Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 1 (June 2018), pp. 199–216

Ultimate boundedness and periodicity results for a certain system Of third-order nonlinear Vector delay differential equations

[1] **Linda D. Oudjedi** and [2] **Moussadek Remili**

Department of Mathematics
University of Oran
1 Ahmed Ben Bella
31000 Oran, Algeria
[1]oudjedi@yahoo.fr; [2]remilimous@gmail.com

Received: March 24, 2017; Accepted: January 27, 2018

Abstract

In the last years, there has been increasing interest in obtaining the sufficient conditions for stability, instability, boundedness, ultimately boundedness, convergence, etc. For instance, in applied sciences some practical problems concerning mechanics, engineering technique fields, economy, control theory, physical sciences and so on are associated with third, fourth and higher order nonlinear differential equations. The problem of the boundedness and stability of solutions of vector differential equations has been widely studied by many authors, who have provided many techniques especially for delay differential equations. In this work a class of third order nonlinear non-autonomous vector delay differential equations is considered by employing the direct technique of Lyapunov as basic tool, where a complete Lyapunov functional is constructed and used to obtain sufficient conditions that guarantee existence of solutions that are periodic, uniformly asymptotically stable, uniformly ultimately bounded and the behavior of solutions at infinity. In addition to being for a more general equation, the obtained results here are new even when our equation is specialized to the forms previously studied and include many recent results in the literature. Finally, an example is given to show the feasibility of our results.

Keywords: Stability; Lyapunov functional; Ultimate boundedness; Periodicity; third-order delay vector differential equations

MSC 2010 No.: 34D05, 34D20, 34C25

1. Introduction

In this paper, we are concerned with the uniform asymptotic stability of solutions of the equation

$$\begin{aligned}&\left[P(X(t))X'(t)\right]'' + A(t)(Q(X(t))X'(t))' + B(t)(R(X(t))X'(t))\\&+C(t)F(X(t-r(t))) = 0,\end{aligned} \tag{1}$$

and the ultimate boundedness and the existence of periodic solutions of the equation

$$\begin{aligned}&\left[P(X(t))X'(t)\right]'' + A(t)(Q(X(t))X'(t))' + B(t)(R(X(t))X'(t))\\&+C(t)F(X(t-r(t))) = H(t, X, X', X''),\end{aligned} \tag{2}$$

in which $X \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$, $H : \mathbb{R}_+ \times \mathbb{R}^{3n} \to \mathbb{R}^n$, P, Q and $R : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, A, B and $C : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, are continuous differentiable functions with P is twice differentiable and $F(0) = 0$, $0 \leq r(t) \leq \gamma$, γ is a positive constant, and $r'(t) \leq \beta_0$, $0 < \beta_0 < 1$ and the dots indicate differentiation with respect to t.

Numerous research activities are concerned with the stability and boundedness of solutions to different functional differential equations, for some related contributions, we refer the reader to Hale (1977) and Tunç (2006a, 2006b, 2006c, 2009, 2014a, 2014b, 2017).

Ezeilo and Tejumola (1966), Afuwape (1983), Meng (1993) studied the ultimately boundedness and existence of periodic solutions of the nonlinear vector differential equation

$$X''' + AX'' + BX' + H(X) = P(t, X, X', X''). \tag{3}$$

Afterward, Feng (1995) established sufficient conditions under which the nonlinear vector differential equation

$$X''' + A(t)X'' + B(t)X' + H(X) = P(t, X, X', X''), \tag{4}$$

has at least unique periodic solution.

Moreover, Omeike (2007) established some sufficient conditions for the ultimate boundedness of the equation (3).

Recently, Omeike (2015) investigated the asymptotic stability of solutions to the following nonlinear third order scalar differential equation with delay for $P \equiv 0$

$$X''' + AX'' + BX' + H(X(t - r(t))) = P(t). \tag{5}$$

Equation (5) is a particular case to our preceding non-autonomous vector differential equation with the deviating argument r if $P(X) = Q(X) = R(X) = C(t) = I$, $A(t) = A$ and $B(t) = B$. On the other hand, we can find the same result for the equation (2) without delay by putting $r = 0$, which is generalization of (3) and (4).

In the case $n = 1$, these problems have been investigated [see Graef et al. (2015a, 2015b), Oudjedi et al. (2014, 2017) and Remili et al. (2014a, 2014b, 2014c, 2016a, 2015, 2016b, 2016c, 2016d, 2016e, 2016f)] for a general scalar delay differential equation. Equation (2) have not been discussed in the literature, yet. The basic reason may be the difficulty to find a suitable Lyapunov function for differential systems of higher order.

The object of the present paper is to provide results for n-dimensional equation (2) following the arguments used in some of the papers mentioned above.

2. Preliminaries

The following notations (see Omeike (2015)) will be useful in subsequent sections. For $x \in \mathbb{R}^n$, $|x|$ is the norm of x. For a given $r > 0, t_1 \in \mathbb{R}$,

$$C(t_1) = \{\phi : [t_1 - r, t_1] \to \mathbb{R}^n/\phi \text{ is continuous}\}.$$

In particular, $C = C(0)$ denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ and for $\phi \in C, \phi = \sup\limits_{-r\leq\theta\leq 0} |\phi(0)|$. C_H will denote the set of ϕ such that $\phi \leq H$. For any continuous function $x(u)$ defined on $-h \leq u < A$, where $A > 0$, and $0 \leq t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t - r, t]$, that is, x_t is an element of C defined by

$$x_t(\theta) = x(t + \theta), -r \leq \theta \leq 0.$$

The following results will be basic to the proofs of Theorems.

Lemma 2.1 (Afuwape (1983), Afuwape (2004), Ezeilio (1966), Tiryaki (1999)).

Let D be a real symmetric positive definite $n \times n$ matrix, then for any X in $\mathbb{R}^n$, we have

$$\delta_d \parallel X \parallel^2 \leq \langle DX, X\rangle \leq \Delta_d \parallel X \parallel^2,$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D, respectively.

Lemma 2.2 (Afuwape (1983), Afuwape (2004), Ezeilio (1966), Tiryaki (1999)).

Let Q, D be any two real $n \times n$ commuting matrices, then

(i) The eigenvalues $\lambda_i(QD)\,(i = 1, 2..., n)$ of the product matrix QD are all real and satisfy

$$\min_{1\leq j,k\leq n} \lambda_j(Q)\,\lambda_k(D) \leq \lambda_i(QD) \leq \max_{1\leq j,k\leq n} \lambda_j(Q)\,\lambda_k(D).$$

(ii) The eigenvalues $\lambda_i(Q + D)\,(i = 1, 2..., n)$ of the sum of matrices Q and D are all real and satisfy

$$\left\{\min_{1\leq j\leq n} \lambda_j(Q) + \min_{1\leq k\leq n} \lambda_k(D)\right\} \leq \lambda_i(Q + D) \leq \left\{\max_{1\leq j\leq n} \lambda_j(Q) + \max_{1\leq k\leq n} \lambda_k(D)\right\}.$$

Lemma 2.3 (Ezeilio (1966), Mahmoud and Tunç (2016), Tiryaki (1999)).

Let $H(X)$ be a continuous vector function with $H(0) = 0$.

$$1)\quad \frac{d}{dt}\left(\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma\right) = \langle H(X), X'\rangle.$$

$$2)\quad \int_0^1 \langle C(t)H(\sigma X), X\rangle d\sigma = \int_0^1 \int_0^1 \sigma[\langle C(t)J_H(\sigma\tau X)X, X\rangle] d\sigma d\tau.$$

Lemma 2.4 (Ezeilio (1966), Mahmoud and Tunç (2016), Tiryaki (1999)).

Let $H(X)$ be a continuous vector function with $H(0)=0$.

$$1)\quad \langle H(X),H(X)\rangle = 2\int_0^1\int_0^1 \sigma\langle J_H(\sigma X)J_H(\sigma\tau X)X,X\rangle d\sigma d\tau.$$

$$2)\quad \langle C(t)H(X),X\rangle = \int_0^1 \langle J_H(\sigma X)C(t)X,X\rangle d\sigma.$$

Lemma 2.5.

Let $H(X)$ be a continuous vector function and that $H(0)=0$. Then,

$$\delta_h \parallel X \parallel^2 \leq \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma \leq \Delta_h \parallel X \parallel^2,$$

where δ_h, Δ_h are the least and the greatest eigenvalues of $J_h(X)$ (Jacobian matrix of H), respectively.

Definition 2.6.

We definite the spectral radius $\rho(A)$ of a matrix A by

$$\rho(A) = \max\{\lambda / \ \lambda \text{ is eigenvalue of } A\}.$$

Lemma 2.7.

For any $A\in\mathbb{R}^{n\times n}$, we have the norm $\|A\| = \sqrt{\rho(A^TA)}$ if A is symmetric then,

$$\|A\| = \rho(A).$$

We shall note all the equivalents norms by the same notation $\|X\|$ for $X\in\mathbb{R}^n$ and $\|A\|$ for a matrix $A\in\mathbb{R}^{n\times n}$.

3. Stability

Consider the functional differential equation

$$x' = f(t,x_t), \ \ x_t(\theta) = x(t+\theta), \quad -r\leq\theta\leq 0, \ \ t\geq 0, \tag{6}$$

where $f: I\times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t,0)=0$,

$$f(t,0)=0, \ C_H := \{\phi\in(C[-r,0], \ \mathbb{R}^n) : \|\phi\|\leq H\},$$

and for $H_1 < H$, there exists $L(H_1)>0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 3.1 (Burton (2005)).

An element $\psi\in C$ is in the $\omega - limit$ set of ϕ, say $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,+\infty)$ and there is a sequence $\{t_n\}, t_n\to\infty$, as $n\to\infty$, with $\|x_{t_n}(\phi)-\psi\|\to 0$ as $n\to\infty$ where $x_{t_n}(\phi) = x(t_n+\theta,0,\phi) \ for \ -r\leq\theta\leq 0$.

Definition 3.2 (Burton (2005)).

A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (6), $x(t, 0, \phi)$, is defined on $[0, \infty)$ and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.

Definition 3.3 (Burton (1985)).

If $\phi \in C_H$is such that the solution $x_t(\phi)$ of (6) with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \;\; as \;\; t \to \infty.$$

Definition 3.4 (Burton (1985)).

Let $V(t, \phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t, 0) = 0$, and such that:

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)|) + W_3(\|\phi\|_2)$ where $\|\phi\|_2 = (\int_{t-r}^{t} \|\phi(s)\|^2 ds)^{\frac{1}{2}}$,

(ii) $\dot{V}_{(6)}(t, \phi) \leq -W_4(|\phi(0)|)$,

where W_i $(i = 1, 2, 3, 4)$ are wedges, then the zero solution of (6) is uniformly asymptotically stable.

4. Assumptions and main results

We shall state here some assumptions which will be used on the functions that appeared in equation (1), and suppose that there are constants $\delta_a, \delta_b, \delta_c$, $\delta_{a'}, \delta_{b'}$, $\delta_{c'}, \delta_p$, δ_f, δ_q, δ_r, Δ_a, Δ_b, Δ_c, $\Delta_{a'}, \Delta_{b'}$, $\Delta_{c'}, \Delta_p, \Delta_q$, Δ_r and Δ_f, such that the matrices A, B, C, P, Q, R and $J_F(X)$ (Jacobian matrix of $F(X)$) are symmetric and positive definite, and furthermore the eigenvalues $\lambda_i(A), \lambda_i(B)$, $\lambda_i(C), \lambda_i(A'), \lambda_i(B'), \lambda_i(C'), \lambda_i(P), \lambda_i(Q), \lambda_i(R)$ and $\lambda_i(J_F(X))(i = 1, 2, ..., n)$ of A, B, C, A', B', C', P, Q, R and $J_F(X)$, respectively satisfy,

$$\begin{aligned}
&0 < \delta_p \leq \lambda_i\left(P\right) \leq \Delta_p, \qquad 0 < \delta_q \leq \lambda_i\left(Q\right) \leq \Delta_q, \qquad 0 < \delta_r \leq \lambda_i\left(R\right) \leq \Delta_r,\\
&0 < \delta_a \leq \lambda_i\left(A\right) \leq \Delta_a, \qquad 0 < \delta_c \leq \lambda_i\left(C\right) \leq \Delta_c, \qquad 0 < \delta_b \leq \lambda_i\left(B\right) \leq \Delta_b,\\
&\quad \delta_{a'} \leq \lambda_i\left(A'\right) \leq \Delta_{a'}, \quad \delta_{c'} \leq \lambda_i\left(C'\right) \leq \Delta_{c'} \leq 0, \quad \delta_{b'} \leq \lambda_i\left(B'\right) \leq \Delta_{b'} \leq 0,\\
&0 < \delta_f \leq \lambda_i\left(J_F\left(X\right)\right) \leq \Delta_f.
\end{aligned}$$

Note that for any matrix M symmetric invertible, we have

$$\Delta_{M^{-1}} = \delta_M^{-1}, \quad \text{and} \quad \delta_{M^{-1}} = \Delta_M^{-1}.$$

For the sake of brevity, we define

$$\begin{aligned}
A_1 &= \frac{1}{2}(1 + \Delta_{p^{-1}}) + \delta_a\delta_q\Delta_{p^{-1}}^2 + \frac{1}{\delta_f\delta_c}\left\|(B(t)R(X) - \delta_b\delta_r I)P^{-1}(X)\right\|^2,\\
A_2 &= \frac{1}{2}(1 + \Delta_{p^{-1}}) + \frac{1}{\delta_f\delta_c}\left\|(A(t)Q(X) - \delta_a\delta_q I)P^{-1}(X)\right\|^2,\\
\Gamma(t) &= B(t)R(X)P^{-1}(X), \qquad \rho(t) = t - r(t),
\end{aligned}$$

and

$$\begin{aligned}\theta_1(t) &= \frac{d}{dt}P^{-1}(X(t)) = -P^{-1}\big(X(t)\big)\big[\frac{d}{dt}P\big(X(t)\big)\big]P^{-1}\big(X(t)\big),\\ \theta_2(t) &= \big[\frac{d}{dt}Q\big(X(t)\big)\big]P^{-1}\big(X(t)\big) + Q(X(t))\theta_1(t),\\ \theta_3(t) &= \big[\frac{d}{dt}R\big(X(t)\big)\big]P^{-1}\big(X(t)\big) + R(X(t))\theta_1(t),\\ \mu(t) &= \int_0^t (\|\theta_1(s)\| + \|\theta_2(s)\| + \|\theta_3(s)\|)ds.\end{aligned}$$

Consider the equivalent system to (1) :

$$\begin{aligned}X' &= P^{-1}(X)Y,\\ Y' &= Z,\\ Z' &= -A(t)\theta_2(t)Y - A(t)Q(X)P^{-1}(X)Z - \Gamma(t)Y\\ &\quad - C(t)F(X) + C(t)\int_{\rho(t)}^t J_F\left(X(s)\right)P^{-1}\left(X(s)\right)Y(s)ds.\end{aligned} \tag{7}$$

The following result is introduced.

Theorem 4.1.

Suppose that $\Delta_c \leq \delta_b$, $\Delta_{b'} \leq \delta_{c'}$ and the assumptions

(i) $\dfrac{\Delta_p\Delta_f}{\delta_r} < \alpha < \delta_a\delta_q$,

(ii) $\dfrac{1}{2}(\alpha + \delta_a\delta_q)\Delta_{a'}\Delta_q\Delta_{p^{-1}} - \delta_b(\alpha\Delta_p^{-1}\delta_r - \Delta_f) < -\epsilon < 0$,

(iii) $\beta < \min\left\{\delta_b\delta_r, \delta_{p^{-1}}\delta_b(\delta_a\delta_q\delta_r\delta_{p^{-1}} - \Delta_f)A_1^{-1}, \dfrac{1}{2}(\delta_a\delta_q - \alpha)A_2^{-1}\right\}$,

(iv) $\displaystyle\int_0^{+\infty}\left\|\frac{d}{ds}\big(P(X(s)) + Q(X(s)) + R(X(s))\big)\right\| ds < +\infty$,

are satisfied, then the zero solution of (7) is uniformly asymptotically stable, if

$$\gamma < \min\left\{\frac{\delta_f\delta_c}{\Delta_f\Delta_c\delta_p^{-1}}, 2\epsilon\delta_{p^{-1}}(1-\beta_0)A_3^{-1}, \frac{(\delta_a\delta_q - \alpha)\delta_{p^{-1}}}{2\Delta_f\Delta_c\delta_p^{-1}}\right\},$$

where

$$A_3 = \Delta_f\Delta_c\delta_p^{-1}(2 + \delta_p^{-1}(\alpha + \delta_a\delta_q)(2-\beta_0) + \beta).$$

Proof:

Let a continuously differentiable Lyapunov functional U defined by

$$U(t, X_t, Y_t, Z_t) = e^{-\frac{\mu(t)}{\upsilon}}V(t, X_t, Y_t, Z_t) = e^{-\frac{\mu(t)}{\upsilon}}V, \tag{8}$$

where

$$\begin{aligned}V &= (\alpha+\delta_a\delta_q)\int_0^1 < C(t)F(\sigma X), X > d\sigma + 2 < C(t)F(X), Y > + < \Gamma(t)Y, Y > \\ &+ \frac{1}{2}(\alpha+\delta_a\delta_q) < A(t)Q(X)P^{-2}(X)Y, Y > +(\alpha+\delta_a\delta_q) < P^{-1}(X)Y, Z > \\ &+ < Z, Z > +\beta\delta_a\delta_q < X, P^{-1}(X)Y > +\beta < X, Z > +\frac{1}{2}\beta\delta_b\delta_r < X, X > \\ &+ \frac{1}{2}\beta < Y, Y > +\omega_0\int_{-r(t)}^{0}\int_{t+s}^{t}\langle Y(\tau), Y(\tau)\rangle\, d\tau ds.\end{aligned}$$

ω_0, υ are some positive constants which will be specified later in the proof. Since

$$\omega_0\int_{-r(t)}^{0}\int_{t+s}^{t}\langle Y(\tau), Y(\tau)\rangle\, d\tau ds$$

is non-negative, and by Lemma 2.3, we have

$$\begin{aligned}V &\geq (\alpha+\delta_a\delta_q)\int_0^1\int_0^1 \sigma < C(t)J_F(\tau\sigma X)X, X > d\tau d\sigma - \left\|C(t)\Gamma^{-\frac{1}{2}}(t)F(X)\right\|^2 \\ &+ \left\|\Gamma^{\frac{1}{2}}(t)Y + C(t)\Gamma^{-\frac{1}{2}}(t)F(X)\right\|^2 + \frac{1}{2}\left\|Z+\alpha P^{-1}(X)Y\right\|^2 \\ &+ \frac{1}{2} < \Big((\alpha+\delta_a\delta_q)A(t)Q(X) - (\alpha^2+\delta_a^2\delta_q^2)I\Big)P^{-2}(X)Y, Y > \\ &+ \frac{1}{2}\beta\|Y\|^2 + \frac{1}{2}\beta(\delta_b\delta_r-\beta)\|X\|^2 + \frac{1}{2}\|\beta X + \delta_a\delta_q P^{-1}(X)Y + Z\|^2,\end{aligned}$$

since

$$\left\|\Gamma^{\frac{1}{2}}(t)Y + C(t)\Gamma^{-\frac{1}{2}}(t)F(X)\right\|^2 \geq 0,$$

and by Lemma 2.4, we have

$$\begin{aligned}V &\geq \int_0^1\int_0^1 \sigma\left\langle\Big[(\alpha+\delta_a\delta_q)C(t) - 2C^2(t)\Gamma^{-1}(t)J_F(\sigma X)\Big]J_F(\tau\sigma X)X, X\right\rangle d\tau d\sigma \\ &+ \frac{1}{2}\left\langle\Big((\alpha+\delta_a\delta_q)A(t)Q(X) - (\alpha^2+\delta_a^2\delta_q^2)I\Big)P^{-2}(X)Y, Y\right\rangle \\ &+ \frac{1}{2}\left\|Z+\alpha P^{-1}(X)Y\right\|^2 + \frac{1}{2}\|\beta X + \delta_a\delta_q P^{-1}(X)Y + Z\|^2 \\ &+ \frac{1}{2}\beta\|Y\|^2 + \frac{1}{2}\beta(\delta_b\delta_r-\beta)\|X\|^2,\end{aligned}$$

under our hypothesis, we get

$$\begin{aligned}V &\geq \frac{1}{2}\Big[\delta_c\delta_f\Big((\alpha+\delta_a\delta_q) - 2\Delta_p\Delta_{r^{-1}}\Delta_f\Big) + \beta(\delta_b\delta_r-\beta)\Big]\|X\|^2 \\ &+ \frac{1}{2}\left\|Z+\alpha P^{-1}(X)Y\right\|^2 + \frac{1}{2}\Big[\alpha(\delta_a\delta_q-\alpha)\delta_{p^{-2}} + \beta\Big]\|Y\|^2 \\ &+ \frac{1}{2}\|\beta X + \delta_a\delta_q P^{-1}(X)Y + Z\|^2,\end{aligned}$$

from conditions (i) and (iii) of Theorem 4.1. We can find a constant k such that

$$V \geq k\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right). \tag{9}$$

By (iv), we obtain

$$\begin{aligned}\mu(t) &\leq \Delta_{p^{-1}}^2(1+\Delta_r+\Delta_q)\int_0^t \left\|\frac{d}{ds}P\big(X(s)\big)\right\| ds\\ &+\Delta_{p^{-1}}\int_0^t (\left\|\frac{d}{ds}R\big(X(s)\big)\right\| + \left\|\frac{d}{ds}Q\big(X(s)\big)\right\|)ds\\ &\leq N<\infty. \qquad (10)\end{aligned}$$

This may be combined with (9) to obtain

$$U \geq K\big(\parallel X\parallel^2+\parallel Y\parallel^2+\parallel Z\parallel^2\big), \qquad (11)$$

where $K = k\exp\left(-\dfrac{N}{\upsilon}\right)$.

The derivative of V along the trajectories of the system (7) is given by

$$\begin{aligned}\frac{d}{dt}V = &-<\Big[(\alpha+\delta_a\delta_q)P^{-1}(X)B(t)R(X)-2C(t)J_F(X)\\ &-\frac{1}{2}(\alpha+\delta_a\delta_q)A'(t)Q(X)P^{-1}(X)\Big]P^{-1}(X)Y,Y>\\ &-<\Big(2A(t)Q(X)-(\alpha+\delta_a\delta_q)I\Big)P^{-1}(X)Z,Z>\\ &-\beta\left[<X,\Gamma(t)Y>-\delta_b\delta_r<X,P^{-1}(X)Y>\right]\\ &+\beta\delta_a\delta_q<P^{-1}(X)Y,P^{-1}(X)Y>+\beta<(I+P^{-1}(x))Y,Z>\\ &-\beta<X,\Big(A(t)Q(X)-\delta_a\delta_q I\Big)P^{-1}(X)Z>-\beta<X,C(t)F(X)>\\ &-\omega_0\left(1-r'(t)\right)\int_{\rho(t)}^t \langle Y(\tau),Y(\tau)\rangle d\tau+\omega_0 r(t)\langle Y,Y\rangle+\psi_1+\psi_2+\psi_3,\end{aligned}$$

where

$$\begin{aligned}\psi_1 = &(\alpha+\delta_a\delta_q)\int_0^1 <C'(t)F(\sigma X),X>d\sigma+2<C'(t)F(X),Y>\\ &+<B'(t)R(X)P^{-1}(X)Y,Y>,\end{aligned}$$

$$\begin{aligned}\psi_2 = &(\alpha+\delta_a\delta_q)<\theta_1(t)Y,Z>+\frac{(\alpha+\delta_a\delta_q)}{2}<A(t)Q(X)P^{-1}(X)\theta_1(t)Y,Y>\\ &+<B(t)\theta_3(t)Y,Y>-\frac{(\alpha+\delta_a\delta_q)}{2}<A(t)\theta_2(t)P^{-1}(X)Y,Y>\\ &+\delta_a\delta_q\beta<X,\theta_1(t)Y>-2<A(t)\theta_2(t)Y,Z>-\beta<X,A(t)\theta_2(t)Y>,\end{aligned}$$

and

$$\begin{aligned}\psi_3 = \int_{\rho(t)}^t \Big\langle\, &C(t)J_F\left(X(s)\right)P^{-1}(X(s))Y(s),2Z(t)\\ &+(\alpha+\delta_a\delta_q)P^{-1}(X(t))Y(t)+\beta X(t)\Big\rangle ds.\end{aligned}$$

We claim that $\psi_1 < 0$, indeed

$$\begin{aligned}
\psi_1 &\leq (\alpha+\delta_a\delta_q)\int_0^1 < C'F(\sigma X), X > d\sigma - \left\| C'^{\frac{1}{2}}Y - C'^{\frac{1}{2}}F(X)\right\|^2 \\
&+ \Delta_{c'}\bigg(\parallel F(X) \parallel^2 + \parallel Y \parallel^2 \bigg) + \Delta_{b'}\delta_r\delta_{p^{-1}} \parallel Y \parallel^2 \\
&\leq (\alpha+\delta_a\delta_q)\int_0^1 < C'F(\sigma X), X > d\sigma \\
&= (\alpha+\delta_a\delta_q)\int_0^1\int_0^1 \sigma < C'J_F(\tau\sigma X)X, X > d\sigma d\tau \\
&\leq \frac{(\alpha+\delta_a\delta_q)}{2}\Delta_{c'}\delta_f \parallel X \parallel^2 < 0.
\end{aligned}$$

By the identity $2\left|\langle U,V\rangle\right| \leq \|U\|^2 + \|V\|^2$, we obtain the following estimates

$$\begin{aligned}
\psi_2 &\leq \Bigg[\bigg(\frac{(\alpha+\delta_a\delta_q)}{2}(1+\frac{\Delta_a\Delta_q}{\delta_p}) + \frac{\beta}{2}\delta_a\delta_q\bigg)\|\theta_1(t)\| \\
&+ \Delta_a\bigg(1+\frac{\alpha+\delta_a\delta_q}{2\delta_p}+\frac{\beta}{2}\bigg)\|\theta_2(t)\| + \Delta_b\|\theta_3(t)\|\Bigg]\frac{V}{k} \\
&\leq \frac{k_1}{k}\Bigg[\|\theta_1(t)\| + \|\theta_2(t)\| + \|\theta_3(t)\|\Bigg]V,
\end{aligned}$$

and

$$\begin{aligned}
\psi_3 &\leq \int_{t-r(t)}^t \bigg(\|2Z(t)\| + (\alpha+\delta_a\delta_q)\left\|P^{-1}(X(t))Y(t)\right\| \\
&+\beta\|X(t)\|\bigg)\left\|C(t)J_F\left(X(s)\right)P^{-1}\left(X(s)\right)Y(s)\right\| ds \\
&\leq \frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}\int_{\rho(t)}^t \Bigg[2\|Z(t)\|^2 + (\alpha+\delta_a\delta_q)\delta_p^{-1}\|Y(t)\|^2 \\
&+\beta\|X(t)\|^2 + (2+(\alpha+\delta_a\delta_q)\delta_p^{-1}+\beta)\|Y(s)\|^2\Bigg] ds \\
&\leq \frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}\Bigg[\gamma\left(2\|Z\|^2 + (\alpha+\delta_a\delta_q)\delta_p^{-1}\|Y\|^2 + \beta\|X\|^2\right) \\
&+(2+(\alpha+\delta_a\delta_q)\delta_p^{-1}+\beta)\int_{\rho(t)}^t \|Y(s)\|^2 ds\Bigg],
\end{aligned}$$

where

$$k_1 = \max\left\{\frac{(\alpha+\delta_a\delta_q)}{2}(1+\frac{\Delta_a\Delta_q}{\delta_p}) + \frac{\beta}{2}\delta_a\delta_q, \Delta_a(1+\frac{\alpha+\delta_a\delta_q}{2\delta_p}+\frac{\beta}{2}), \Delta_b\right\}.$$

From (i) and (ii) of Theorem 4.1 and Lemma 2.4, we obtain

$$\begin{aligned}
\frac{d}{dt}V \leq & \frac{k_1}{k}\Big[\|\theta_1(t)\|+\|\theta_2(t)\|+\|\theta_3(t)\|\Big]V-\frac{\beta}{2}\big(\delta_f\delta_c-\Delta_f\Delta_c\delta_p^{-1}\gamma\big)\|X\|^2\\
&-\left[\epsilon\delta_{p^{-1}}-\left(\omega_0+\frac{1}{2}(\alpha+\delta_a\delta_q)\Delta_f\Delta_c\delta_p^{-2}\right)\gamma\right]\|Y\|^2\\
&-\frac{1}{2}\left[(\delta_a\delta_q-\alpha)\delta_{p^{-1}}-2\Delta_f\Delta_c\delta_p^{-1}\gamma\right]\|Z\|^2\\
&-\left[\delta_b\delta_{p^{-1}}(\delta_a\delta_q\delta_r\delta_{p^{-1}}-\Delta_f)-\beta A_1\right]\|Y\|^2-\left[\frac{\delta_{p^{-1}}}{2}(\delta_a\delta_q-\alpha)-\beta A_2\right]\|Z\|^2\\
&-\frac{\beta}{4\delta_f\delta_c}\left[\delta_f\delta_c\|X\|+2\left\|(B(t)R(X)-\delta_b\delta_r I)P^{-1}(X)Y\right\|\right]^2\\
&-\frac{\beta}{4\delta_f\delta_c}\left[\delta_f\delta_c\|X\|+2\left\|(A(t)Q(X)-\delta_a\delta_q I)P^{-1}(X)Z\right\|\right]^2\\
&-\left[\omega_0(1-\beta_0)-\frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}(2+(\alpha+\delta_a\delta_q)\delta_p^{-1}+\beta)\right]\int_{\rho(t)}^t\|Y(s)\|^2\,ds.
\end{aligned}$$

Choosing

$$\omega_0=\frac{\Delta_f\Delta_c\delta_p^{-1}(2+(\alpha+\delta_a\delta_q)\delta_p^{-1}+\beta)}{2(1-\beta_0)},$$

and by (iii) we get

$$\begin{aligned}
\frac{d}{dt}V \leq & \frac{k_1}{k}\Big[\|\theta_1(t)\|+\|\theta_2(t)\|+\|\theta_3(t)\|\Big]V-\frac{\beta}{2}\big(\delta_f\delta_c-\Delta_f\Delta_c\delta_p^{-1}\gamma\big)\|X\|^2\\
&-\left[\epsilon\delta_{p^{-1}}-\frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}\gamma\left(\frac{2+(\alpha+\delta_a\delta_q)\delta_p^{-1}+\beta}{1-\beta_0}+(\alpha+\delta_a\delta_q)\delta_p^{-1}\right)\right]\|Y\|^2\\
&-\frac{1}{2}\left[(\delta_a\delta_q-\alpha)\delta_{p^{-1}}-2\Delta_f\Delta_c\delta_p^{-1}\gamma\right]\|Z\|^2.
\end{aligned}$$

Using (8), (9), (10) and taking $\upsilon=\dfrac{k}{k_1}$ we see at once that

$$\begin{aligned}
\frac{d}{dt}U &= e^{-\frac{k_1\mu(t)}{k}}\left(\frac{d}{dt}V-\frac{k_1(||\theta_1(t)||+||\theta_2(t)||+||\theta_3(t)||)}{k}V\right)\\
&\leq e^{-\frac{k_1N}{k}}\Bigg[-\frac{\beta}{2}\big(\delta_f\delta_c-\Delta_f\Delta_c\delta_p^{-1}\gamma\big)\|X\|^2\\
&\quad-\left\{\epsilon\delta_{p^{-1}}-\frac{1}{2}\Delta_f\Delta_c\delta_p^{-1}\gamma\left(\frac{2+(\alpha+\delta_a\delta_q)\delta_p^{-1}+\beta}{1-\beta_0}+(\alpha+\delta_a\delta_q)\delta_p^{-1}\right)\right\}\|Y\|^2\\
&\quad-\left\{\frac{1}{2}\left((\delta_a\delta_q-\alpha)\delta_{p^{-1}}-2\Delta_f\Delta_c\delta_p^{-1}\gamma\right)\right\}\|Z\|^2\Bigg].
\end{aligned}$$

To conclude, if we choose γ so that

$$\gamma<\min\left\{\frac{\delta_f\delta_c}{\Delta_f\Delta_c\delta_p^{-1}},2\epsilon\delta_{p^{-1}}(1-\beta_0)A_3^{-1},\frac{(\delta_a\delta_q-\alpha)\delta_{p^{-1}}}{2\Delta_f\Delta_c\delta_p^{-1}}\right\},$$

we will have the desired inequality

$$\frac{d}{dt}U(t,X_t,Y_t,Z_t)\leq-\xi\Big(\|X\|^2+\|Y\|^2+\|Z\|^2\Big). \tag{12}$$

This shows that the zero solution of system (7) is uniformly asymptotically stable. ■

Example 4.2.

As a special case of the equation (1)

$$(P(X(t))X'(t))'' + A(t)(Q(X(t))X'(t))' + B(t)R(X(t))X'(t) + C(t)F(X(\rho(t))) = 0,$$

where

$$X(t) = \begin{pmatrix} x\,(t) \\ y\,(t) \end{pmatrix},$$

and

$$F\big(X(\rho(t))\big) = \begin{pmatrix} \frac{1}{4}\arctan x(\rho(t)) + \frac{1}{4}x(\rho(t)) \\ 0.16y(\rho(t)) \end{pmatrix}, \quad J_F\,(X) = \begin{pmatrix} \frac{1}{4(1+x^2)} + \frac{1}{4} & 0 \\ 0 & 0.16 \end{pmatrix},$$

$$P(X(t)) = \begin{pmatrix} \frac{\sin(x(t))}{1+x^2(t)} + 2 & 0 \\ 0 & \frac{\cos(y(t))}{1+y^2(t)} + 2 \end{pmatrix}, \quad A(t) = \begin{pmatrix} \frac{e^{\sin t}}{4} + \frac{10}{4} & 0 \\ 0 & \frac{\cos t}{2} + 2 \end{pmatrix},$$

$$Q(X(t)) = \begin{pmatrix} \frac{e^{\sin(x(t))}}{1+x^2(t)} + 2 & 0 \\ 0 & \frac{9\cos(y(t))}{10+y^2(t)} + \frac{41}{10} \end{pmatrix}, \quad B(t) = \begin{pmatrix} \frac{e^{-t^2}+3}{2} & 0 \\ 0 & \frac{\sin t}{2} + 1 \end{pmatrix},$$

$$R(X(t)) = \begin{pmatrix} \frac{10e^{-x^2(t)}}{2+x^2(t)} + 75 & 0 \\ 0 & \frac{100\sin(y(t))}{4+y^2(t)} + 50 \end{pmatrix}, \quad C(t) = \begin{pmatrix} e^{-2t} + 5 & 0 \\ 0 & e^{-t} + 5 \end{pmatrix}.$$

Clearly, $P\,(X)\,, Q\,(X)\,, R\,(X)\,, A, B, C$ and $J_F\,(X)$ are diagonal matrices, hence they are symmetric and commute pairwise. Then, by an easy calculation, we obtain eigenvalues of the matrices $P, Q, R,\ A,\ B,\ C$ and $J_F(X)$ as follows:

$$\delta_p = 1 \le \lambda_1\,(P(X(t))) = \tfrac{\sin x}{1+x^2} + 2, \qquad \lambda_2\,(P(X(t))) = \tfrac{\cos x}{1+x^2} + 2 \le 3 = \Delta_p,$$

$$\delta_q = 2 \le \lambda_1\,(Q(X(t))) = \tfrac{e^{\sin x}}{1+x^2} + 2, \qquad \lambda_2\,(Q(X(t))) = \tfrac{9\cos y}{10+y^2} + \tfrac{41}{10} \le 5 = \Delta_q,$$

$$\delta_r = 25 \le \lambda_1\,(R(X(t))) = \tfrac{100\sin y}{4+y^2} + 50, \qquad \lambda_2\,(R(X(t))) = \tfrac{10e^{-x^2}}{2+x^2} + 75 \le 80 = \Delta_r,$$

$$\delta_a = 1.5 \le \lambda_1\,(A(t)) = \tfrac{1}{2}\cos t + 2, \qquad \lambda_2\,(A(t)) = \tfrac{e^{\sin t}}{4} + \tfrac{10}{4} \le 3.1795 = \Delta_a,$$

$$\delta_b = 0.5 \le \lambda_1\,(B(t)) = \tfrac{\sin t}{2} + 1, \qquad \lambda_2\,(B(t)) = \tfrac{e^{-t^2}}{2} + \tfrac{3}{2} \le 2 = \Delta_b,$$

$$\delta_c = 5 \le \lambda_1\,(C(t)) = e^{-2t} + 5, \qquad \lambda_2\,(C(t)) = e^{-3t} + 5 \le 6 = \Delta_c,$$

$$\delta_f = 0.16 = \lambda_1\,(J_F(X))\,, \qquad \lambda_2\,(J_F(X)) = \tfrac{1}{4(1+x^2)} + \tfrac{1}{4} \le \tfrac{1}{2} = \Delta_f.$$

A simple computation gives

$$\lambda_1\left(A'(t)\right) = -\frac{1}{2}\sin t, \quad \lambda_2\left(A'(t)\right) = \frac{\cos t}{4}e^{\sin t},$$
$$\lambda_1\left(B'(t)\right) = -\frac{\cos t}{2}, \quad \lambda_2\left(B'\right) = -te^{-t^2},$$
$$\lambda_1\left(C'(t)\right) = -2e^{-2t}, \quad \lambda_2\left(C'(t)\right) = -e^{-t}.$$

A trivial verification shows that P, Q and R are nonsingular matrices and we have

$$\frac{d}{dt}P(X(t)) = \begin{pmatrix} \left(\frac{\cos(x(t))}{1+x^2(t)} - \frac{2x(t)\sin(x(t))}{(1+x^2(t))^2}\right)x'(t) & 0 \\ 0 & \left(\frac{-\sin(y(t))}{1+y^2(t)} - \frac{2y(t)\cos(y(t))}{(1+y^2(t))^2}\right)y'(t) \end{pmatrix},$$

$$\frac{d}{dt}Q(X(t)) = \begin{pmatrix} (\frac{\cos(x(t))e^{\sin(x(t))}}{1+x^2(t)} - \frac{2x(t)e^{\sin(x(t))}}{(1+x^2(t))^2})x'(t) & 0 \\ 0 & (\frac{-9\sin(y(t))}{10+y^2(t)} - \frac{18y(t)\cos(y(t))}{(10+y^2(t))^2})y'(t) \end{pmatrix},$$

and

$$\frac{d}{dt}R(X(t)) = \begin{pmatrix} \frac{-20x(t)e^{-x^2(t)}(3+x^2(t))}{(2+x^2(t))^2}x'(t) & 0 \\ 0 & (\frac{100\cos(y(t))}{4+y^2(t)} - \frac{200y(t)\sin(y(t))}{(4+y^2(t))^2})y'(t) \end{pmatrix}.$$

For $t \in [0, +\infty)$ a straightforward calculation give

$$\begin{aligned}
\int_0^t \|\frac{d}{ds}P(X(s))\|ds &= \int_0^t \left|(\frac{\cos(x(s))}{1+x^2(s)} - \frac{2x(s)\sin(x(s))}{(1+x^2(s))^2})x'(s)\right| ds \\
&\quad + \int_0^t \left|(\frac{-\sin(y(s))}{1+y^2(s)} - \frac{2y(s)\cos y(s)}{(1+y^2(s))^2})y'(s)\right| ds \\
&\leq \int_{\omega_1(t)}^{\omega_2(t)} \left|(\frac{\cos u}{1+u^2} - \frac{2u\sin u}{(1+u^2)^2})du\right| \\
&\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \left|(\frac{-\sin v}{1+v^2} - \frac{2v\cos v}{(1+v^2)^2})dv\right| \\
&< \int_{-\infty}^{+\infty} \frac{1+u^2+2|u|}{(1+u^2)^2}du + \int_{-\infty}^{+\infty} \frac{1+u^2+2|u|}{(1+u^2)^2}du \\
&= 2\pi,
\end{aligned}$$

$$\begin{aligned}
\int_0^t \|\frac{d}{ds}(Q(X(s)))\|ds &= \int_0^t \left|(\frac{\cos(x(s))e^{\sin(x(s))}}{1+x^2(s)} - \frac{2x(s)e^{\sin(x(s))}}{(1+x^2(s))^2})x'(s)\right| ds \\
&\quad + \int_0^t \left|(\frac{-9\sin(y(s))}{10+y^2(s)} - \frac{18y(s)\cos(y(s))}{(10+y^2(s))^2})y'(s)\right| ds \\
&\leq \int_{\omega_1(t)}^{\omega_2(t)} \left|(\frac{\cos u e^{\sin u}}{1+u^2} - \frac{2ue^{\sin u}}{(1+u^2)^2})du\right| \\
&\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \left|(\frac{-9\sin v}{10+v^2} - \frac{18v\cos v}{(10+v^2)^2})dv\right| \\
&< \int_{-\infty}^{+\infty}(\frac{e}{1+u^2} + \frac{2e\,|u|}{(1+u^2)^2})du \\
&\quad + \int_{-\infty}^{+\infty}(\frac{9}{10+v^2} + \frac{18\,|v|}{(10+v^2)^2})dv \\
&= (e + \frac{9}{\sqrt{10}})\pi,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \|\frac{d}{ds}(R(X(s)))\|ds &= \int_0^t \left|\frac{-20x(s)e^{-x^2(s)}(3+x^2(s))}{(2+x^2(s))^2}x'(s)\right| ds \\
&\quad + \int_0^t \left|(\frac{100\cos(y(s))}{4+y^2(s)} - \frac{200y(s)\sin(y(s))}{(4+y^2(s))^2})y'(s)\right| ds \\
&\leq \int_{\omega_1(t)}^{\omega_2(t)} \left|(\frac{-20ue^{-u^2}(3+u^2)}{(2+u^2)^2})du\right| \\
&\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \left|(\frac{100\cos v}{4+v^2} - \frac{200v\sin v}{(4+v^2)^2})dv\right| \\
&< \int_{-\infty}^{+\infty} \frac{60\,|u|}{(2+u^2)^2}du + \int_{-\infty}^{+\infty}\left(\frac{100}{4+v^2} + \frac{200\,|v|}{(4+v^2)^2}\right)dv \\
&= 50\pi,
\end{aligned}$$

where

$$\omega_1(t) = \min\{x(0), x(t)\}, \qquad \omega_2(t) = \max\{x(0), x(t)\},$$

and

$$\varphi_1(t) = \min\{y(0), y(t)\}, \qquad \varphi_2(t) = \max\{y(0), y(t)\}.$$

By taking $\alpha = 2.5$ it follows easily that

$$0.06 = \frac{\Delta_p \Delta_f}{\delta_r} < \alpha < \delta_a \delta_q = 3,$$

and

$$\frac{1}{2}(\alpha + \delta_a\delta_q)\Delta_{a'}\Delta_q\Delta_{p^{-1}} - \delta_b(\alpha\Delta_p^{-1}\delta_r - \Delta_f) = -1.65 < 0.$$

We take $r\,(t) = \exp\left(-t^2\right)$, then, $0 \leq r(t) \leq \gamma$, $(\gamma > 0)$, and $r'\,(t) = -2t\exp\left(-t^2\right) \leq \beta_0$ for $0 < \beta_0 < 1$. Thus, all the conditions of Theorem 4.1 are satisfied.

5. Boundedness and the existence of periodic solutions

First, consider a system of delay differential equations

$$x' = F(t, x_t)\ ,\ x_t(\theta) = x(t+\theta)\ ,\ \ -r \le \theta \le 0\ ,\ t \ge 0, \tag{13}$$

where $F : \mathbb{R} \times C_H \longrightarrow \mathbb{R}^n$ is a continuous mapping and takes bounded set into bounded sets. The following lemma is a well-known result obtained by Burton (1985).

Lemma 5.1 (Burton (1985)).

Let $V(t,\phi) : \mathbb{R} \times C_H \longrightarrow \mathbb{R}$ be a continuous and local Lipschitz in ϕ. If
$(i)\ W(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2\left(\int_{t-r(t)}^{t} W_3(|x(s)|)\,ds\right),$
$(ii)\ V'_{(13)} \le W_3(|x(s)|) + M$ for some $M > 0,$ where $W(r), W_i\,(i = 1, 2, 3)$ are wedges,
then the solutions of (13) are uniformly bounded and uniformly ultimately bounded for bound B.

If (13) is a periodic system with period T, we have the following result.

Lemma 5.2 (Li Senlin and Wen Lizhi (1987)).

Suppose that, for $\alpha > 0$, there exists $L(\alpha) > 0$ such that $|f(t, x_t)| \le L(\alpha)$, for $t \in [-T, 0]$ and $\|x_t\| \le \alpha$, and suppose that the solutions of (13) are bounded and ultimately bounded for bound B, then, there exists a periodic solution of (13) of period T.

To study the boundedness and the existence of periodic solutions of (2), we would need to write (2) in the form

$$\begin{aligned}
&X' = P^{-1}(X)Y,\\
&Y' = Z,\\
&Z' = -A(t)\theta_2(t)Y - A(t)Q(X)P^{-1}(X)Z - B(t)R(X)P^{-1}(X)Y - C(t)F(X)\\
&\qquad + C(t)\int_{\rho(t)}^{t} J_F(X(s))\,P^{-1}(X(s))\,Y(s)ds + H\Big(t, X, P^{-1}(X)Y, \theta_1(t)Y + P^{-1}(X)Z\Big).
\end{aligned} \tag{14}$$

Thus, our main theorem in this section is stated with respect to (14) as follows.

Theorem 5.3.

One assumes that all the assumptions of Theorem 4.1 and the assumption

$$\|H(t, X, Y, Z)\| \le h_1(t) + h_2(t)(\|X\| + \|Y\| + \|Z\|) \tag{15}$$

hold, where $h_1(t)$ and $h_2(t)$ are continuous functions and there exist $H_0, \epsilon > 0$ such that

$$h_1(t) \le H_0 \quad h_2(t) \le \epsilon.$$

Then all solutions of system (14) are uniformly bounded and uniformly ultimately bounded.

Proof:

Along any solution $(X(t), Y(t), Z(t))$ of (14), we have

$$\frac{d}{dt}U_{(14)} = \frac{d}{dt}U_{(7)} + < \beta X + (\alpha + \delta_a\delta_q)P^{-1}(X)Y + 2Z, H(t, X, P^{-1}(X)Y, X'') > .$$

From (12), we obtain

$$\frac{d}{dt}U_{(14)} \leq -\xi\bigg(\|X\|^2 + \|Y\|^2 + \|Z\|^2 \bigg) + \kappa_1(\|X\| + \|Y\| + \|Z\|)\|H(t, X, P^{-1}(X)Y, X'')\|,$$

where $\kappa_1 = \max\{\beta, (\alpha + \delta_a\delta_q)\delta_p^{-1}, 2\}$.

Choosing $\epsilon < 3^{-1}\kappa_1^{-1}\xi$, then, there exists $\kappa_2 = \xi - 3\kappa_1\epsilon > 0$.

In view of (15) we have

$$\frac{d}{dt}U_{(14)} \leq -\frac{\kappa_2}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \frac{3}{2}\kappa_1^2 H_0^2 \kappa_2^{-1}, \tag{16}$$

since

$$\frac{\kappa_2}{2}\left\{\left(\|X\| - \kappa_1 H_0\kappa_2^{-1}\right)^2 + \left(\|Y\| - \kappa_1 H_0\kappa_2^{-1}\right)^2 + \left(\|Z\| - \kappa_1 H_0\kappa_2^{-1}\right)^2\right\} \geq 0,$$

for all X, Y and Z. From estimate (16), hypothesis (ii) of Lemma 5.1 is satisfied. Also from estimates (11) and by the fact that $W(t, \phi) \leq W_2(\|\phi\|) + W_3(\int_{\rho(t)}^t W_4(\phi(s))ds)$, is easily verified, then condition (i) of Lemma 5.1 follows. This completes the proof of the theorem. ∎

The following theorem being a consequence of Theorem 5.3 and Lemma 5.2.

Theorem 5.4.

If hypotheses of Theorem 5.3 be satisfied and A, B, C, H are periodic functions of period T, then there exists a periodic solution of system (14) with the period T.

Proof:

It only remains to verify using the assumptions of Theorem 5.3 that the conditions of Lemma 5.2 follow easily. ∎

6. Conclusion

Lyapunov's method has proved to be a popular and useful technique in the study of the stability and boundedness of solutions of higher order non-linear differential equations. In this paper we investigate the asymptotic stability of the zero solution and ultimate boundedness of solutions for certain third order non-linear non-autonomous vector differential equations with delay. Sufficient conditions were obtained for the existence of at least one periodic solution of the equation.

Acknowledgement:

The authors of this paper would like to express their sincere appreciation to the main editor and the anonymous referees for their valuable comments and suggestions which have led to an improvement in the presentation of the paper.

REFERENCES

Afuwape, A. U. (1983). Ultimate boundedness results for a certain system of third order non-linear differential equations, J. Math. Anal. Appl., Vol. 97, No. 1, pp. 140–150.

Afuwape, A. U. Omeike, M. O. (2004). Further ultimate boundedness of solutions of some system of third-order nonlinear ordinary differential equations, Acta Univ. Palacki. Olo-muc. Fac. rer. nat. Math. Vol. 43, pp. 7–20.

Burton, T. A. (1985). *Stability and periodic solutions of ordinary and functional differential equations*, Mathematics in Science and Engineering, 178. Academic Press. Inc. Orlando. FL.

Burton, T. A. (2005). *Volterra Integral and Differential Equations*, Mathematics in Science and Engineering. 202, 2nd edition.

Ezeilo, J. O. C. Tejumola, H. O. (1966). Boundedness and periodicity of solutions of a certain system of third-order nonlinear differential equations, Ann. Math. Pura Appl. 74, pp. 283–316.

Feng, C. (1995). On the existence of periodic solutions for a certain system of third order nonlinear differential equations, Ann. Differential Equations, 11, no. 3, pp. 264–269.

Graef, J. R. Beldjerd, D. and Remili, M. (2015a). On stability, ultimate boundedness, and existence of periodic solutions of certain third order differential equations with delay, Pan American Mathematical Journal 25 , pp. 82–94.

Graef, J. R. Oudjedi, L. D. and Remili, M. (2015b). Stability and square integrability of solutions of nonlinear third order differential equations, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 22, pp. 313–324.

Hale, J. K. (1977). *Theory of Functional Differential Equations*, Springer Verlag, New York.

Mahmoud, A. M. and Tunç, C. (2016). Stability and boundedness of solutions of a certain n-dimensional nonlinear delay differential system of third-order, Adv. Pure. Appl. Math.7(1), pp. 1–11.

Meng, F. W. (1993). Ultimate boundedness results for a certain system of third-order nonlinear differential equations, J. Math. Anal. Appl. 177, pp. 496–509.

Omeike, M. O. (2007). Ultimate Boundedness Results for a Certain Third Order Nonlinear Matrix Differential Equations, Acta Univ. Palacki. Olomuc. Fac. rer. nat. Mathematica 46, pp. 65–73.

Omeike, M. O. (2015). Stability and Boundedness of Solutions of a Certain System of Third order Nonlinear Delay Differential Equations, Acta Univ. Palacki. Olomuc. Fac. rer. nat. Mathematica 54, 1, pp. 109–119.

Oudjedi, L., Beldjerd, D. and Remili, M. (2014). On the Stability of Solutions for non-autonomous delay differential equations of third-order, Differential Equations and Control Processes 1, pp.

22–34.

Oudjedi, L. D. and Remili, M. (2017). Boundedness and stability in third order nonlinear vec tor differential equations with multiple deviating arguments, Journal of the Association of Arab Universities for Basic and Applied Sciences, 24, pp. 176–183.

Remili, M. and Beldjerd, D. (2014a). On the asymptotic behavior of the solutions of third order delay differential equations, Rend. Circ. Mat. Palermo, Vol 63, No 3, pp. 447–455.

Remili, M. and Beldjerd, D. (2015). A boundedness and stability results for a kind of third order delay differential equations, Applications and Applied Mathematics, Vol 10, Issue 2, pp. 772–782.

Remili, M. and Beldjerd, D. (2017). Stability and ultimate boundedness of solutions of some third order differential equations with delay, Journal of the Association of Arab Universities for Basic and Applied Sciences 23, pp. 90–95.

Remili, M. and Oudjedi, D. L. (2014b). Stability and boundedness of the solutions of nonautonomous third order differential equations with delay. Acta Univ. Palacki. Olomuc. Fac. rer. nat. Mathematica Vol 53, No. 2, pp. 139–147.

Remili, M. and Oudjedi, L. D. (2016a). Boundedness and stability in third order nonlinear differential equations with bounded delay, Analele Universităţii Oradea Fasc. Matema-tica, Tom XXIII, Issue No. 1, pp. 135–143.

Remili, M. and Oudjedi, L. D. (2016b). Boundedness and stability in third order nonlinear differential equations with multiple deviating arguments, Archivum Mathematicum, Vol. 52, No. 2, pp. 79–90.

Remili, M. and Oudjedi, L. D. (2016c). Stability of the solutions of nonlinear third order differential equations with multiple deviating arguments, Acta Univ. Sapientiae. Mathematica, 8, 1, pp. 150–165.

Remili, M. and Oudjedi, L. D. (2016d). On asymptotic stability of solutions to third order nonlinear delay differential equation, Filomat, 30, 12, pp. 3217–3226.

Remili, M. Oudjedi, L. D. and Beldjerd, D. (2016e). On the qualitative behaviors of solutions to a kind of nonlinear third order differential equation with delay, Communications in Applied Analysis 20, pp. 53–64.

Remili, M. and Oudjedi, L. D. (2014c). Uniform Stability and Boundedness of a Kind of Third Order Delay Differential Equations, Bull. Comput. Appl. Math. Vol.2, no.1, pp. 25–35.

Remili, M. and Oudjedi, L. D. (2016f). Uniform ultimate boundedness and asymptotic beha viour of third order nonlinear delay differential equation, Afrika Matematika. Vol. 27, Issue 7, pp. 1227–1237.

Li Senlin and Wen Lizhi. (1987). *Functional Differential Equations*, Hunan Science and Tech nology press.

Tiryaki, A. (1999). Boundedness and periodicity results for a certain system of third order nonlinear differential equations, Indian J. Pure Appl. Math. 30, 4, pp. 361–372.

Tunç, C. and GÂÍozen, M. (2014a). Convergence of Solutions to a Certain Vector Differential Equation of Third Order, Abstract and Applied Analysis, vol. 2014, Article ID 424512, 6 pages.

Tunç, C. (2006a). New ultimate boundedness and periodicity results for certain third order nonlinear vector differential equations, Math. J. Okayama Univ. 48, pp. 159–172.

Tunç, C. (2006b). On the boundedness of solutions of certain nonlinear vector differential equations of third order, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 49, 97, no. 3, pp. 291–300.
Tunç, C. (2014b). On the qualitative properties of differential equations of third order with retarded argument, Proyecciones 33, no. 3, pp. 325–347.
Tunç, C. (2009). On the stability and boundedness of solutions of nonlinear vector differential equations of third order, Nonlinear Anal. 70, no. 6, pp. 2232–2236.
Tunç, C. (2017). Stability and boundedness in delay system of differential equations of third order, Journal of the Association of Arab Universities for Basic and Applied Sciences, 22, pp. 76–82.
Tunc, C. and Ates, M. (2006c). Stability and boundedness results for solutions of certain third order nonlinear vector differential equations, Nonlinear Dynam. 45, 3-4 pp. 273–281.