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Representation and Decomposition of an Intuitionistic Fuzzy Matrix Using Some (α, α') Cuts

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Abstract

The aim of this paper is to study the properties of various (α, α') cuts on Intuitionistic Fuzzy Matrices. Here we introduce different kinds of cuts on Intuitionistic Fuzzy Sets. We discuss some properties of the cuts with some other existing operators on Intuitionistic Fuzzy Matrix. Finally some representation and decomposition of an Intuitionistic Fuzzy Matrix using (α, α') cuts are given.

Keywords: Intuitionistic fuzzy sets; Intuitionistic fuzzy matrix; Intuitionistic fuzzy value; Reflexive matrix; Cut matrix; Irreflexive; Symmetric

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1. Introduction

There have been theories evolved over the years to deal with the various types of uncertainties. These evolved theories are put into practice and when found to be wanting are improved upon, paving the way for new theories to handle the tricky uncertainties. Probability theory is one such important theory concerned with the analysis of random phenomena. Zadeh (1965) came out with the concept of Fuzzy Set which is indeed an extension of the classical notion of set. Fuzzy Set has been found to be an effective tool to deal with fuzziness. However, it often falls short of the expected standard when describing the neutral state. As a result, a new concept namely Intuitionistic Fuzzy Set (IFS) was worked out and the same was introduced in 1983 by Atanassov (1983, 1986). Using the concept of IFS, Im et al. (2001, 2003a) studied Intuitionistic Fuzzy Matrix (IFM).

IFM generalizes the Fuzzy Matrix introduced by Thomson (1977) and has been useful in dealing with areas such as decision making, relational equations, clustering analysis etc,. A number of authors (1985, 1980, 1977) have effectively presented impressive results using Fuzzy Matrix. Bustince (1996) and Meenakshi (2010) show the importance IFM in the discussion of Intuitionistic fuzzy relation. Z.S.Xu (2012c) and Zhang (2012d) studied Intuitionistic Fuzzy Value and also IFMs. He defined intuitionistic fuzzy similarity relation and also utilize it in clustering analysis.

A lot of research activities have been carried out over the years on IFMs in Im et al. (2003a), Pal et al. (2002), Murugadas (2011a), Pradhan (2014). The period of powers of Square IFMs is discussed at length along with some of the results for the equivalence IFM by Jeong and Park (2005) while Pal et al. (2002) made a comprehensive study and neatly developed IFM in 2002. Another researcher namely Mondal (2013b) attempted a study on the similarity relations, invertibility and eigenvalues of IFM. In (2003b), a research was carried out on how a transitive IFM decomposed into a sum of nilpotent IFM and symmetric IFM by Jeong et al. Murugadas and Lalitha (2016) decomposed an IFM into a product of idempotent IFM and rectangular IFM.

It is well known the cut set is an important concept in theory of fuzzy sets and systems, which plays a significant role in fuzzy topology, fuzzy algebra, fuzzy analysis, fuzzy optimization, fuzzy logic and so on. The cut sets are the bridge between the fuzzy sets and classical sets. Li (2007) and Xue (2011b) gave the concepts of upper cut sets and lower cut sets of IFSs and they also discussed the decomposition theorem, representation theorem of IFSs by using cut sets. Since then different researchers in Barbhuiya (2015a), Huang (2013a), Yuan et al. (2009) have contributed significantly for the development of cut sets. Some cut sets and their properties are discussed by Shyamala and Pal (2004). Zhang and Zheng (2011c) introduced various cuts on fuzzy matrices and several properties are investigated. An IFM is expressed as a sum of its (α, α') cuts by Murugadas (2011a). In the same manner here we introduce some (α, α') cuts. We investigate various properties of the above cuts on IFM. Finally we obtain different kinds of decomposition and representation of an IFM, which is known as the resolution of IFM.

2. Preliminaries

Definition 2.1. (Atanassov (1983))

An IFS A in E (universal set) is defined as an object of the following form $A = \{(x, \mu_A(x), \gamma_A(x))/x \in E\}$, where the functions $\mu_A(x) : E \to [0, 1]$ and $\gamma_A(x) : E \to [0, 1]$ define the membership and non membership function of the element $x \in E$ respectively and for every $x \in E : 0 \leq \mu_A(x) + \gamma_A(x) \leq 1$.

Definition 2.2. (Atanassov (1983))

For $(x, x'), (y, y') \in$ IFS, define $(x, x') \vee (y, y') = (\max\{x, y\}, \min\{x', y'\})$, $(x, x') \wedge (y, y') = (\min\{x, y\}, \max\{x', y'\})$, and $(x, x')^c = (x', x)$.

Definition 2.3. (Muthuraji and Sriram (2015b))

For $(x, x'), (y, y') \in$ IFS, define $(x, x') \oplus (y, y') = \{(x+y) \wedge 1, (x'+y'-1) \vee 0\}$ and $(x, x') \odot (y, y') = \{(x + y - 1) \vee 0, (x' + y') \wedge 1\}$.

Definition 2.4. (Xu (2012c), Zhang (2012d))

The two tuple $(\mu(x_i), \gamma(x_i)) = (x, x')$ called an Intuitionistic fuzzy value such that $0 \leq x+x' \leq 1$ and $x, x' \in [0, 1]$.

Definition 2.5. (Xu (2012c))

Let $Z = (z_{ij})_{m\times n}$ be a matrix with all its elements are Intuitionistic Fuzzy values then Z is called an IFM. Hereafter F_{mn} denotes Set of all IFMs of order $m \times n$.

Definition 2.6. (Murugadas (2011a), Pal et al. (2002))

For any two elements $A = [(a_{ij}, a'_{ij})], B = [(b_{ij}, b'_{ij})] \in F_{mn}$ define $A \vee B = [(a_{ij}, a'_{ij}) \vee (b_{ij}, b'_{ij})]$ and $A \wedge B = [(a_{ij}, a'_{ij}) \wedge (b_{ij}, b'_{ij})]$.

Definition 2.7. (Murugadas (2011a))

For all $i = 1, 2, ...m., j = 1, 2, ...n., A = [(a_{ij}, a'_{ij})]$, we have

(1) If $(a_{ij}, a'_{ij}) = (1, 0)$ when $i = j$ otherwise $(a_{ij}, a'_{ij}) = (0, 1)$, then the matrix A is said to be an Identity matrix denoted by I_n.
If $(a_{ij}, a'_{ij}) = (1, 0)$ for all i, j, then A is said to an Universal matrix denoted by J.
If $(a_{ij}, a'_{ij}) = (0, 1)$ for all i, j, then A is said to be Zero matrix denoted by O.
(2) If $A \geq I_n$, then A is called reflexive.
(3) If $(a_{ij}, a'_{ij}) = (0, 1)$ when $i = j$, then A is called irreflexive.
(4) $A^c = (a'_{ij}, a_{ij})$ for all i, j.
(5) If $A \leq B$, then $a_{ij} \leq b_{ij}$ and $a'_{ij} > b'_{ij}$ for all i, j in which A and B are comparable.
(6) If A is symmetric, then $(a_{ij}, a'_{ij}) = (a_{ji}, a'_{ji})$ for all i, j.
(7) $\Delta A = \begin{cases} (a_{ij}, a'_{ij}), & \text{if } (a_{ij}, a'_{ij}) \geq (a_{ji}, a'_{ji}) \\ (0, 1), & \text{if } (a_{ij}, a'_{ij}) < (a_{ji}, a'_{ji}) \end{cases}$

Definition 2.8. (Muthuraji and Sriram (2015b))

For any $A, B \in F_{mn}$, we define
$A \oplus B = \{(a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0\}$
$A \odot B = \{(a_{ij} + b_{ij} - 1) \vee 1, (a'_{ij} + b'_{ij}) \wedge 1\}$.

3. (α, α') Cuts and Some Properties

Here we define (α, α') cuts for IFSs and IFMs also we studied some of its properties.

Definition 3.1.

For any $(x, x'), (\alpha, \alpha') \in IFS$ define

(1) $(x, x')^{(\alpha,\alpha')} = \begin{cases} (1,0), & \text{if } (x,x') \geq (\alpha,\alpha'), \\ (0,1), & \text{otherwise.} \end{cases}$

(2) $(x, x')_{(\alpha,\alpha')} = \begin{cases} (x,x'), & \text{if } (x,x') \geq (\alpha,\alpha'), \\ (0,1), & \text{otherwise.} \end{cases}$

(3) $(x, x')_{\alpha,\alpha'} = \begin{cases} (1,0), & \text{if } (x,x') \geq (\alpha,\alpha'), \\ (x,x'), & \text{otherwise.} \end{cases}$

(4) $(x, x')^{[\alpha,\alpha']} = \begin{cases} (1,0), & \text{if } \alpha + x \geq 1 \text{ and } \alpha' + x' < 1, \\ (0,1), & \text{otherwise.} \end{cases}$

(5) $(x, x')_{[\alpha,\alpha']} = \begin{cases} (x,x'), & \text{if } \alpha + x \geq 1 \text{ and } \alpha' + x' < 1, \\ (0,1), & \text{otherwise.} \end{cases}$

(6) $(x, x')^{<\alpha,\alpha'>} = \begin{cases} (1,0), & \text{if } \alpha + x \geq 1 \text{ and } \alpha' + x' < 1, \\ & \text{or } x \geq \alpha \text{ and } x' < \alpha', \\ (0,1), & \text{otherwise.} \end{cases}$

(7) $(x, x')_{<\alpha,\alpha'>} = \begin{cases} (x,x'), & \text{if } \alpha + x \geq 1 \text{ and } \alpha' + x' < 1, \\ & \text{or } x \geq \alpha \text{ and } x' < \alpha', \\ (0,1), & \text{otherwise.} \end{cases}$

(8) $(x, x')_{|\alpha,\alpha'|} = \begin{cases} (\alpha,\alpha'), & \text{if } (x,x') \geq (\alpha,\alpha'), \\ (x,x'), & \text{if } (x,x') < (\alpha,\alpha') \text{ and otherwise.} \end{cases}$

Consider $A \in F_{mn}$, $(\alpha, \alpha') \in IFS$. Now we extend the above definitions to IFMs as follows,

(1) $[A]^{(\alpha,\alpha')} = [(a_{ij}, a'_{ij})^{(\alpha,\alpha')}]$.
(2) $[A]_{(\alpha,\alpha')} = [(a_{ij}, a'_{ij})_{(\alpha,\alpha')}]$.
(3) $[A]_{\alpha,\alpha'} = [(a_{ij}, a'_{ij})_{\alpha,\alpha'}]$.
(4) $[A]_{[\alpha,\alpha']} = [(a_{ij}, a'_{ij})_{[\alpha,\alpha']}]$.
(5) $[A]^{[\alpha,\alpha']} = [(a_{ij}, a'_{ij})^{[\alpha,\alpha']}]$.
(6) $[A]_{<\alpha,\alpha'>} = [(a_{ij}, a'_{ij})_{<\alpha,\alpha'>}]$.
(7) $[A]^{<\alpha,\alpha'>} = [(a_{ij}, a'_{ij})^{<\alpha,\alpha'>}]$.

(8) $[A]_{|\alpha,\alpha'|} = [(a_{ij}, a'_{ij})_{|\alpha,\alpha'|}]$.

Proposition 3.1.

For any two IFMs $A, B \in F_{mn}$ and $A \geq B$. We have the following results,

(i) $A^{(\alpha,\alpha')} \geq B^{(\alpha,\alpha')}$.
(ii) $A_{(\alpha,\alpha')} \geq B_{(\alpha,\alpha')}$.
(iii) $A_{\alpha,\alpha'} \geq B_{\alpha,\alpha'}$.
(iv) $A^{[\alpha,\alpha']} \geq B^{[\alpha,\alpha']}$.
(v) $A_{[\alpha,\alpha']} \geq B_{[\alpha,\alpha']}$.
(vi) $A^{<\alpha,\alpha'>} \geq B^{<\alpha,\alpha'>}$.
(vii) $A_{<\alpha,\alpha'>} \geq B_{<\alpha,\alpha'>}$.
(viii) If all the entries of the A are comparable with (α, α'), $A_{|\alpha,\alpha'|} \geq B_{|\alpha,\alpha'|}$

Proof:

(i) Consider any ij^{th} element of $A^{(\alpha,\alpha')}$ as $(a_{ij}, a'_{ij})^{(\alpha,\alpha')}$
Case 1:
If $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha')$, then $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (1, 0)$.

Sub Case 1.1:
If $(a_{ij}, a'_{ij}) \geq (b_{ij}, b_{ij}) \geq (\alpha, \alpha')$, then $(b_{ij}, b'_{ij})^{(\alpha,\alpha')} = (1, 0)$.

Sub Case 1.2:
If $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha') \geq (b_{ij}, b_{ij})$, then $(b_{ij}, b'_{ij})^{(\alpha,\alpha')} = (0, 1)$. Thus, $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} \geq (b_{ij}, b'_{ij})^{(\alpha,\alpha')}$.

Case 2:
If $(a_{ij}, a'_{ij}) < (\alpha, \alpha')$, then $(b_{ij}, b'_{ij}) \leq (a_{ij}, a'_{ij}) \leq (\alpha, \alpha')$. Hence, $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (b_{ij}, b'_{ij})^{(\alpha,\alpha')} = (0, 1)$.

Case 3:
If the entries of the matrix A are not comparable to (α, α'), then $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (0, 1)$.

Sub Case 3.1:
If the entries of the matrix B are also not comparable to (α, α') or $(b_{ij}, b'_{ij}) \leq (\alpha, \alpha')$, then $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (b_{ij}, b'_{ij})^{(\alpha,\alpha')}$.

Sub Case 3.2:
If $(b_{ij}, b'_{ij}) \geq (\alpha, \alpha')$, then from Subcase 1.1, $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (b_{ij}, b'_{ij})^{(\alpha,\alpha')} = (1, 0)$. Suppose the entries of the matrix B are not comparable to (α, α') we get $A \geq B$ whenever the entries of the matrix A are comparable or not. Hence, from Case (1), (2), and (3), we have $A^{(\alpha,\alpha')} \geq B^{(\alpha,\alpha')}$ when $A \geq B$.

(ii) Clear from (i).

(iii) Also clear from (i).

(iv) **Case 1:**

If $\alpha + a_{ij} \geq 1$ and $\alpha' + a'_{ij} < 1$, then $(a_{ij}, a'_{ij})^{[\alpha,\alpha']} = (1, 0)$. Since $a_{ij} \geq b_{ij}, \alpha + b_{ij} \geq 1$ or $\alpha + b_{ij} < 1$.

Sub Case 1.1:

If $\alpha + b_{ij} \geq 1$ and $\alpha' + b'_{ij} < 1$, then $(b_{ij}, b'_{ij})^{[\alpha,\alpha']} = (1, 0)$.

Sub Case 1.2:

If $\alpha + b_{ij} < 1$ and $\alpha' + b'_{ij} > 1$ and

$$\alpha + b_{ij} < 1 \text{ and } \alpha' + b'_{ij} < 1, \text{ then } (b_{ij}, b'_{ij})^{[\alpha,\alpha']} = (0, 1).$$

In this case $(a_{ij}, a'_{ij})^{[\alpha,\alpha']} \geq (b_{ij}, b'_{ij})^{[\alpha,\alpha']}$.

Case 2: If $\alpha + a_{ij} \leq 1$ and $\alpha' + a'_{ij} > 1$ and

$$\alpha + a_{ij} \leq 1 \text{ and } \alpha' + a'_{ij} < 1, \text{ then } (a_{ij}, a'_{ij})^{[\alpha,\alpha']} = (0, 1).$$

Since $a_{ij} \geq b_{ij}$ gives $\alpha + b_{ij} \leq 1 \Rightarrow (b_{ij}, b'_{ij})^{[\alpha,\alpha']} = (0, 1)$. In this case $(a_{ij}, a'_{ij})^{[\alpha,\alpha']} = (b_{ij}, b'_{ij})^{[\alpha,\alpha']}$. Hence, $(a_{ij}, a'_{ij})^{[\alpha,\alpha']} \geq (b_{ij}, b'_{ij})^{[\alpha,\alpha']}$.

(v) Similar to (iv).

(vi) We can write $A_{<\alpha,\alpha'>}$ and $A^{<\alpha,\alpha'>}$ in terms of $A_{(\alpha,\alpha')}, A_{[\alpha,\alpha']}, A^{(\alpha,\alpha')}$ and $A^{[\alpha,\alpha']}$ as follows: $A_{<\alpha,\alpha'>} = A_{(\alpha,\alpha')} \cup A_{[\alpha,\alpha']}$ and $A^{<\alpha,\alpha'>} = A^{(\alpha,\alpha')} \cup A^{[\alpha,\alpha']}$. Now from (i) we have $A_{<\alpha,\alpha'>} \geq B_{<\alpha,\alpha'>}$.

(vii) From (ii) it is clear from the above.

(viii) **Case 1:**

If $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha')$ then $(a_{ij}, a'_{ij})_{|\alpha,\alpha'|} = (\alpha, \alpha')$. Now $(b_{ij}, b'_{ij})_{|\alpha,\alpha'|} = (\alpha, \alpha')$ when $(a_{ij}, a'_{ij}) \geq (b_{ij}, b'_{ij}) \geq (\alpha, \alpha')$ and $(b_{ij}, b'_{ij})_{|\alpha,\alpha'|} = (b_{ij}, b'_{ij}) \leq (\alpha, \alpha') = (a_{ij}, a'_{ij})_{|\alpha,\alpha'|}$ when $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha') \geq (b_{ij}, b'_{ij})$.

Case 2:

If $(a_{ij}, a'_{ij}) \leq (\alpha, \alpha')$, then $(a_{ij}, a'_{ij})_{|\alpha,\alpha'|} = (a_{ij}, a'_{ij})$ and $(b_{ij}, b'_{ij})_{|\alpha,\alpha'|} = (b_{ij}, b'_{ij})$ gives $A_{|\alpha,\alpha'|} \geq B_{[\alpha,\alpha']}$.

Case 3:

This inequality is not valid when (b_{ij}, b'_{ij}) is not comparable with (α, α') since the value of $(b_{ij}, b'_{ij})_{|\alpha,\alpha'|} = (b_{ij}, b'_{ij})$ when $(a_{ij}, a'_{ij})_{|\alpha,\alpha'|}$ may be either (α, α') or (a_{ij}, a'_{ij}) and in this case (α, α') is not comparable with (b_{ij}, b'_{ij}).

Proposition 3.2.

Consider any two elements $(\alpha, \alpha'), (\beta, \beta') \in$ IFS such that $(\alpha, \alpha') \geq (\beta, \beta')$ and $A \in F_{mn}$. We have

(i) $A^{(\alpha,\alpha')} \leq A^{(\beta,\beta')}$.
(ii) $A_{(\alpha,\alpha')} \leq A_{(\beta,\beta')}$.
(iii) $A_{\alpha,\alpha'} \leq A_{\beta,\beta'}$.
(iv) $A^{[\alpha,\alpha']} \geq A^{[\beta,\beta']}$.
(v) $A_{[\alpha,\alpha']} \geq A_{[\beta,\beta']}$.
(vi) $A_{|\alpha,\alpha'|} \geq A_{|\beta,\beta'|}$.

Proof:

(i) Consider any ij^{th} element of $A^{(\alpha,\alpha')}$ as $(a_{ij}, a'_{ij})^{(\alpha,\alpha')}$.
Case 1:
When $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha') \geq (\beta, \beta') \Rightarrow (a_{ij}, a'_{ij}) \geq (\beta, \beta')$, i.e., $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (1, 0) = (a_{ij}, a'_{ij})^{(\beta,\beta')}$, when $(a_{ij}, a_{ij}) < (\alpha, \alpha')$ and $(a_{ij}, a'_{ij}) \leq (\beta, \beta') \leq (\alpha, \alpha')$, then $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (0, 1)$ and $(a_{ij}, a'_{ij})^{(\beta,\beta')} = (0, 1)$. Otherwise $(\beta, \beta') \leq (a_{ij}, a'_{ij}) \leq (\alpha, \alpha')$ gives $(a_{ij}, a'_{ij})^{(\beta,\beta')} = (1, 0)$. In this case, $A^{(\alpha,\alpha')} \leq A^{(\beta,\beta')}$.

Case 2:
Suppose for some i, j, (a_{ij}, a'_{ij}) is not comparable to (α, α'). we have $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (0, 1)$. Moreover $(a_{ij}, a'_{ij})^{(\beta,\beta')} = (0, 1)$ or $(1, 0)$. On the other hand, if (a_{ij}, a'_{ij}) is not comparable to (β, β'), then $(a_{ij}, a'_{ij})^{(\alpha,\alpha')} = (0, 1) = (a_{ij}, a'_{ij})^{(\beta,\beta')}$ since $(\alpha, \alpha') \geq (\beta, \beta')$.
(ii) similar to (i).
(iii) similar to (i).
(iv) Since $\alpha \geq \beta$ and $\alpha' < \beta', \alpha + a_{ij} \geq 1 \Rightarrow \beta + a_{ij} \geq 1$ or $\beta + a_{ij} \leq 1$. Similarly $\alpha' + a'_{ij} < 1 \Rightarrow \beta' + a'_{ij} \leq 1$ or $\beta' + a'_{ij} > 1$. Hence, $(a_{ij}, a_{ij})^{[\alpha,\alpha']} = (1, 0) \Rightarrow (a_{ij}, a_{ij})^{[\beta,\beta']} = (1, 0)$ or $(a_{ij}, a'_{ij})^{[\beta,\beta']} = (0, 1)$. When $\alpha + a_{ij} \leq 1$, it is clear $\beta + a_{ij} \leq 1$ and $\beta' + a'_{ij} \leq 1$ or ≥ 1. Therefore in this case, $(a_{ij}, a'_{ij})^{[\alpha,\alpha']} = (a_{ij}, a'_{ij})^{[\beta,\beta']} = (0, 1)$. In general $A^{[\alpha,\alpha']} \geq A^{[\beta,\beta']}$.
(v) From (iv) it is clear.
(vi) **Case 1:**
If $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha') \geq (\beta, \beta')$ then $(a_{ij}, a'_{ij})_{|\alpha,\alpha'|} = (\alpha, \alpha') \geq (\beta, \beta') = (a_{ij}, a'_{ij})_{|\beta,\beta'|}$.

Case 2:
If $(a_{ij}, a'_{ij}) < (\alpha, \alpha')$ then $(a_{ij}, a'_{ij})_{|\alpha,\alpha'|} = (a_{ij}, a'_{ij})$ and
$(a_{ij}, a'_{ij})^{|\beta,\beta'|} = (a_{ij}, a'_{ij})$ when $(a_{ij}, a'_{ij}) < (\beta, \beta') < (\alpha, \alpha')$. Also $(a_{ij}, a'_{ij})_{|\alpha,\alpha'|} = (a_{ij}, a'_{ij})$ and $(a_{ij}, a'_{ij})_{|\beta,\beta'|} = (\beta, \beta') < (a_{ij}, a'_{ij})$ when $(\beta, \beta') < (a_{ij}, a'_{ij}) < (\alpha, \alpha')$. From above we have $A_{|\alpha,\alpha'|} \geq A_{|\beta,\beta'|}$.

Case 3:
If (a_{ij}, a'_{ij}) is not comparable with (α, α') for any i, j, then $(a_{ij}, a'_{ij})_{|\alpha,\alpha'|} = (a_{ij}, a'_{ij})$.

Sub Case 3.1:
If $(a_{ij}) < \alpha$ and $(a'_{ij}) < \alpha'$. Now $(a_{ij}, a'_{ij})_{|\beta,\beta'|} = (\beta, \beta')$ when $(a_{ij}) \geq \beta$ and $(a'_{ij}) < \beta'$ and $(a_{ij}, a'_{ij})_{|\beta,\beta'|} = (a_{ij}, a'_{ij})$ when$(a_{ij}) < \beta$ and $(a'_{ij}) < \beta'$ since $(\alpha, \alpha') \geq (\beta, \beta')$.

Sub Case 3.2:
If $(a_{ij}) \geq \alpha$ and $(a'_{ij}) \geq \alpha'$, then $(a_{ij}) \geq \beta$ and $(a'_{ij}) \geq \beta'$ or $< \beta'$. Therefore, $(a_{ij}, a'_{ij})_{|\beta,\beta'|} = (\beta, \beta')$ or $(a_{ij}, a'_{ij}) \leq (a_{ij}, a'_{ij})_{|\alpha,\alpha'|}$. Similarly we can prove the inequality when (β, β') is not comparable with (a_{ij}, a'_{ij}). Hence, from the above three cases we have $A_{|\alpha,\alpha'|} \geq A_{|\beta,\beta'|}$.

Proposition 3.3.

For any two IFMs $A, B \in F_{mn}$, we have the following inequalities,

(i) $(A \oplus B)^{(\alpha,\alpha')} \geq A^{(\alpha,\alpha')} \oplus B^{(\alpha,\alpha')}$.
(ii) $(A \oplus B)_{(\alpha,\alpha')} \geq A_{(\alpha,\alpha')} \oplus B_{(\alpha,\alpha')}$.
(iii) $(A \oplus B)_{\alpha,\alpha'} \geq A_{\alpha,\alpha'} \oplus B_{\alpha,\alpha'}$.
(iv) $(A \oplus B)^{[\alpha,\alpha']} \geq A^{[\alpha,\alpha']} \oplus B^{[\alpha,\alpha']}$.
(v) $(A \oplus B)_{[\alpha,\alpha']} \geq A_{[\alpha,\alpha']} \oplus B_{[\alpha,\alpha']}$.

Proof:

(i) Consider the ij^{th} element of $(A \oplus B)^{(\alpha,\alpha')}$ is
$[(a_{ij}, a'_{ij}) \oplus (b_{ij}, b'_{ij})]^{(\alpha,\alpha')}$ as (c_{ij}, c'_{ij}). Now $(c_{ij}, c'_{ij}) = [(a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0]^{(\alpha,\alpha')}$

$$= \begin{cases} (1,0), & \text{if } [(a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1 \vee 0)] \geq (\alpha, \alpha'), \\ (0,1), & \text{if } [(a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1 \vee 0)] < (\alpha, \alpha'), \\ & \text{otherwise.} \end{cases}$$

$$= \begin{cases} (1,0), & \text{if } a_{ij} + b_{ij} \geq \alpha \text{ and } a'_{ij} + b'_{ij} - 1 < \alpha', \\ (0,1), & \text{if } a_{ij} + b_{ij} < \alpha \text{ and } a'_{ij} + b'_{ij} - 1 \geq \alpha', \\ & \text{otherwise.} \end{cases}$$

Assume (d_{ij}, d'_{ij}) as the ij^{th} element of $(a_{ij}, a'_{i_j})^{(\alpha,\alpha')} \oplus (b_{ij}, b'_{ij})^{(\alpha,\alpha')}$, i.e. $A^{(\alpha,\alpha')} \oplus B^{(\alpha,\alpha')}$.

Case 1:
If $(a_{ij} + b_{ij}) \geq \alpha$ and $a'_{ij} + b'_{ij} - 1 < \alpha'$.

Sub Case 1.1:
If $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha')$ and $(b_{ij}, b'_{ij}) \geq (\alpha, \alpha')$ then $(d_{ij}, d'_{ij}) = (1,0) \oplus (1,0) = (1,0)$.

Sub Case 1.2:
If $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha')$ and $(b_{ij}, b'_{ij}) < (\alpha, \alpha')$ then $(d_{ij}, d'_{ij}) = (1,0) \oplus (0,1) = (1,0)$.

Sub Case 1.3:
If $(a_{ij}, a'_{ij}) < (\alpha, \alpha')$ and $(b_{ij}, b'_{ij}) \geq (\alpha, \alpha')$ then $(d_{ij}, d'_{ij}) = (0,1) \oplus (1,0) = (1,0)$.

Sub Case 1.4:
If $(a_{ij}, a'_{ij}) < (\alpha, \alpha')$ and $(b_{ij}, b'_{ij}) < (\alpha, \alpha')$ then $(d_{ij}, d'_{ij}) = (0, 1) \oplus (0, 1) = (0, 1)$. In this case $(c_{ij}, c'_{ij}) = (1, 0) \geq (d_{ij}, d'_{ij})$.

Case 2:
If $(a_{ij} + b_{i_j}) < \alpha$ and $(a'_{ij} + b'_{ij} - 1) \geq \alpha'$, since $a_{ij} \leq (a_{ij} + b_{ij}) \leq \alpha$ and $b_{ij} \leq (a_{ij} + b_{ij}) \leq \alpha$ and $a'_{ij} \geq a'_{ij} + (b'_{ij} - 1) \geq \alpha'$ and $b'_{ij} \geq b'_{ij} + (a_{ij} - 1) \geq \alpha'$, i.e., $(a_{ij}, a'_{ij}) < (\alpha, \alpha')$ and $(b_{ij}, b'_{ij}) < (\alpha, \alpha')$, $(d_{ij}, d'_{ij}) = (0, 1) \oplus (0, 1) = (0, 1) = (c_{ij}, c'_{ij})$.

Case 3:
When $(a_{ij} + b_{ij}) \geq \alpha$ and $a'_{ij} + b'_{ij} - 1 \geq \alpha'$,$(c_{ij}, c'_{ij}) = (0, 1)$.

Sub Case 3.1:
If (a_{ij}, a'_{ij}) and (b_{ij}, b'_{ij}) are not comparable to (α, α') then $(d_{ij}, d'_{ij}) = (0, 1)$.

Sub Case 3.2:
If for some i, j either (a_{ij}, a'_{ij}) or (b_{ij}, b'_{ij}) is comparable to (α, α') and which is of the form $(a_{ij}, a'_{ij}) < (\alpha, \alpha')$ or $(b_{ij}, b'_{ij}) < (\alpha, \alpha')$, then $(d_{ij}, d'_{ij}) = (0, 1)$. In this case there is no possibility for $(a_{ij}) \geq \alpha$ and $a'_{ij} < \alpha'$ or $b_{ij} \geq \alpha$ and $b'_{ij} < \alpha'$ since $a'_{ij} + b'_{ij} - 1 \geq \alpha'$ gives both $a'_{ij}$and$b'_{ij} \geq \alpha'$. Hence in this case $(c_{ij}, c'_{ij}) = (0, 1) = (d_{ij}, d'_{ij})$.

Case 4:
Suppose $(a_{ij} + b_{ij}) \leq \alpha$ and $a'_{ij} + b'_{ij} - 1 \leq \alpha'$. Then $(c_{ij}, c'_{ij}) = (0, 1) = (d_{ij}, d'_{ij})$, since both (a_{ij}) and $(b_{ij}) \leq \alpha$. Hence, from the above four cases we can conclude $(A \oplus B)^{(\alpha, \alpha')} \geq A^{(\alpha, \alpha')} \oplus B^{(\alpha, \alpha')}$.

(ii)
$$(c_{ij}, c'_{ij}) = \begin{cases} (a_{ij} + b_{ij}) \wedge 1, \quad (a'_{ij} + b'_{ij} - 1) \vee 0 \\ \qquad\qquad \text{if } [(a_{ij} + b_{ij}) \wedge 1, (a_{ij} + b'_{ij} - 1) \vee 0] \geq (\alpha, \alpha'), \\ (0, 1), \qquad \text{if } [(a_{ij} + b_{ij}) \wedge 1, (a_{ij} + b'_{ij} - 1) \vee 0] \leq (\alpha, \alpha'). \end{cases}$$
$$= \begin{cases} (1, 0), \ \text{if } a_{ij} + b_{ij} \geq 1 \text{ and } a'_{ij} + b_{ij} - 1 < 0, \\ (a_{ij} + b_{ij}), (a'_{ij} + b'_{ij} - 1) \ , \text{if} \\ \qquad (\alpha, \alpha') \leq [(a_{ij} + b_{ij}), (a'_{ij} + b'_{ij} - 1)] \leq (1, 0), \\ (0, 1), \ \text{if } [(a_{ij} + b_{ij}), (a'_{ij} + b'_{ij} - 1)] < (\alpha, \alpha'). \end{cases}$$

Case 1:
If $(a_{ij} + b_{ij}) \geq 1$ and $a'_{ij} + b'_{ij} - 1 < 0$,

Sub Case 1.1:
$(d_{ij}, d'_{ij}) = (1, 0)$.

Sub Case 1.2:
$(d_{ij}, d'_{ij}) = (a_{ij}, a'_{ij}) \oplus (0, 1) = (a_{ij}, a'_{ij})$.

Sub Case 1.3:
$(d_{ij}, d'_{ij}) = (b_{ij}, b'_{ij})$.

Sub Case 1.4:
$(d_{ij}, d'_{ij}) = (0, 1)$ and $(c_{ij}, c'_{ij}) = (1, 0) \geq (d_{ij}, d'_{ij})$.

Case 2:
If $(\alpha, \alpha') < [(a_{ij} + b_{ij}), (a'_{ij} + b'_{ij} - 1)] < (1, 0)$,

Sub Case 2.1:
$(d_{ij}, d'_{ij}) = [(a_{ij} + b_{ij}), 0]$.

Sub Case 2.2:
$(d_{ij}, d'_{ij}) = (a_{ij}, a'_{ij}) \oplus (0, 1) = (a_{ij}, a'_{ij})$.

Sub Case 2.3:
$(d_{ij}, d'_{ij}) = (b_{ij}, b'_{ij})$.

Sub Case 2.4:
$(d_{ij}, d'_{ij}) = (0, 1)$. In this case $(c_{ij}, c'_{ij}) = [(a_{ij} + b_{ij}), (a'_{ij} + b'_{ij} - 1)] \geq (d_{ij}, d'_{ij})$.

Case 3:
$(c_{ij}, c'_{ij}) = (d_{ij}, d_{ij}) = (0, 1)$. Hence, we conclude $(c_{ij}, c'_{ij}) \geq (d_{ij}, d_{ij})$, $(A \oplus B)_{(\alpha,\alpha')} \geq A_{(\alpha,\alpha')} \oplus B_{(\alpha,\alpha')}$.

(iii) Proof is similar to (i) and (ii).

(iv) Assume $(A \oplus B)^{[\alpha,\alpha']} = C$ and $A^{[\alpha,\alpha']} \oplus B^{[\alpha,\alpha']} = D$. Now,

$$A^{[\alpha,\alpha']} = \begin{cases} (1,0), & \text{if } \alpha + a_{ij} \geq 1 \text{ and } \alpha' + a'_{ij} < 1, \\ (0,1), & \text{otherwise.} \end{cases}$$

$$B^{[\alpha,\alpha']} = \begin{cases} (1,0), & \text{if } \alpha + b_{ij} \geq 1 \text{ and } \alpha' + b'_{ij} < 1, \\ (0,1), & \text{otherwise.} \end{cases}$$

$$\begin{aligned}(A \oplus B)^{[\alpha,\alpha']} &= (c_{ij}, c'_{ij})^{[\alpha,\alpha']} \\ &= \begin{cases} (1,0), & \text{if } \alpha + a_{ij} + b_{ij} \geq 1 \text{ and } \alpha' + a'_{ij} + b'_{ij} < 2, \\ (0,1), & \text{otherwise.} \end{cases}\end{aligned}$$

Case 1:
If $\alpha + a_{ij} + b_{ij} \geq 1$ and $\alpha' + a'_{ij} + b'_{ij} < 2$ then $(c_{ij}, c'_{ij})^{[\alpha,\alpha']} = (1, 0)$.

Sub Case 1.1:
$\alpha + a_{ij} \geq 1, \alpha + b_{ij} \geq 1$ and $\alpha' + a'_{ij} < 1, \alpha' + b'_{ij} < 1$. Now $(d_{ij}, d'_{ij}) = (1, 0) \oplus (1, 0) = (1, 0)$.

Sub Case 1.2:
When $\alpha + a_{ij} \geq 1, \alpha + b_{ij} < 1$ and $\alpha' + a'_{ij} < 1, \alpha' + b'_{ij} > 1$, $(d_{ij}, d'_{ij}) = (1, 0) \oplus (0, 1) = (1, 0)$.

Sub Case 1.3:
When $\alpha+a_{ij} < 1, \alpha+b_{ij} > 1$ and $\alpha'+a'_{ij} \geq 1, \alpha'+b'_{ij} < 1$, $(d_{ij}, d'_{ij}) = (0,1)\oplus(1,0) = (1,0)$.

Sub Case 1.4:
$\alpha + a_{ij} < 1, \alpha + b_{ij} < 1$ and $\alpha' + a'_{ij} \geq 1, \alpha' + b'_{ij} \geq 1$. $(d_{ij}, d'_{ij}) = (0,1) \oplus (0,1) = (0,1)$. In this case $(c_{ij}, c'_{ij}) \geq (d_{ij}, d_{ij})$.

Case 2:
If $\alpha + a_{ij} + b_{ij} < 1$ and $\alpha' + a'_{ij} + b'_{ij} > 2$, or $\alpha + a_{ij} + b_{ij} < 1$ and $\alpha' + a'_{ij} + b'_{ij} < 2$, $(c_{ij}, c'_{ij})^{[\alpha,\alpha']} = (0,1)$. Since $\alpha + a_{ij} + b_{ij} < 1 \Rightarrow \alpha + a_{ij} < 1$ and $\alpha + b_{ij} < 1$. Now whatever be the values of $\alpha' + a'_{ij} + b'_{ij}$ the value of $(d_{ij}, d'_{ij}) = (0,1) \oplus (0,1) = (0,1)$. In this case $(c_{ij}, c'_{ij}) = (d_{ij}, d'_{ij})$. From the above two cases we conclude in general $(A \oplus B)^{[\alpha,\alpha']} \geq A^{[\alpha,\alpha']} \oplus B^{[\alpha,\alpha']}$.

(v) Similar to (iv).

Proposition 3.4.

If $A, B \in F_{mn}$ then we have

(i) $(A \odot B)^{(\alpha,\alpha')} \leq A^{(\alpha,\alpha')} \odot B^{(\alpha,\alpha')}$.
(ii) $(A \odot B)_{(\alpha,\alpha')} \leq A_{(\alpha,\alpha')} \odot B_{(\alpha,\alpha')}$.
(iii) $(A \odot B)_{\alpha,\alpha'} \leq A_{\alpha,\alpha'} \odot B_{\alpha,\alpha'}$.
(iv) $(A \odot B)^{[\alpha,\alpha']} \leq A^{[\alpha,\alpha']} \odot B^{[\alpha,\alpha']}$.
(v) $(A \odot B)_{[\alpha,\alpha']} \leq A_{[\alpha,\alpha']} \odot B_{[\alpha,\alpha']}$.

Proof:

(i) $(A \odot B)^{(\alpha,\alpha')} = [(a_{ij} + b_{ij} - 1) \vee 0, (a'_{ij} + b'_{ij}) \wedge 1]^{(\alpha,\alpha')}$.

Let $(c_{ij}, c'_{ij}) = ij^{th}$ element of $(A \odot B)^{(\alpha,\alpha')}$

$$= \begin{cases} (1,0), & \text{if } (a_{ij} + b_{ij} - 1, a'_{ij} + b'_{ij}) \geq (\alpha, \alpha'), \\ (0,1), & \text{if } (a_{ij} + b_{ij} - 1, a'_{ij} + b'_{ij}) < (\alpha, \alpha'). \end{cases}$$

Case 1:

If $(a_{ij} + b_{ij} - 1) \geq \alpha$ and $a'_{ij} + b'_{ij} < \alpha'$, then $(c_{ij}, c'_{ij}) = (1,0)$. $a_{ij} \geq a_{ij} + b_{ij} - 1 \geq \alpha$ and $a'_{ij} < a'_{ij} + b'_{ij} < \alpha' \Rightarrow (a_{ij}, a'_{ij}) \geq (\alpha, \alpha')$. Similarly $(b_{ij}, b'_{ij}) \geq (\alpha, \alpha')$ gives $(d_{ij}, d'_{ij}) = A^{(\alpha,\alpha')} \odot B^{(\alpha,\alpha')} = (1,0) \odot (1,0) = (1,0)$

Case 2:

If $a_{ij} + b_{ij} - 1 < \alpha$ and $a'_{ij} + b'_{ij} > \alpha'$, then $(c_{ij}, c'_{ij}) = (0,1)$.

Sub Case 2.1:

When $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha')$ and $(b_{ij}, b'_{ij}) < (\alpha, \alpha')$, $(d_{ij}, d'_{ij}) = (1,0) \odot (0,1) = (0,1)$.

Sub Case 2.2:

When $(a_{ij}, a'_{ij}) < (\alpha, \alpha')$ and $(b_{ij}, b'_{ij}) \geq (\alpha, \alpha')$, $(d_{ij}, d'_{ij}) = (0, 1)$.

Sub Case 2.3:

When $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha')$ and $(b_{ij}, b'_{ij}) \geq (\alpha, \alpha')$, $(d_{ij}, d'_{ij}) = (1, 0) \odot (1, 0) = (1, 0)$. Hence, $(c_{ij}, c'_{ij}) \leq (d_{ij}, d'_{ij})$; that is, $(A \odot B)^{(\alpha,\alpha')} \leq A^{(\alpha,\alpha')} \odot B^{(\alpha,\alpha')}$.

Case 3:

In dual way it is clear from of Proposition 3.3.

Case 4:

Similar to Proposition 3.3. Proofs of (ii) and (iii) evident from (i). In dual of Proposition 3.3 we can easily prove (iv) and (v) of Proposition 3.4.

Proposition 3.5

Let $A \in F_{mn}$ then we have

(i) $(A^c)^{(\alpha,\alpha')} \leq [A^{(\alpha',\alpha)}]^c$.
(ii) $(A^c)_{(\alpha,\alpha')} \leq [A_{(\alpha',\alpha)}]^c$.
(iii) $(A^c)_{\alpha,\alpha'} \geq [A_{\alpha',\alpha}]^c$.
(iv) $(A^c)^{[\alpha,\alpha']} \leq [A^{[\alpha',\alpha]}]^c$.
(v) $(A^c)_{[\alpha,\alpha']} \leq [A_{[\alpha',\alpha]}]^c$.

Proof:

(i) **Case 1:**

$A = (a_{ij}, a'_{ij}) \Rightarrow A^c = (a'_{ij}, a_{ij})$ and $(A^c)^{(\alpha,\alpha')} = \begin{cases} (1,0) & \text{if } (a'_{ij}, a_{ij}) \geq (\alpha, \alpha') \\ (0,1) & \text{if } (a'_{ij}, a_{ij}) < (\alpha, \alpha') \end{cases}$ when $a'_{ij} \geq \alpha, a_{ij} < \alpha' \Rightarrow a_{ij} < \alpha'$ and $a'_{ij} \geq \alpha \Rightarrow (a_{ij}, a'_{ij}) < (\alpha', \alpha) \Rightarrow (a_{ij}, a'_{ij})^{(\alpha',\alpha)} = (0, 1)$ $\Rightarrow (a_{ij}, a'_{ij})^{(\alpha',\alpha)} = [A^{(\alpha',\alpha)}]^c = (0, 1)^c = (1, 0)$. Therefore $(A^c)^{(\alpha,\alpha')} = [A^{(\alpha',\alpha)}]^c$ when $(a'_{ij}, a_{ij}) < (\alpha, \alpha') \Rightarrow a'_{ij} < \alpha$ and $a_{ij} \geq \alpha' \Rightarrow a_{ij} \geq \alpha'$ and $a'_{ij} < \alpha \Rightarrow (a_{ij}, a'_{ij}) \geq (\alpha', \alpha)$ $\Rightarrow (a_{ij}, a'_{ij})^{(\alpha',\alpha)} = (1, 0) \Rightarrow [A^{(\alpha',\alpha)}]^c = (0, 1)$.

Case 2:

If (a_{ij}, a'_{ij}) is not comparable to (α, α') and (a'_{ij}, a_{ij}) is comparable then from case (1) we have $(A^c)^{(\alpha,\alpha')} = [A^{(\alpha',\alpha)}]^c$. Suppose (a'_{ij}, a_{ij}) is also not comparable then $(A^c)^{(\alpha,\alpha')} = (0, 1)$ but $[A^{(\alpha',\alpha)}]^c = (0, 1)$ or $(1, 0)$. Hence $(A^c)^{(\alpha,\alpha')} \leq [A^{(\alpha',\alpha)}]^c$.

(ii) $[A^c]_{(\alpha,\alpha')} = (a'_{ij}, a_{ij})_{(\alpha,\alpha')} = \begin{cases} (a'_{ij}, a_{ij}), & \text{when } (a'_{ij}, a_{ij}) \geq (\alpha, \alpha') \\ (0, 1) & \text{otherwise.} \end{cases}$

when $(a'_{ij}, a_{ij}) \geq (\alpha, \alpha') \Rightarrow a'_{ij} \geq \alpha$ and $a_{ij} < \alpha' \Rightarrow a_{ij} < \alpha'$ and $a'_{ij} \geq \alpha \Rightarrow$ $[(a_{ij}, a'_{ij})_{(\alpha',\alpha)}]^c = (0, 1)^c = (1, 0)$ when $(a'_{ij}, a_{ij}) < (\alpha, \alpha') \Rightarrow a'_{ij} < \alpha$ and $a_{ij} > \alpha'$ $\Rightarrow (a_{ij}, a'_{ij}) \geq (\alpha', \alpha) \Rightarrow [(a_{ij}, a'_{ij})_{(\alpha',\alpha)}]^c = (a'_{ij}, a_{ij})$. Hence $[A^c]_{(\alpha,\alpha')} \leq [A_{(\alpha',\alpha)}]^c$. For incomparable entries the proof is similar to (i).

(iii) **Case 1:**
$[A^c]_{\alpha,\alpha'} = \begin{cases}(1,0), & \text{if } (a'_{ij},a_{ij}) \geq (\alpha,\alpha'),\\ (a'_{ij},a_{ij}), & \text{if } (a'_{ij},a_{ij}) < (\alpha,\alpha')\end{cases}$ when $(a'_{ij},a_{ij}) \geq (\alpha,\alpha') \Rightarrow (a_{ij},a'_{ij}) < (\alpha',\alpha) \Rightarrow [A_{\alpha',\alpha}]^c = (a_{ij},a_{ij})^c = (a'_{ij},a_{ij})$
when $(a'_{ij},a_{ij}) < (\alpha,\alpha') \Rightarrow (a_{ij},a'_{ij}) > (\alpha',\alpha) \Rightarrow [A_{\alpha',\alpha}]^c = (1,0)^c = (0,1)$.

Case 2:
$[A^c]_{\alpha,\alpha'} = (a'_{ij},a_{ij})$ but $[A_{\alpha',\alpha}]^c = (0,1)$ or (a'_{ij},a_{ij}). Hence $[A^c]_{\alpha,\alpha'} \geq [A_{\alpha',\alpha}]^c$. Proofs of (iv) and (v) are similar to (i) and (ii).

Definition 3.2.

When the matrices A and A^T are comparable for any IFM $A \in F_n$, we define
$$\Delta_1 A = \begin{cases}(1,0), & \text{if } (a_{ij},a'_{ij}) \geq (a_{ji},a'_{ji}),\\ (a_{ij},a'_{ij}), & \text{if } (a_{ij},a'_{ij}) < (a_{ji},a'_{ji}),\end{cases}$$
and $\nabla_1 A = A \vee A^T$.

Proposition 3.7.

Let $A \in F_{mn}$. We have

(i) $\Delta_1 A_{\alpha,\alpha'} = [\Delta_1 A]_{\alpha,\alpha'}$.
(ii) $\Delta_1 \Delta A = \Delta\Delta_1 A$.
(iii) $[A_{(\alpha,\alpha')}]_{\alpha,\alpha'} = [A_{\alpha,\alpha'}]_{(\alpha,\alpha')} = A^{(\alpha,\alpha')}$.

Proof:

(i) **Case 1:**
If $(a_{ij},a'_{ij}) \geq (a_{ji},a'_{ji})$ then $\Delta_1(a_{ij},a'_{ij}) = (1,0)$ and $[\Delta_1(a_{ij},a'_{ij})]_{\alpha,\alpha'} = (1,0)$.

Sub Case 1.1:
If $(a_{ij},a'_{ij}) \geq (\alpha,\alpha')$ then $A_{\alpha,\alpha'} = (1,0)$ and $\Delta_1(a_{ij},a'_{ij})_{\alpha,\alpha'} = (1,0)$ since $(1,0) \geq (a_{ji},a'_{ji})_{\alpha,\alpha'}$.

Sub Case 1.2:
If $(a_{ij},a'_{ij}) < (\alpha,\alpha')$ then $(a_{ij},a'_{ij})_{\alpha,\alpha'} = (a_{ij},a'_{ij})$ and $\Delta_1(a_{ij},a'_{ij})_{\alpha,\alpha'} = (1,0)$ since $(a_{ij},a'_{ij}) \geq (a_{ji},a'_{ji})$.

Sub Case 1.3:
For some i,j the entries of the matrix A are not comparable with (α,α'), we have $(a_{ij},a'_{ij})_{\alpha,\alpha'} = (a_{ij},a'_{ij})$ and $\Delta_1(a_{ij},a'_{ij})_{\alpha,\alpha'} = (1,0)$. In this case $[\Delta_1 A]_{\alpha,\alpha'} = \Delta_1 A_{\alpha,\alpha'}$. The above equality is also true when $(a_{ji},a'_{ji}) = (0,1)$.

Case 2:
If $(a_{ij},a'_{ij}) \leq (a_{ji},a'_{ji})$ then $\Delta_1(a_{ij},a'_{ij}) = (a_{ij},a'_{ij})$ and $[\Delta_1(a_{ij},a'_{ij})]_{\alpha,\alpha'} = (a_{ij},a'_{ij})_{\alpha,\alpha'} =$

$$\begin{cases}(1,0), & \text{if } (a_{ij},a'_{ij}) \geq (\alpha,\alpha'), \\ (a_{ij},a'_{ij}), & \text{otherwise.}\end{cases}.$$

Sub Case 2.1:
When $(a_{ij},a'_{ij}) \geq (\alpha,\alpha'), \Delta_1(a_{ij},a'_{ij})_{\alpha,\alpha'} = (1,0)$ since $(\alpha,\alpha') \leq (a_{ij},a'_{ij}) \leq (a_{ji},a'_{ji})$.

Sub Case 2.2:
When $(a_{ij},a'_{ij}) < (\alpha,\alpha'), \Delta_1(a_{ij},a'_{ij})_{\alpha,\alpha'} = (a_{ij},a'_{ij})$ since $(a_{ji},a'_{ji})_{\alpha,\alpha'} = (1,0)$ if $(a_{ij},a'_{ij}) \leq (\alpha,\alpha') \leq (a_{ji},a'_{ji})$ and $(a_{ji},a'_{ji})_{\alpha,\alpha'} = (a_{ji},a'_{ji})$ if $(a_{ij},a'_{ij}) \leq (a_{ji},a'_{ji}) \leq (\alpha,\alpha')$. In this case $[\Delta_1 A]_{\alpha,\alpha'} = \Delta_1 A_{\alpha,\alpha'}$.

Case 3:
When (a_{ij},a'_{ij}) and (a_{ji},a'_{ji}) are not comparable to (α,α'). $\Delta_1(a_{ij},a'_{ij}) = (a_{ij},a'_{ij})$ gives $[\Delta_1 A]_{(\alpha,\alpha')} = (a_{ij},a'_{ij})$. Also $(a_{ji},a'_{ji})_{\alpha,\alpha'} = (a_{ji},a'_{ji}) \geq (a_{ij},a'_{ij}) = (a_{ij},a'_{ij})_{\alpha,\alpha'}$. Hence $\Delta_1 A_{\alpha,\alpha'} = (a_{ij},a'_{ij})$. If (a_{ji},a'_{ji}) is comparable when (a_{ij},a'_{ij}) is not comparable with (α,α') then the value of $(a_{ji},a'_{ji})_{\alpha,\alpha'}$ may be either $(1,0)$ or (a_{ji},a'_{ji}) which are greater than $(a_{ij},a'_{ij}) = (a_{ij},a'_{ij})_{\alpha,\alpha'}$. From the above we conclude $\Delta_1 A_{\alpha,\alpha'} = (a_{ij},a'_{ij}) = [\Delta_1 A]_{\alpha,\alpha'}$.

(ii) Now we have to prove $(\Delta\Delta_1)A = (\Delta_1\Delta)A$.

Case 1:
If $(a_{ij},a'_{ij}) \geq (a_{ji},a'_{ji})$ then $\Delta A = (a_{ij},a'_{ij})$. $\Delta_1\Delta A = \Delta(a_{ij},a'_{ij}) = (1,0)$. Now $\Delta_1 A = (1,0)$ and $\Delta\Delta_1 A = \Delta(1,0) = (1,0)$ since $(1,0) \geq (a_{ji},a'_{ji})$ $\Delta\Delta_1 = \Delta_1\Delta$.

Case 2:
If $(a_{ij},a'_{ij}) < (a_{ji},a'_{ji})$, then $\Delta A = (0,1)$ and $\Delta_1\Delta A = \Delta_1(0,1) = (0,1)$ and $\Delta_1 A = (a_{ij},a'_{ij})$ gives $\Delta\Delta_1 A = \Delta(a_{ij},a'_{ij}) = (0,1)$. From the above two cases $\Delta\Delta_1 = \Delta_1\Delta$.

(iii) From the definition of cuts it is clear that $[A_{(\alpha,\alpha')}]_{\alpha,\alpha'} = [A_{\alpha,\alpha'}]_{(\alpha,\alpha')} = A^{(\alpha,\alpha')}$.

Corollary 3.1.

For an IFM $A \in F_{mn}$, we have

(i) A is reflexive or symmetric $\Rightarrow A_{(\alpha,\alpha')}, A^{(\alpha,\alpha')}, A_{[\alpha,\alpha']}, A_{\alpha,\alpha'}$ and $A^{[\alpha,\alpha']}$ are also the same.
(ii) A is irreflexive and (α,α') is not equal $(0,1) \Rightarrow A_{(\alpha,\alpha')}, A^{(\alpha,\alpha')}, A_{\alpha,\alpha'}$ are irreflexive.
(iii) A is irreflexive and (α,α') not equal to $(1,0) \Rightarrow A_{[\alpha,\alpha']}, A^{[\alpha,\alpha']}$ are irreflexive.

Corollary 3.2.

(i) For all $(\alpha,\alpha') \in IFS, J^{(\alpha,\alpha')} = J_{(\alpha,\alpha')} = J_{\alpha,\alpha'} = J^{[\alpha,\alpha']} = J_{[\alpha,\alpha']} = J$ where J is the universal matrix.
(ii) $O^{(0,1)} = O_{0,1} = O^{[1,0]} = J$.
(iii) $O_{(0,1)} = O^{[0,1]} = O_{[0,1]} = O^{(1,0)} = O_{(1,0)} = O_{1,0} = O_{[1,0]} = O$.
(iv) For all (α,α') other than $(0,1)$ and $(1,0)$, $O^{(\alpha,\alpha')} = O_{(\alpha,\alpha')} = O_{\alpha,\alpha'} = O^{[\alpha,\alpha']} = O_{[\alpha,\alpha']} = O$.
(v) If I_n is the identify matrix then $\forall(\alpha,\alpha') \in IFS$, $I^{(\alpha,\alpha')} = I_{(\alpha,\alpha')} = I_{\alpha,\alpha'} = I^{[\alpha,\alpha']} = I_{[\alpha,\alpha']} = I$. Except $I^{(0,1)} = I_{0,1} = I^{[1,0]} = J$.

4. Representation and Decomposition of an IFM

Using (α, α') cuts defined in section 3, any IFM can be represented as a linear combination of their cuts. In the same manner we can decompose an IFM using some (α, α') cuts.

Proposition 4.1

Let $A \in F_{mn}$ and $S = \{\text{elements of } A\}$. Then, the following results hold.

(i) $A = \bigvee_{(\alpha,\alpha')\in S} \{A_{(\alpha,\alpha')}\}$.

(ii) $A = \bigvee_{(\alpha,\alpha')\in S} \{(\alpha, \alpha') \bigwedge A^{(\alpha,\alpha')}\}$.

(iii) $A = \bigvee_{(\alpha,\alpha')\in S} \{(\alpha, \alpha') \bigwedge A_{(\alpha,\alpha')}\}$.

(iv) $A = [\Delta_1 A] \bigwedge [\nabla_1 A]$.

(v) $A = A_{|\alpha,\alpha'|} \bigvee A_{(\alpha,\alpha')}$.

Proof:

The proofs of (i) to (iii) are clear from their definitions.

(vi) $\Delta_1 A = \begin{cases} (1,0), & \text{if } (a_{ij}, a'_{ij}) \geq (a_{ji}, a'_{ji}), \\ (a_{ij}, a'_{ij}), & \text{if } (a_{ij}, a'_{ij}) < (a_{ji}, a'_{ji}), \end{cases}.$

$\nabla_1 A = (a_{ij}, a'_{ij}) \bigvee (a_{ji}, a'_{ji})$.

Case 1:

If $(a_{ij}, a'_{ij}) \geq (a_{ji}, a'_{ji})$ then $\Delta_1 A = (1, 0)$ and $\nabla_1 A = (a_{ij}, a'_{ij})$. Hence $[\Delta, A] \bigwedge [\nabla_1 A] = (a_{ij}, a'_{ij})$

Case 2:

If $(a_{ij}, a'_{ij}) < (a_{ji}, a'_{ji})$ then $\Delta_1 A = (a_{ij}, a'_{ij})$ and $\nabla_1 A = (a_{ji}, a'_{ji})$. Hence $[\Delta, A] \bigwedge [\nabla_1 A] = (a_{ij}, a'_{ij})$. So $A = [\Delta_1 A] \bigwedge [\nabla_1 A]$.

(vii) $A_{|\alpha,\alpha'|} = \begin{cases} (\alpha, \alpha'), & \text{if } (a_{ij}, a'_{ij}) \geq (\alpha, \alpha'), \\ (a_{ij}, a'_{ij}), & \text{if } (a_{ij}, a'_{ij}) < (\alpha, \alpha'). \end{cases}.$

Case 1:

If $(a_{ij}, a'_{ij}) \geq (\alpha, \alpha')$, then $A_{|\alpha,\alpha'|} = (\alpha, \alpha')$ and $A_{(\alpha,\alpha')} = (a_{ij}, a'_{ij})$. Hence $A_{(\alpha,\alpha')} \bigvee A_{|\alpha,\alpha'|} = (a_{ij}, a'_{ij})$

Case 2:

If $(a_{ij}, a'_{ij}) < (\alpha, \alpha')$ or both are incomparable, then $A_{|\alpha,\alpha'|} = (a_{ij}, a'_{ij})$ and $A_{(\alpha,\alpha')} = (0, 1)$. Hence $A_{(\alpha,\alpha')} \bigvee A_{|\alpha,\alpha'|} = (a_{ij}, a'_{ij})$. Therefore $A = A_{|\alpha,\alpha'|} \bigvee A_{(\alpha,\alpha')}$.

We illustrate the above by an example as follows.

Consider an IFM $A = \begin{bmatrix} (0.6,0.3) & (0.4,0.2) & (0.0,1.0) \\ (0.2,0.3) & (1.0,0.0) & (0.7,0.1) \\ (0.5,0.2) & (0.2,0.5) & (0.1,0.2) \end{bmatrix}$

$S = \Big\{ (0.6,0.3),(0.4,0.2),(0.0,1.0),(0.2,0.3),(1.0,0.0),(0.7,0.1),(0.5,0.2),(0.2,0.5),(0.1,0.2)$

$$A_{(0.6,0.3)} = \begin{bmatrix} (0.6,0.3) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (1.0,0.0) & (0.7,0.1) \\ (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}, \quad (0.6,0.3) \wedge A_{(0.6,0.3)} = \begin{bmatrix} (0.6,0.3) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (0.6,0.3) & (0.6,0.3) \\ (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}$$

$$A_{(0.4,0.2)} = \begin{bmatrix} (0.0,1.0) & (0.4,0.2) & (0.0,1.0) \\ (0.0,1.0) & (1.0,0.0) & (0.7,0.1) \\ (0.5,0.2) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}, \quad (0.4,0.2) \wedge A_{(0.4,0.2)} = \begin{bmatrix} (0.0,1.0) & (0.4,0.2) & (0.0,1.0) \\ (0,0,1.0) & (0.4,0.2) & (0.4,0.2) \\ (0.4,0.2) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}$$

$$A_{(0.0,1.0)} = \begin{bmatrix} (0.6,0.3) & (0.4,0.2) & (0.0,1.0) \\ (0.2,0.3) & (1.0,0.0) & (0.7,0.1) \\ (0.5,0.2) & (0.2,0.5) & (0.1,0.2) \end{bmatrix}, \quad (0.0,1.0) \wedge A_{(0.0,1.0)} = \begin{bmatrix} (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}$$

$$A_{(0.2,0.3)} = \begin{bmatrix} (0.6,0.3) & (0.4,0.2) & (0.0,1.0) \\ (0.2,0.3) & (1.0,0.0) & (0.7,0.1) \\ (0.5,0.2) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}, \quad (0.2,0.3) \wedge A_{(0.2,0.3)} = \begin{bmatrix} (0.2,0.3) & (0.2,0.3) & (0.0,1.0) \\ (0.2,0.3) & (0.2,0.3) & (0.2,0.3) \\ (0.2,0.3) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}$$

$$A_{(1.0,0.0)} = \begin{bmatrix} (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (1.0,0.0) & (0.0,1.0) \\ (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}, \quad (1.0,0.0) \wedge A_{(1.0,0.0)} = \begin{bmatrix} (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (1.0,0.0) & (0.0,1.0) \\ (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}$$

$$A_{(0.7,0.1)} = \begin{bmatrix} (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (1.0,0.0) & (0.7,0.1) \\ (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}, \quad (0.7,0.1) \wedge A_{(0.7,0.1)} = \begin{bmatrix} (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (0.7,0.1) & (0.7,0.1) \\ (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}$$

$$A_{(0.5,0.2)} = \begin{bmatrix} (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (1.0,0.0) & (0.7,0.1) \\ (0.5,0.2) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}, \quad (0.5,0.2) \wedge A_{(0.5,0.2)} = \begin{bmatrix} (0.0,1.0) & (0.0,1.0) & (0.0,1.0) \\ (0.0,1.0) & (0.5,0.2) & (0.5,0.2) \\ (0.5,0.2) & (0.0,1.0) & (0.0,1.0) \end{bmatrix}$$

$$A_{(0.2,0.5)} = \begin{bmatrix} (0.6,0.3) & (0.4,0.2) & (0.0,1.0) \\ (0.2,0.3) & (1.0,0.0) & (0.7,0.1) \\ (0.5,0.2) & (0.2,0.5) & (0.0,1.0) \end{bmatrix}, \quad (0.2,0.5) \wedge A_{(0.2,0.5)} = \begin{bmatrix} (0.2,0.5) & (0.2,0.5) & (0.0,1.0) \\ (0.2,0.5) & (0.2,0.5) & (0.2,0.5) \\ (0.2,0.5) & (0.2,0.5) & (0.0,1.0) \end{bmatrix}$$

$$A_{(0.1,0.2)} = \begin{bmatrix} (0.0,1.0) & (0.4,0.2) & (0.0,1.0) \\ (0.0,1.0) & (1.0,0.0) & (0.7,0.1) \\ (0.5,0.2) & (0.0,1.0) & (0.1,0.2) \end{bmatrix}, \quad (0.1,0.2) \wedge A_{(0.1,0.2)} = \begin{bmatrix} (0.0,1.0) & (0.1,0.2) & (0.0,1.0) \\ (0.0,1.0) & (0.1,0.2) & (0.1,0.2) \\ (0.1,0.2) & (0.0,1.0) & (0.1,0.2) \end{bmatrix}$$

Now $\bigvee_{(\alpha,\alpha')\in S} \left[A_{(\alpha,\alpha')}\right] = \bigvee_{(\alpha,\alpha')\in S} \left[(\alpha,\alpha') \bigwedge A_{(\alpha,\alpha')}\right] = A = \begin{bmatrix} (0.6,0.3) & (0.4,0.2) & (0.0,1.0) \\ (0.2,0.3) & (1.0,0.0) & (0.7,0.1) \\ (0.5,0.2) & (0.2,0.5) & (0.1,0.2) \end{bmatrix}$.

Similarly we can verify $A = \bigvee_{(\alpha,\alpha')\in S} \left[(\alpha,\alpha') \bigwedge A^{(\alpha,\alpha')}\right]$.

5. Conclusion

In this paper we introduce various cuts on IFSs and extend the above into IFMs. The properties of cut matrices are investigated in different cases. Finally we decompose any IFM in terms of their cut matrices also we express an IFM as the sum(max) of its own cut matrices.

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