^2 e^{2r/(r+b_n)} + \left(e^{r/(r+b_n)}-1\right)^2\right\} \\ &\le 2e^{2r/(r+b_n)}\left\{\left(\frac{a_n}{r+b_n}-1\right)^2 + \left(\frac{r}{r+b_n}\right)^2\right\}. \end{aligned}$$

Hence, from (18) we have,

$$\nu_r(x) S_n^*\left(\frac{(t-x)^2}{\nu_r(t)}; r; x\right) \le \left\{2x^2 e^{2r/(r+b_n)}\left[\left(\frac{a_n}{r+b_n}-1\right)^2 + \left(\frac{r}{r+b_n}\right)^2\right] + \frac{a_n x}{(r+b_n)^2}e^{r/(r+b_n)}\right\} e^{(c_n - r)x}, \tag{19}$$

which yields the result. ■

3. Approximation Theorems

In this section we will give our main results on the approximation of the operators $S_n^*(f; r; x)$.

Theorem 3.1.

Let $r > 0$ and $x_0 \in R_0$ be a point of continuity of f. Then for $f \in B_r$, we have

$$\lim_{n\to\infty} S_n^*\left(f;r,x_0\right)=f(x_0). \tag{20}$$

Proof:

Let $x_0>0$. From (7) and (10) we obtain

$$S_n^*(f;r;x_0)-f(x_0)=\sum_{k=0}^{\infty} p_k(a_nx_0)\left(f\left(\frac{k}{r+b_n}\right)-f(x_0)\right), \qquad n\in N.$$

From the hypothesis of the theorem, there exists a number $\delta=\delta(\varepsilon;x_0)>0$ such that,

$$\left|f\left(\frac{k}{r+b_n}\right)-f(x_0)\right|<\frac{\varepsilon}{2}, \tag{21}$$

for $\left|\frac{k}{(r+b_n)}-x_0\right|<\delta$. On the other hand, the linearity of S_n^* implies

$$\begin{aligned}\nu_r\left(x_0\right)\left|S_n^*\left(f;r;x_0\right)-f\left(x_0\right)\right| &\le \sum_{|k/(r+b_n)-x_0|<\delta} \nu_r\left(x_0\right)p_k(a_nx_0)\left|f\left(\frac{k}{r+b_n}\right)-f\left(x_0\right)\right| \\ &\quad+\sum_{|k/(r+b_n)-x_0|\ge\delta} \nu_r\left(x_0\right)p_k(a_nx_0)\left|f\left(\frac{k}{r+b_n}\right)-f\left(x_0\right)\right| \\ &= I_1+I_2.\end{aligned}$$

From (21), we have for all $n\in N$

$$\begin{aligned} I_1 &= \sum_{|k/(r+b_n)-x_0|<\delta} \nu_r\left(x_0\right)p_k(a_nx_0)\left|f\left(\frac{k}{r+b_n}\right)-f\left(x_0\right)\right| \\ &< \frac{\varepsilon}{2}\sum_{k=0}^{\infty}\nu_r\left(x_0\right)p_k(a_nx_0)=\frac{\varepsilon}{2}. \end{aligned} \tag{22}$$

If $\left|\frac{k}{(r+b_n)}-x_0\right|\ge\delta$, then $\frac{1}{\delta^2}\left(\frac{k}{r+b_n}-x_0\right)^2\ge 1$. For $f\in B_r$ we also have

$$\left|f\left(\frac{k}{r+b_n}\right)-f(x_0)\right|\le\|f\|_r\left(e^{rk/(r+b_n)}+e^{rx_0}\right),\ k\in N_0, n\in N.$$

Hence, we can write

$$
\begin{aligned}
I_2 &= \sum_{|k/(r+b_n)-x_0|\geq\delta} \nu_r(x_0)\, p_k(a_n x_0) \left| f\left(\frac{k}{r+b_n}\right) - f(x_0) \right| \\
&\leq \|f\|_r \frac{1}{\delta^2} \nu_r(x_0) \sum_{|k/(r+b_n)-x_0|\geq\delta} p_k(a_n x_0) \left(e^{rk/(r+b_n)} + e^{rx_0}\right) \left(\left(\frac{k}{r+b_n}\right) - x_0\right)^2 \\
&\leq \|f\|_r \frac{1}{\delta^2} \left[\nu_r(x_0)\, S_n^*\left(e^{rt}(t-x_0)^2; r; x_0\right) + S_n^*\left((t-x_0)^2; r; x_0\right)\right].
\end{aligned}
$$

From the last identity in (10) and from Lemma 2.5, we eventually have

$$
\begin{aligned}
I_2 \leq \|f\|_r \frac{1}{\delta^2} &\left\{ \left(\frac{a_n}{r+b_n} - 1\right)^2 \left(1 + 2e^{2r/(r+b_n)}\right) x_0^2 \right. \\
&\left. + 2e^{2r/(r+b_n)} \left(\frac{r}{r+b_n}\right)^2 x_0^2 + \frac{a_n}{(r+b_n)^2} (e^{r/(r+b_n)} + 1) x_0 \right\}.
\end{aligned} \tag{23}
$$

Hence, for fixed positive x_0, δ and $\|f\|_r$ there exists $n_0 \in N$ and for all $n > n_0$

$$
\sum_{|k/(r+b_n)-x_0|\geq\delta} \nu_r(x_0)\, p_k(a_n x_0) \left| f\left(\frac{k}{r+b_n}\right) - f(x_0) \right| < \frac{\varepsilon}{2}. \tag{24}
$$

From (22) and (24) we obtain

$$
\nu_r(x_0) \left| S_n^*(f; r; x_0) - f(x_0) \right| < \varepsilon\ . \tag{25}
$$

which yields

$$
\lim_{n\to\infty} S_n^*(f; r, x_0) = f(x_0). \qquad \blacksquare
$$

Now we establish the next theorem.

Theorem 3.2.

Let C_r^2 be the space of functions such that $f, f'' \in C_r$. If $r > 0$ and $f \in C_r^2$ then for all $x \in R_0, n \in N$ we have

$$\begin{aligned}\|\{S_n^*(f(t);r;x)-f\}\Psi\|_r &\le \|f'\|_r\left|\frac{a_n}{r+b_n}-1\right| \\ &+\|f''\|_r\left\{\left(\frac{a_n}{r+b_n}-1\right)^2\left(1+2e^{2r/(r+b_n)}\right)\right. \\ &\left.+2e^{2r/(r+b_n)}\left(\frac{r}{r+b_n}\right)^2+\frac{a_n}{(r+b_n)^2}(e^{r/(r+b_n)}+1)\right\}, \end{aligned} \tag{26}$$

where

$$\psi(x) := (1+x^2)^{-1}, \qquad x \in R_0. \tag{27}$$

Proof:

Let $x \in R_0$ be a fixed point. Using Taylor's formula for $f \in C_r^2$ and $t \in R_0$, we have

$$f(t) = f(x) + f^{'}(x)(t-x) + \int_x^t (t-u)\, f^{''}(u)\, du.$$

Applying the operator $S_n^*(f;r,x)$ to the both side of the above equality, we get

$$S_n^*(f(t);r,x) = f(x) + f^{'}(x)S_n^*((t-x);r,x) + S_n^*\left(\int_x^t (t-u)\, f^{''}(u)\, du; r, x\right), n \in N. \tag{28}$$

For the integral term above, we can write

$$\left|\int_x^t (t-u)\, f''(u)\, du\right| \le \left\|f^{''}\right\|_r\left(\frac{1}{\nu_r(t)}+\frac{1}{\nu_r(x)}\right)(t-x)^2. \tag{29}$$

Multiplying both side of (28) by $\nu_r(x)$ and then using (29), we get,

$$\begin{aligned}\nu_r(x)\left|S_n^*(f(t);r;x)-f(x)\right| &\le \left\|f^{'}\right\|_r\left|S_n^*((t-x);r;x)\right| + \\ &+\left\|f^{''}\right\|_r\left\{\nu_r(x)\,S_n^*\left(\frac{(t-x)^2}{\nu_r(t)};r;x\right)+S_n^*\left((t-x)^2;r;x\right)\right\}.\end{aligned}$$

By using (10) and Lemma 2.5, we finally get

$$\nu_r(x)\left|S_n^*(f(t);r,x)-f(x)\right| \le \|f'\|_r \left|\frac{a_n}{r+b_n}-1\right| x + \|f''\|_r \left\{ \left(\frac{a_n}{r+b_n}-1\right)^2 \left(1+2e^{2r/(r+b_n)}\right)x^2 + 2e^{2r/(r+b_n)}\left(\frac{r}{r+b_n}\right)^2 x^2 + \frac{a_n}{(r+b_n)^2}(e^{r/(r+b_n)}+1)x \right\}, \tag{30}$$

from which can write

$$\nu_r(x)\left|\{S_n^*(f(t);r;x)-f\}\Psi\right| \le \|f'\|_r \left|\frac{a_n}{r+b_n}-1\right| \frac{x}{1+x^2} + \|f''\|_r \left\{ \left(\frac{a_n}{r+b_n}-1\right)^2 \left(1+2e^{2r/(r+b_n)}\right)\frac{x^2}{1+x^2} + 2e^{2r/(r+b_n)}\left(\frac{r}{r+b_n}\right)^2 \frac{x^2}{1+x^2} + \frac{a_n}{(r+b_n)^2}(e^{r/(r+b_n)}+1)\frac{x}{1+x^2} \right\}. \tag{31}$$

Taking the supremum of both sides, we get the desired result. ■

Theorem 3.3.

Let $r>0$ be a fixed number. For all $n\in N, x\in R_0$ and $f\in C_r$, we have

$$\left\|\left(S_n^*(f_h;r,.)-f_h\right)\Psi\right\|_r \le 5e^{r\delta_{n,r}}\omega_1(f;C_r,\delta_{n,r}) + (207+9e^2)\omega_2(f;C_r,\delta_{n,r}), \tag{32}$$

where

$$\delta_{n,r} = \max_{n\in N, r>0}\left\{\delta_{n,r}^1, \delta_{n,r}^2\right\}$$

with

$$\delta_{n,r}^1 = \left|\frac{a_n}{r+b_n}-1\right|, \qquad \delta_{n,r}^2 = \left(\frac{r^2+b_n}{(r+b_n)^2}\right). \tag{33}$$

Proof:

Let $x\in R_0$ and f_h be the second order Steklov mean of $f\in C_r$, i.e,

$$f_h(x) := \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left\{2f(x+s+t)-f(x+2(s+t))\right\}ds\,dt,\ x\in R_0, h>0.$$

Notice that

$$f(x) - f_h(x) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \Delta_{s+t}^2 ds dt.$$

$$f_h'(x) = \frac{1}{h^2} \int_0^{h/2} \left[8\Delta_{h/2} f(x+s) - 2\Delta_h f(x+2s) \right] ds.$$

$$f_h''(x) = \frac{1}{h^2} \left[8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x) \right].$$

From the definition of norm and the modulus of continuity and smoothness of $f \in C_r$, we get

$$\|f - f_h\|_r \leq \omega_2(f; C_r; h), \tag{34}$$

$$\left\|f_h'\right\|_r \leq 5e^{rh} \frac{1}{h} \omega_1(f; C_r; h), \tag{35}$$

$$\left\|f_h''\right\|_r \leq \frac{9}{h^2} \omega_2(f; C_r; h). \tag{36}$$

The above inequalities imply that if $f \in C_r$, the Steklov mean f_h, f_h' and f_h'' belongs to C_r. By the linearity of the operator we can write,

$$\begin{aligned} \nu_r(x) |S_n^*(f; r, x) - f(x)| &\leq \nu_r(x) \{ |S_n^*(f - f_h; r, x)| + |S_n^*(f_h; r, x) - f_h(x)| \\ &\quad + |f_h(x) - f(x)| \}, \end{aligned} \tag{37}$$

for $x > 0, h > 0$ and $n \in \mathbb{N}$, from which we have,

$$\begin{aligned} &\|(S_n^*(f; r, .) - f) \Psi\|_r \\ &\leq \|S_n^*(f - f_h; r, .) \Psi\|_r + \|(S_n^*(f_h; r, .) - f_h) \Psi\|_r + \|(f_h - f) \Psi\|_r. \end{aligned} \tag{38}$$

By (16), (27) and (34), we get

$$\|S_n^*(f - f_h; r, .) \Psi\|_r \leq \|f - f_h\|_r \leq \omega_2(f; C_r; h), \tag{39}$$

for $n \in N$ and $h > 0$. Applying Theorem 3.2, using (35) and (36), we get,

$$
\begin{aligned}
\left\|\left(S_n^*\left(f_h;r,.\right)-f_h\right)\Psi\right\|_r \leq & \left|\frac{a_n}{r+b_n}-1\right|\left\|f_h^{'}\right\|_r+\left\|f_h^{''}\right\|_r\left\{\left(\frac{a_n}{r+b_n}-1\right)^2\left(1+2e^{2r/(r+b_n)}\right)\right. \\
& \left.+2e^{2r/(r+b_n)}\left(\frac{r}{r+b_n}\right)^2+\frac{a_n}{(r+b_n)^2}(e^{r/(r+b_n)}+1)\right\} \\
& \leq 5e^{rh}\frac{1}{h}\left|\frac{a_n}{(r+b_n)}-1\right|\omega_1\left(f;C_r,h\right) \\
& +\frac{9}{h^2}\left\{\left(\frac{a_n}{r+b_n}-1\right)^2\left(1+2e^{2r/(r+b_n)}\right)\right. \\
& \left.+2e^{2r/(r+b_n)}\left(\frac{r}{r+b_n}\right)^2+\frac{a_n}{(r+b_n)^2}(e^{r/(r+b_n)}+1)\right\}\omega_2\left(f;C_r,h\right), \qquad (40)
\end{aligned}
$$

for $n \in N$ and $h > 0$. By making some computations, we can rewrite (40) as,

$$
\begin{aligned}
\left\|\left(S_n^*\left(f_h;r,.\right)-f_h\right)\Psi\right\|_r \leq & 5e^{rh}\frac{1}{h}\left|\frac{a_n}{r+b_n}-1\right|\omega_1\left(f;C_r,h\right) \\
& +\frac{9}{h^2}\left\{\left(\frac{a_n}{r+b_n}-1\right)^2\left(1+2e^{2r/(r+b_n)}\right)\right\}\omega_2\left(f;C_r,h\right) \\
& +\frac{198}{h^2}\left(\frac{r^2+a_n}{(r+b_n)^2}\right)\omega_2\left(f;C_r,h\right) \\
& =A_1+A_2, \qquad (41)
\end{aligned}
$$

where A_1 is the first two terms and A_2 is the last term on the right hand side of the above inequality. Now, for A_1, setting $h=\delta_{n,r}^1$ where $\delta_{n,r}^1=\left|\frac{a_n}{r+b_n}-1\right|$, and for A_2, setting $h=\delta_{n,r}^2$ where $\delta_{n,r}^2=\left(\frac{r^2+a_n}{(r+a_n)^2}\right)$ for fixed $n \in N$ and $r>0$, we get

$$
\begin{aligned}
\left\|\left(S_n\left(f_h;r,.\right)-f_h\right)\Psi\right\|_r \leq & 5e^{r\delta_{n,r}^1}\omega_1\left(f;C_r,\delta_{n,r}^1\right)+9(1+e^2)\omega_2\left(f;C_r,\delta_{n,r}^1\right) \\
& +198\omega_2\left(f;C_r,\delta_{n,r}^2\right).
\end{aligned}
$$

Finally, taking

$$
\delta=\max_{n\in N,r>0}\left\{\delta_{n,r}^1,\delta_{n,r}^2\right\}
$$

we get the inequality (32). ∎

Theorem 3.3 implies the following.

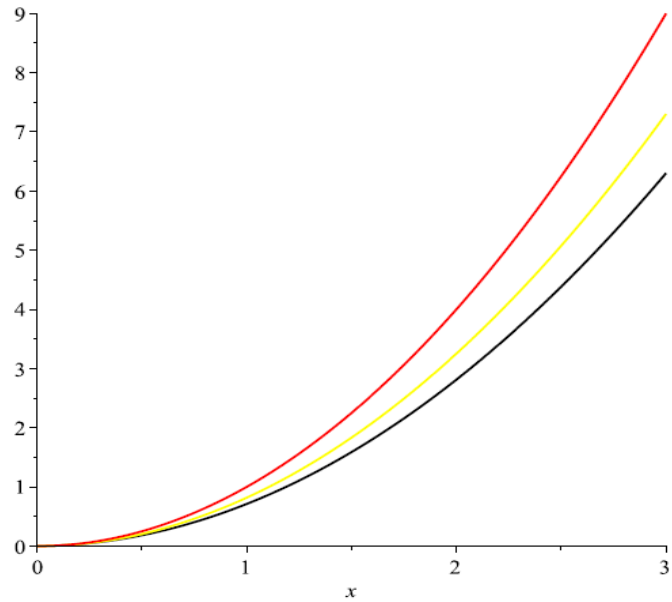

Figure 1. Curves for the error estimates of $S_n(f;r;x)$ (black) and $S_n^*(f;r;x)$ (yellow) with $f(x) = x^2$ (red).

Corollary 3.4.

Let $f \in C_r$, $r > 0$. Then for the operator S_n^*, we have

$$\lim_{n\to\infty} S_n^*(f;r;x) - f(x) = 0, \qquad x \in \mathbb{R}_0.$$

The convergence is uniform on every interval $[x_1, x_2], x_2 > x_1 > 0$.

In the following example we show a comparison for the error estimates of the operators $S_n(f;r;x)$ and $S_n^*(f;r;x)$ by using the software "MAPLE".

Example 3.5.

Choosing $f(x) = x^2$, we compute the error estimations of Szász-Mirakyan operators $S_n(f;r;x)$ given in (6) and generalized Szász-Mirakyan operators $S_n^*(f;r;x)$ given in (9). Here we take $a_n = 2n, b_n = 2n+1, r = 8, n = 40$.

Table 1. Error Estimates of $S_n(f;r;x)$ and $S_n^*(f;r;x)$ for $x = 1, 2, 3$.

x	Error bound for $S_n(f;r;x)$	Error bound for $S_n^*(f;r;x)$
1	0.2881944	0.1819215
2	1.1875000	0.7478854
3	2.6979167	1.6978917

4. Conclusion

In this paper, using the method given by Rempulska and Walczak, we give theorems on the degree of approximation of generalized Szász–Mirakyan operators in exponential weighted space of function of one variable. As is seen from the graph, when examining the error bounds for two operators, with an appropriate choice of the sequences (a_n) and (b_n), satisfying the conditions given in (9), we obtain better convergence results.

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