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# Resonance in the Motion of a Geocentric Satellite due to Poynting-Robertson Drag

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## Abstract

The problem of resonance in a geocentric Satellite under the combined gravitational forces of the Sun and the Earth due to Poynting-Robertson (P-R) drag has been discussed in this paper with the assumption that all three bodies, the Earth, the Sun and the Satellite, lie in an ecliptic plane. Our approach differs from conventional ones as we have placed evaluated velocity of the Satellite in equations of motion. We observed five resonance points commensurable between the mean motion of the Satellite and the average angular velocity of the Earth around the Sun, out of which two resonances occur only due to velocity dependent terms of P-R drag. Amplitudes and time periods are periodic with respect to the angle (angle between direction of vernal equinox and the direction of the Sun) which have been evaluated graphically in this paper. We have also found that amplitude as well as time-period decreases as orbital angle of the Earth around the Sun increases in the first quadrant.

**Keywords:** Three-body problem; Ecliptic Plane; Resonance; Poynting-Robertson drag

**MSC 2010 No.:** 37M10, 37M05, 70F07, 34C60

## 1. Introduction

The problem of resonance in the Solar system is one of the important problem and it plays a significant role in dynamical system. Resonance happens when any two or more frequencies are commensurable. Out of five general types of resonances, the most obvious type in a planetary system is mean motion resonance, which occurs when orbital periods of two Satellites or planets are close to a ratio of small integers.

Poynting (1903); Robertson (1937) investigated that the radiation pressure, the Doppler shift of the incident radiation and the Poynting drag are the three terms which generally constitutes the radiation force on a particle exerted by a radiating body. Frick and Garber (1962) have studied the in-plane perturbations of a geosynchronous Satellite under the gravitational forces of the Moon, the Sun and the oblate Earth. They have assumed that all the three bodies (Sun, Moon, Earth) lie in the plane of ecliptic and have also assumed that orbital plane of the Satellite and reference plane coincide with the Earth's equatorial plane. Brouwer (1963) has discussed the resonance in the motion of an artificial satellite caused by Solar radiation pressure. Bhatnagar and Gupta (1977) examined resonance in the motion of an artificial Earth's Satellite caused by Solar-Radiation pressure. They expanded Hamiltonian and the generating function in the power series of a small parameter, which depends on Solar Radiation pressure. Bhatnagar and Mehra (1986) have also discussed the motion of geosynchronous Satellite under the combined gravitational effects of Sun, Moon and the oblate Earth with radiation pressure of the Moon. They have shown that orbital plane of the synchronous Satellite rotates with an angular velocity lying between $0.042^0$ per year and $0.058^0$ per year. They also observed that regression period increases as both orbital inclination and the altitude increase. The radial deviation and the tangential deviation have been determined in the in-plane motion of a geosynchronous Satellite under the gravitational effects of the Sun, Moon and the oblate Earth by Bhatnagar (1990).

Ragos (1995) numerically studied the existence and stability of equilibrium points for particles moving in the vicinity of two massive bodies which exert light radiation pressure. Ragos and Vrahatis (1995) have discussed the photo-gravitational circular restricted three body problem including the P-R effect to describe the effect in the vicinity of two massive radiating bodies. A modified bisection method is used to compute the position of the equilibrium and thereby establishing the stability. Liou and Zook (1995) have explored the effect of radiation pressure, P-R drag, and Solar wind drag on the dust grains trapped in the mean motion resonances with the Sun and Jupiter in the restricted three body problem (R3BP) with negligible dust mass. They especially examined the evolution of dust grain in the 1:1 resonances. Kushvah (2009) investigated the effect of P-R drag on linear stability of equilibrium points in the generalized photo gravitational Chermnykh's problem when a bigger primary is radiating and a smaller primary is an oblate spheroid. It was found that when P-R effect is taken into account, these points were unstable in a linear sense.

Some results on the dynamics of the regularised R3BP with dissipation have been discussed by Alessandra Celletti et al. (2011). They found that a large fraction of test particles with initial conditions in the interior region collides with the Sun. They also found out many interesting and important results. By taking two bodies, one luminous and another non-luminous in elliptical three-body problem Jagdish et al. (2012) studied the motion of infinitesimal mass around seven equilibrium points. There exists conditional stability around triangular points. Lhotka et al. (2014) surveyed the stability of motion to the Lagrangian equilibrium points $L_4$ and $L_5$ in the framework of the spatial elliptic restricted three body problem, subject to the radial component of P-R drag. Yadav and Aggarwal (2013a, 2013b, 2014, 2015), in the series of papers, discussed the resonances in a geo-centric synchronous Satellite under the gravitational forces of the Moon, the Sun and the Earth including its equatorial ellipticity. The amplitude and the time period of the oscillation was determined by using the procedure of Brown and Shook (1933).

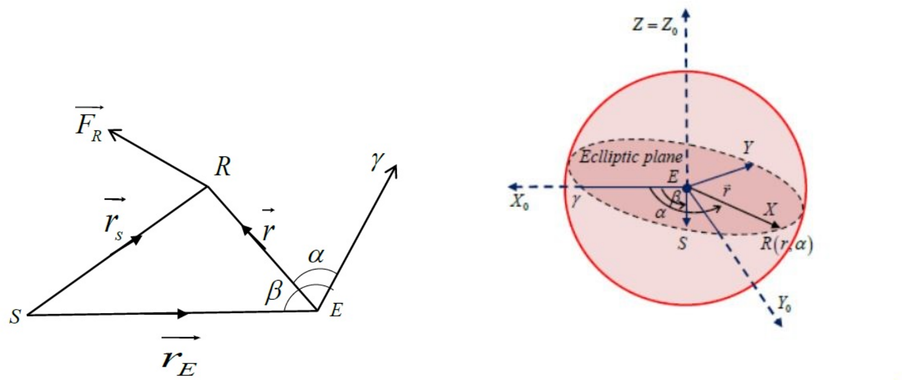
Jain and Aggarwal (2015) investigated the existence of non-collinear liberation points and their stability in the circular restricted three body problem in which they considered the smaller primary as an oblate spheroid and bigger one a point mass including the effect of dissipative forces, especially Stokes drag. Other studies regarding resonance or P-R drag may be seen in Mehra et al. (2016), Lhotka et al. (2016) and Pushparaj and Sharma (2017).

In majority, authors have discussed only two of the three i.e. P-R drag, three-body problem or resonance. We have taken into account all of the three above, which makes our work different from others. This paper aims to find the resonance in the motion of geocentric Satellite under Poynting-Robertson drag of the three-body problem. Diligent scrutinization of equations of motion in Sect. 2 of this paper unveils that there are five points $R_i$'s, $i = 1 - 5$, of resonance in the motion of the orbiting satellite due to the presence of $m$ and $\dot{\beta}$ where $m$ is the mean motion of the satellite and $\dot{\beta}$ the average angular velocity of the sun. Amplitudes and time periods at resonance points have been evaluated in Sect. 3. This section also explores the variation of amplitude and time period with respect to $\beta$ and different values of $q$. The paper ends with the Sect. 4 where the discussions and conclusions are presented.

## 2. Statement of the problem and Equations of motion

Let $S$ represent the Sun, $E$ the Earth and $R$ the Satellite with their masses $M_S$, $M_E$ and $M_R$ respectively. The Satellite moves around the Earth in an ecliptic plane with angular velocity $\vec{\omega}^*$ and the system is also revolving with the same angular velocity $\vec{\omega}^*$. Let $\vec{r}_{_E}$, $\vec{r}_s$ and $\vec{r}$ represent the vectors from Sun and Earth, Sun and Satellite and Earth and Satellite respectively; $\gamma$ be the vernal equinox and $c_0$ the velocity of light. Let $X$, $Y$, $Z$ be the co-ordinate system of the Satellite with origin at the center of the Earth with unit vectors $\hat{I}$, $\hat{J}$ and $\hat{K}$ along the co-ordinates axes respectively. Let $X_0$, $Y_0$ and $Z_0$ be another set of co-ordinate system in the same plane, with origin at the center of the Earth, and unit vectors $\hat{I_0}$ ,$\hat{J_0}$ and $\hat{K_0}$ along the co-ordinate axes respectively (Figure 1(b)). Let $\vec{F}_R$ be the Poynting-Robertson drag per unit mass acting on the Satellite due to radiating body (Sun) in the arbitrary direction as shown in Figure 1(a), given by

$$M_R\vec{F}_R = \vec{f_1} + \vec{f_2} + \vec{f_3}, \quad (\text{Ragos}, 1995)$$




(a) Without co-ordinate axis (b) With co-ordinate axis

**Figure 1.** Configuration of the three body problem

where

$$\vec{f_1} = F\frac{\vec{r_s}}{r_s}(\text{radiation pressure}),$$
$$\vec{f_2} = -F\frac{(\vec{v}\cdot\vec{r_s})}{c_0}\frac{\vec{r_s}}{r_s}(\text{doppler shift owing to the motion}),$$
$$\vec{f_3} = -F\frac{\vec{v}}{c_0},$$
$$= \text{force due to the absorption and re-emission of part of the incident radiation},$$
$$\vec{v} = \text{velocity of } R,$$
$$F = \text{the measure of the radiation pressure.}$$
$$\alpha = \text{the angle between direction of vernal equinox and the direction of the Satellite},$$
$$\beta = \text{the angle between direction of vernal equinox and the direction of the Sun},$$

The relative motion of the Satellite with respect to the Earth can be obtained by

$$\ddot{\vec{r}} = \ddot{\vec{r}}_s - \ddot{\vec{r}}_E,$$
$$= \frac{\vec{F}_{SR} + \vec{F}_{ER} + \vec{F}_R\vec{M}_R}{M_R} - \frac{\vec{F}_{SE}}{M_E},$$

where

$$\vec{F}_{SR} = -G\frac{M_s M_R}{{r_s}^3}\vec{r}_s,$$
$$\vec{F}_{SE} = -G\frac{M_s M_E}{r_E^3}\vec{r}_E,$$
$$\vec{F}_{ER} = -G\frac{M_R M_E}{r^3}\vec{r},$$
$$G = \text{Gravitational constant.}$$

Thus,

$$\ddot{\vec{r}} = -qF_g \frac{\vec{r}_s}{r_s} - \frac{GM_E}{r^3}\vec{r} + \frac{GM_s}{r_{_E}^3}\vec{r}_{_E} - (1-q)F_g \left( \frac{(\vec{v}\cdot\vec{r}_s)}{c_0}\frac{\vec{r}_s}{r_s} - \frac{\vec{v}}{c_0} \right),$$

where

$$q = 1 - \frac{F}{F_g}, \quad F_g = \frac{GM_s}{r_s^2}, \quad q = 1-p, \quad p = \frac{F}{F_g}.$$

$$\dot{\beta}^2 = \frac{GM_s}{r_E^3},$$

also,

$$\vec{r} = r\hat{I}, \quad \vec{r}_{_E} = r_{_E}\hat{r}_{_E}, \quad \hat{r}_{_E} = \cos\beta\hat{I}_\circ + \sin\beta\hat{J}_\circ,$$
$$\vec{r}_{_E} = r_{_E}\cos\beta\hat{I}_\circ + r_{_E}\sin\beta\hat{J}_\circ.$$

Using these values in the equation of motion of the Satellite with respect to the Earth in vector form can be written as

$$\begin{aligned}\ddot{\vec{r}} = {} & -qGM_s\frac{\vec{r}_s}{r_s^3} - \frac{GM_E}{r^3}\vec{r} + \dot{\beta}^2 r_{_E}(\cos\beta\hat{I}_\circ + \sin\beta\hat{J}_\circ) \\ & - (1-q)F_g\left[\frac{(\vec{v}\cdot\vec{r}_s)}{c_0}\frac{\vec{r}_s}{r_s} + \frac{\vec{v}}{c_0}\right].\end{aligned} \tag{1}$$

In the rotating frame of reference with angular velocity $\vec{\omega}^*$ of the satellite about the center of the earth, we get

$$\begin{aligned}\ddot{\vec{r}} = {} & \frac{d^2r}{dt^2}\hat{I} + 2\frac{dr}{dt}(\vec{\omega^*}\times\hat{I}) + r\left(\frac{d\vec{\omega^*}}{dt}\times\hat{I}\right) \\ & + r\left[(\vec{\omega^*}\cdot\hat{I})\vec{\omega^*} - (\vec{\omega^*}\cdot\vec{\omega^*})\hat{I}\right],\end{aligned} \tag{2}$$

where

$$\vec{\omega}^* = \dot{\alpha}\hat{K}.$$

Taking dot products of Equations (1) and (2) with $\hat{I}$, and $\hat{J}$ respectively, and equating the respective coefficients, we get the equations of motion of the Satellite in the synodic coordinate system (Bhatnagar and Mehra (1986))

$$\begin{aligned}\frac{d^2r}{dt^2} - r\dot{\alpha}^2 + \frac{GM_E}{r^2} = {} & \dot{\beta}^2 r_{_E}\cos(\alpha-\beta) - qGM_s\frac{\vec{r}_s\cdot\hat{I}}{r_s^3} \\ & - (1-q)\frac{GM_s}{r_s^2}\left[\frac{(\vec{v}\cdot\vec{r}_s)}{c_0 r_s}(\vec{r}_s\cdot\hat{I}) + \frac{(\vec{v}\cdot\hat{I})}{c_0}\right],\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{d(r^2\dot{\alpha})}{dt} = {} & -\dot{\beta}^2 r_{_E} r\sin(\alpha-\beta) - qGM_s r\frac{\vec{r}_s\cdot\hat{J}}{r_s^3} \\ & - (1-q)\frac{rGM_s}{r_s^2}\left[\frac{(\vec{v}\cdot\vec{r}_s)}{c_0 r_s}(\vec{r}_s\cdot\hat{J}) + \frac{(\vec{v}\cdot\hat{J})}{c_0}\right].\end{aligned} \tag{4}$$

Equations (3) and (4) are the required equations of motion of the Satellite in polar form. These equations are not integrable, therefore we follow the perturbation technique and replace $r$ and $\dot{\alpha}$, by their steady state values $r_0$ and $\dot{\alpha}_0$, also we may take $\alpha = \dot{\alpha}_0 t$ and $\beta = \dot{\beta} t$. Putting the steady state values in the R.H.S of Equations (3) and (4), we get

$$\begin{aligned}\frac{d^2r}{dt^2} - r\dot{\alpha}^2 + \frac{GM_E}{r^2} &= \dot{\beta}^2 r_E \cos(\dot{\alpha}_0 - \dot{\beta})t - qGM_s \frac{\vec{r}_s \cdot \hat{I}}{r_s^3} \\ &\quad -(1-q)\frac{GM_s}{r_s^2}\left[\frac{(\vec{v}\cdot\vec{r}_s)(\vec{r}_s\cdot\hat{I})}{c_0 r_s} + \frac{(\vec{v}\cdot\hat{I})}{c_0}\right], \end{aligned} \tag{5}$$

$$\begin{aligned}\frac{d(r^2\dot{\alpha})}{dt} &= -\dot{\beta}^2 r_E r_0 \sin(\dot{\alpha}_0 - \dot{\beta})t - qGM_s r_0 \frac{\vec{r}_s \cdot \hat{J}}{r_s^3} \\ &\quad - (1-q) r_0 \frac{GM_s}{r_s^2}\left[\frac{(\vec{v}\cdot\vec{r}_s)}{c_0 r_s}(\vec{r}_s\cdot\hat{J}) + \frac{(\vec{v}\cdot\hat{J})}{c_0}\right]. \end{aligned} \tag{6}$$

Now,

$$\begin{aligned}\vec{v} &= \frac{\partial \vec{r}_s}{\partial t} + \vec{\omega}^* \times \vec{r}_s, \\ &= r_E \dot{\beta} \sin(\dot{\alpha}_0 - \dot{\beta})t\hat{I} + r_E \dot{\beta}\cos(\dot{\alpha}_0 - \dot{\beta})t\hat{J} + \dot{\alpha}_0 r_0 \hat{J}.\end{aligned}$$

**Table 1.** Relation between coordinate system

| | $I_0$ | $J_0$ | $K_0$ |
|---|---|---|---|
| $I$ | $\cos\alpha$ | $\sin\alpha$ | 0 |
| $J$ | $-\sin\alpha$ | $\cos\alpha$ | 0 |
| $K$ | 0 | 0 | 1 |

Substituting above values in Equation (5), transformations (Table 1) and taking $r^2\dot{\alpha}$= constant , $r = \frac{1}{u}$ we have

$$\begin{aligned}\frac{d^2u}{d\alpha^2} + u &= \frac{GM_E}{r_0^4 \dot{\alpha_0}^2} - \frac{r_E \dot{\beta}^2 u^2 \cos(\dot{\alpha}_0 - \dot{\beta})t}{\dot{\alpha}_0^2} \\ &\quad + \frac{r_E GM_s q u^2 \cos(\dot{\alpha}_0 - \dot{\beta})t}{r_s^3 \dot{\alpha}_0^2} + \frac{GM_s q u}{r_s^3 \dot{\alpha}_0^2} \\ &\quad - \frac{GM_s(1-q)u r_E^2 (\dot{\alpha}_0 - \dot{\beta})\sin 2(\dot{\alpha}_0 - \dot{\beta})t}{2c_0 r_s^3 \dot{\alpha}_0^2} \\ &\quad - \frac{GM_s(1-q) r_E (\dot{\alpha}_0 - \dot{\beta})\sin(\dot{\alpha}_0 - \dot{\beta})t}{c_0 r_s^3 \dot{\alpha}_0^2} \\ &\quad + \frac{GM_s(1-q)u^2 r_E \dot{\beta}\sin(\dot{\alpha}_0 - \dot{\beta})t}{c_0 r_s^2 \dot{\alpha}_0^2}. \end{aligned} \tag{7}$$

The solution of unperturbed system

$$\frac{d^2u}{d\alpha^2} + u = \frac{GM_E}{r_0^4 \dot{\alpha_0}^2},$$

is given by

$$\frac{l}{r} = 1 + e\cos(\alpha - \omega),$$

where

$$
\begin{aligned}
&r^2\dot{\alpha} = \text{constant},\\
&l = a(1-e^2),\\
&e, \omega = \text{constants of integration},\\
&u = \frac{1 + e\cos(\alpha-\omega)}{a(1-e^2)}.
\end{aligned}
$$

Let us consider

$$\alpha - \omega = f = \dot{\alpha}_0 t = mt(\text{say}).$$

Since $e < 1$, we have

$$(1 + e\cos mt)^{h_1} \approx 1 + h_1 e\cos mt.$$

On simplifying Equation (7) we get

$$
\begin{aligned}
\frac{d^2u}{dt^2} + m^2 u = {}& P_1 + P_2\cos mt + P_3\cos\dot{\beta}t + P_4\sin\dot{\beta}t + P_5\cos(m-\dot{\beta})t\\
&+ P_6\sin(m-\dot{\beta})t + P_7\sin 2(m-\dot{\beta})t + P_8\cos(2m-\dot{\beta})t\\
&+ P_9\sin(2m-\dot{\beta})t + P_{10}\sin(m-2\dot{\beta})t + P_{11}\sin(3m-2\dot{\beta})t.
\end{aligned} \tag{8}
$$

The solution of Equation (8) is given by

$$
\begin{aligned}
u = {}& A\cos(mt-\xi) + \frac{P_1}{m^2} + \frac{P_2 t\sin mt}{2m} + \frac{P_3\cos\dot{\beta}t}{m^2-(\dot{\beta})^2}\\
&+ \frac{P_4\sin\dot{\beta}t}{m^2-(\dot{\beta})^2} + \frac{P_5\cos(m-\dot{\beta})t}{m^2-(m-\dot{\beta})^2} + \frac{P_6\sin(m-\dot{\beta})t}{m^2-(m-\dot{\beta})^2}\\
&+ \frac{P_7\cos(2m-2\dot{\beta})t}{m^2-(2m-2\dot{\beta})^2} + \frac{P_8\cos(2m-\dot{\beta})t}{m^2-(2m-\dot{\beta})^2}\\
&+ \frac{P_9\sin(2m-\dot{\beta})t}{m^2-(2m-\dot{\beta})^2} + \frac{P_{10}\sin(m-2\dot{\beta})t}{m^2-(m-2\dot{\beta})^2}\\
&+ \frac{P_{11}\sin(3m-2\dot{\beta})t}{m^2-(3m-2\dot{\beta})^2},
\end{aligned} \tag{9}
$$

where $P_i$'s are are given in the Appendix A. It is clear that the motion becomes indeterminate if any one of the denominator vanishes in equation (9), which are the points at which resonance occurs. It is found that resonance occurs at five points $m = \dot{\beta}$, $m = 2\dot{\beta}$, $3m = \dot{\beta}$, $2m = \dot{\beta}$, $3m = 2\dot{\beta}$. The 2:1 resonance repeated twice, 1:1 resonance occurs thrice while other resonances occur only once. Out of all resonances, the 3:2 and 1:2 resonances occurs only due to P-R drag. We have also found out the amplitude and time period at these resonance points.

## 3. Time period and amplitude at $m = \dot{\beta}$

In our problem, the solution of Equation (8) is periodic and known which is a condition of Brown and Shook (1933). So we follow the same to determine time periods and amplitudes at $n = \dot{\beta}$. It is recommended to obtain the solution of Equation (8) when that of

$$\frac{d^2u}{dt^2} + m^2u = 0, \tag{10}$$

is periodic and is known. The solution of Equation (10) is

$$u = v\cos p,$$

where

$$\begin{aligned} p &= mt + \bar{\epsilon}, \\ m &= \frac{v_1}{v} = \text{function of } v; \end{aligned} \tag{11}$$

$v$, $v_1$ and $\bar{\epsilon}$ are arbitrary constants. As we are probing the resonance in the motion of the Satellite at the point $m = \dot{\beta}$, the resulting Equation $(9)$ can be written as

$$\frac{d^2u}{dt^2} + m^2u = L\bar{A}\cos mt = L\psi',$$

where

$$\begin{aligned} L &= \frac{(1-q)GM_s e r_{_E}^2(m-\dot{\beta})}{4car_s^3(1-e^2)} = \text{constant}, \\ \bar{A} &= -1, \\ \psi' &= \frac{\partial\psi}{\partial u} = \bar{A}\cos m't, \qquad \psi = u\bar{A}\cos mt, \\ \psi &= \frac{v\bar{A}}{2}\{\cos(m't+p) + \cos(m't-p)\}. \end{aligned} \tag{12}$$

Then,

$$\frac{dv}{dt} = \frac{L}{V}\frac{\partial u}{\partial p}\psi' = \frac{L}{V}\frac{\partial\psi}{\partial p}, \tag{13}$$

$$\frac{dp}{dt} = m - \frac{L}{V}\frac{\partial u}{\partial v}\psi' = m - \frac{L}{V}\frac{\partial\psi}{\partial v}, \tag{14}$$

where

$$\begin{aligned} V &= \frac{\partial}{\partial v}\left(m\frac{\partial u}{\partial p}\right)\frac{\partial u}{\partial p} - m\frac{\partial^2u}{\partial p^2}\frac{\partial u}{\partial v} \\ &= \text{a function of } v \text{ only.} \end{aligned} \tag{15}$$

Since $m$ and $V$ are functions of $v$ only, we can put Equations (13) and (14) into canonical form with new variables defined by,

$$dv_1 = V dv, \tag{16}$$

$$dB_1 = -m dv_1 = -mV dv, \tag{17}$$

Equations (16) and (17) can be put in the form

$$\frac{dv_1}{dt} = \frac{\partial}{\partial v}(B_1 + L\psi), \quad \frac{dp}{dt} = -\frac{\partial}{\partial v}(B_1 + L\psi). \tag{18}$$

Differentiating Equation (14) with respect to $t$ and substituting the expression for $\frac{dp}{dt}$ and $\frac{dv}{dt}$, we have

$$\frac{d^2v}{dt^2} = \frac{L}{V}\left(\frac{\partial m}{\partial v}\frac{\partial \psi}{\partial p} - m\frac{\partial^2 \psi}{\partial p \partial v} - \frac{\partial^2 \psi}{\partial v \partial t}\right) + \frac{L^2}{V^2}\left(\frac{\partial^2 \psi}{\partial p \partial v}\frac{\partial \psi}{\partial v} - V\frac{\partial}{\partial v}\left(\frac{1}{V}\frac{\partial \psi}{\partial v}\right)\frac{\partial \psi}{\partial p}\right). \tag{19}$$

Since the last expression of Equation (19) has the factor $L^2$ it may, in general, be neglected in a first approximation. In Equation (12) we find $p$ and $t$ are present in $\psi$ as sum of the periodic terms with argument

$$p' = p - mt.$$

In our case, the affected term is

$$\psi = \frac{v\bar{A}\cos p'}{2}. \tag{20}$$

Equation (19) for $p'$ is then

$$\frac{d^2p'}{dt^2} + (m - m')^2\frac{L}{V}\frac{\partial}{\partial v}\left(\frac{1}{m - m'}\frac{\partial \psi}{\partial p'}\right) = 0$$

or

$$\frac{d^2p'}{dt^2} - (m - m')^2\frac{L}{2V}\frac{\partial}{\partial v}\left(\frac{v\bar{A}}{m - m'}\right)\sin p' = 0. \tag{21}$$

At first approximation, we put

$$v = v_0, \quad m = m_0, \quad V = V_0.$$

Then Equation (21) can be written as

$$\frac{d^2p'}{dt^2} - (m - m')^2\frac{L}{2V}\frac{\partial}{\partial v}\left(\frac{v\bar{A}}{m - m'}\right)\sin p' = 0. \tag{22}$$

If the oscillations are small, then Equation (22) may be put in the form

$$\frac{d^2p'}{dt^2} - (m - m')^2\frac{L}{2V}\frac{\partial}{\partial v}\left(\frac{v\bar{A}}{m - m'}\right)p' = 0,$$

or

$$\frac{d^2p'}{dt^2} + c_1^2 p' = 0, \tag{23}$$

where

$$c_1 = \sqrt{\frac{(1-q)GM_s er_E(m_0 - \dot{\beta})}{8car_s^3(1-e^2)}}\sqrt{\frac{\sqrt{v}_1}{V_0 v_0}}, \tag{24}$$

$$\begin{aligned} V_0 = (V)_0 &= \left\{\frac{\partial}{\partial v}(m\frac{\partial y}{\partial p})\frac{\partial y}{\partial p} - m\frac{\partial^2 y}{\partial p^2}\frac{\partial y}{\partial v}\right\}_0, \\ &= (\sqrt{v}_1\cos^2(mt + \bar{\epsilon})_0), \\ &= \sqrt{v}_1\cos^2(\dot{\beta}_0 t + \bar{\epsilon}_0). \\ &= \sqrt{v}_1\cos^2(\beta t + \bar{\epsilon}_0), \end{aligned}$$

$$p' = A\sin(c_1 t + \lambda_0) \tag{25}$$

is a solution of (23), where

$$A = \frac{\sqrt{v}_2}{c_1},$$
$$v_2, \lambda_0 = \text{contants of integration},$$
$$v' = v - m't.$$

The equation for $v$ gives

$$v = m't + A\sin(c_1 t + \lambda). \tag{26}$$

Using Equations (13), (20) and (25) the equation for $v$ gives

$$v = v_0 + L\bar{A}\left(\frac{q}{V}\right)_0 \frac{A}{c_1}\cos(c_1 t + \lambda), \tag{27}$$

where $v_0$ is determined from $m_0 = m'$. Since $m_0$ is a known function of $v_0$, the amplitude $A$ and the time period $T$ are given by

$$A = \frac{\sqrt{v}_2}{c_1}, \quad T = \frac{2\pi}{c_1},$$

where $c_2$ is an arbitrary constant,

$$c_1 = \frac{\sqrt{(1-q)GM_s e m_0 r_{_E}^2 (m_0 - \dot{\beta})}}{\sqrt{8car_s^3(1-e^2)}\cos(\beta + \bar{\epsilon}_0)}.$$

Using Equation (10), $v_0$ may be written as

$$v_0 = \frac{\sqrt{v}_1}{m_0},$$

we may choose the constants of integration $v_1 = 1, v_2 = 1, \epsilon_0 = 0$. (Yadav and Aggarwal (2013)).

The amplitude '$A$' and time period '$T$' are given by

$$A = \frac{2\sqrt{2car_s^3(1-e^2)}}{\sqrt{(1-q)GM_s e m_0 r_{_E}^2(m_0 - \dot{\beta})}}\cos\beta,$$
$$T = \frac{4\pi\sqrt{2car_s^3(1-e^2)}}{\sqrt{(1-q)GM_s e m_0 r_{_E}^2(m_0 - \dot{\beta})}}\cos\beta.$$

In the same manner we have calculated amplitudes and time periods at other points. Thereafter two cases arise:

**Case 1** If we take only Solar radiation pressure as perturbing force, then there are only three points $R_1(m = \dot{\beta})$, $R_2(3m = \dot{\beta})$, $R_3(2m = \dot{\beta})$ at which resonance occurs. At critical point $m = \dot{\beta}$ we get amplitude $A_3$ and time-period $T_3$. Corresponding amplitudes and time-periods of our findings are given in table 2 below.

**Case 2** In addition to the above, if we consider velocity dependent terms of P-R drag, then five points $R_1(m = \dot{\beta})$, $R_2(3m = \dot{\beta})$, $R_3(2m = \dot{\beta})$, $R_4(3m = 2\dot{\beta})$, $R_5(m = 2\dot{\beta})$ of resonance occur where three points of resonance are same as in case 1, and $1:2$ and $3:2$ resonances occur only due to velocity dependent terms of P-R drag. But amplitudes and time-periods at all resonance points are not same as in the case of Solar radiation pressure. Corresponding amplitudes and time-periods of the above are given in table 3, where $A_i$'s and $T_i$'s are are given in the Appendix A and Appendix B respectively.

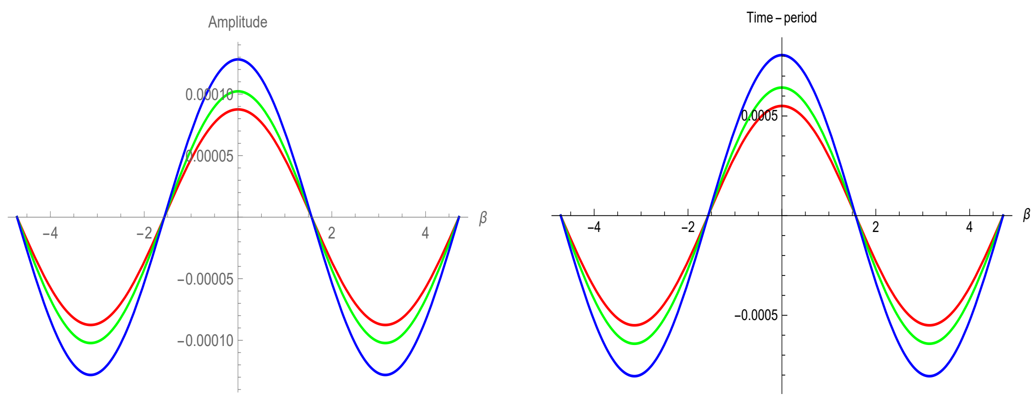
**Table 2.** Resonance Points with only radiation pressure as perturbing force

| | Resonance | Amplitude($A_i$) | Time Period($T_i$) |
|---|---|---|---|
| 1 | $m = \dot{\beta}$ | $A_3$ | $T_3$ |
| 2 | $2m = \dot{\beta}$ | $A_7$ | $T_7$ |
| 3 | $3m = \dot{\beta}$ | $A_6$ | $T_6$ |

**Table 3.** Resonance Points for velocity dependent terms of P-R drag

| | Resonance | Amplitude ($A_i$) | Time Period ($T_i$) |
|---|---|---|---|
| 1 | $m = \dot{\beta}$ | $A_1, A_2$ | $T_1, T_2$ |
| 2 | $2m = \dot{\beta}$ | $A_8, A_9$ | $T_8, T_9$ |
| 3 | $3m = \dot{\beta}$ | $A_5$ | $T_5$ |
| 4 | $m = 2\dot{\beta}$ | $A_4$ | $T_4$ |
| 5 | $3m = 2\dot{\beta}$ | $A_{10}$ | $T_{10}$ |

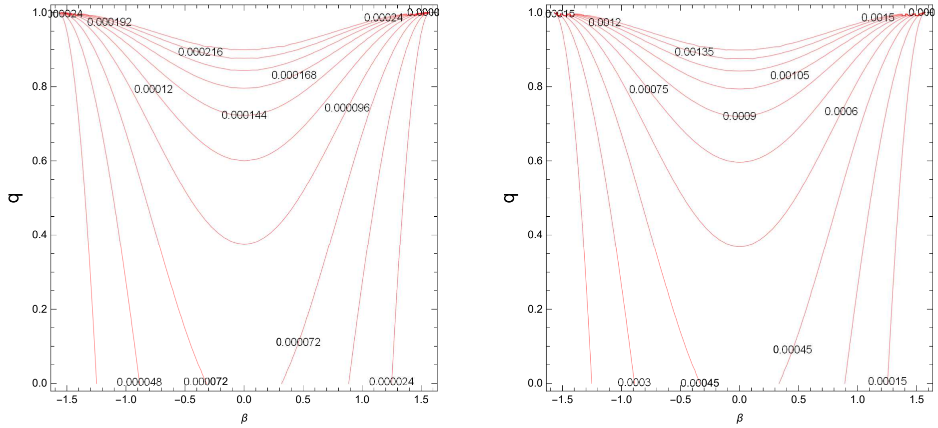
## 4. Discussions and Conclusions




(a) (b)

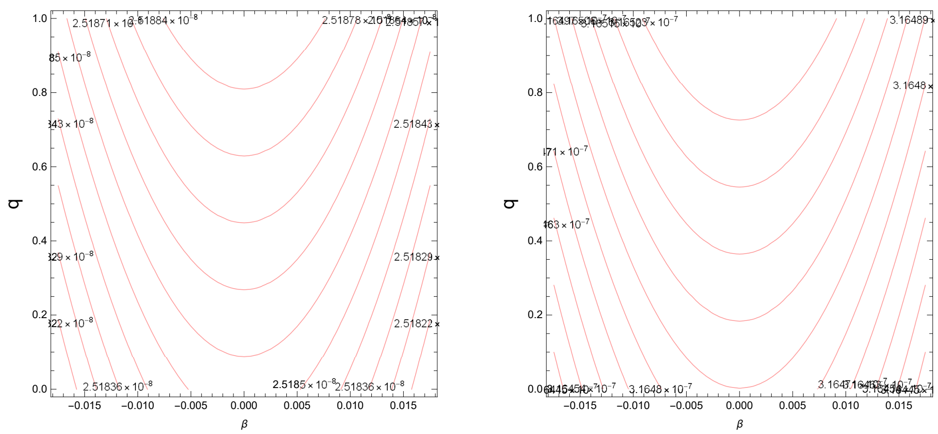
**Figure 2.** (a) Variation in Amplitudes for $0 < \beta < 90^0$ at $q1 = 0.25(Red)$, $q2 = 0.45(Green)$, $q3 = 0.65(Blue)$, (b) Variation of Time-periods for $0 < \beta < 90^0$ at $q1 = 0.25(Red)$, $q2 = 0.45(Green)$, $q3 = 0.65(Blue)$

In the present study the resonance in the motion of the Satellite in the Sun-Earth-Satellite system have been studied under the influence of P-R drag effect by using the method of Brown and Shook (1933). Firstly, we have derived the equations of motion of the geocentric Satellite in vector form and further this equation is converted to polar form to obtain two system of equations, so that we can apply the well known method discussed by Brown and Shook. Next, it is found that there are five points $R_1(m = \dot{\beta})$, $R_2(3m = \dot{\beta})$, $R_3(2m = \dot{\beta})$, $R_4(3m = 2\dot{\beta})$, $R_5(m = 2\dot{\beta})$ at which resonances occur where $m \approx \dot{\alpha_0}$ is average angular velocity of a Satellite and $\dot{\beta}$ is average angular velocity of the Earth. The 2:1 resonance occurs twice, 1:1 resonance occurs thrice while 1:2, 3:1 and 3:2 resonances occur only once. There are two resonance points 3:2 and 1:2 occur only due



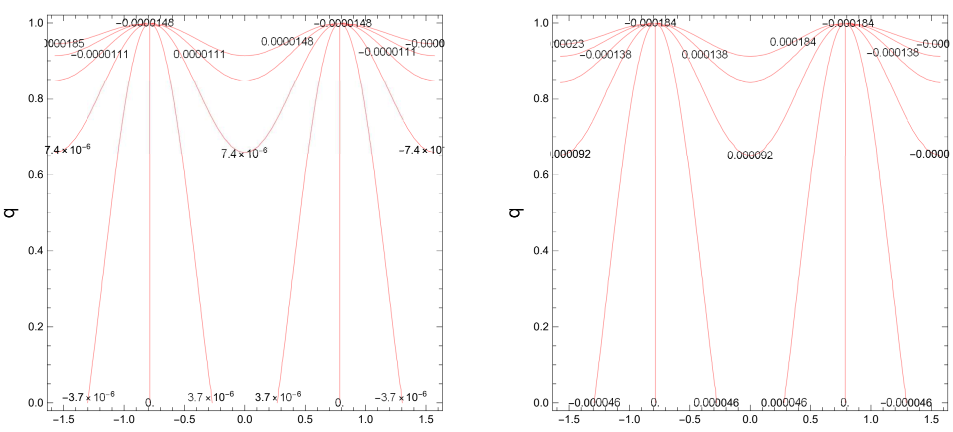

(a) (b)

**Figure 3.** (a) Variation in Amplitude 'A' with respect to $\beta$, $-90^0 < \beta < 90^0$ and $q$ $(0 < q < 1)$ at resonance 1:1, (b) Time-period 'T' for $-90^0 < \beta < 90^0$ and $q$ $(0 < q < 1)$ at resonance 1:1




(a) (b)

**Figure 4.** (a) Variation in Amplitude 'A' with respect to $\beta$, $-1^0 < \beta < 1^0$ and $q$ $(0 < q < 1)$at resonance 1:1, (b) Time-period 'T' for $-1^0 < \beta < 1^0$ and $q$ $(0 < q < 1)$ at resonance 1:1




(a) (b)

**Figure 5.** (a) Variation in Amplitude 'A' with respect to $\beta$, $-90^0 < \beta < 90^0$ and $q$ $(0 < q < 1)$ at resonance 1:2, (b) Time-period 'T' for $-90^0 < \beta < 90^0$ and $q$ $(0 < q < 1)$ at resonance 1:2

to velocity dependent terms of P-R drag. If we ignore perturbing force then resonance will occur only at three points in the equation of motion of a Satellite. We have shown the effect of P-R drag on amplitudes and time-periods. Further, we have evaluated the amplitudes and time periods of the Satellite numerically by using, data mentioned in Appendix A. From Figure 2, we observe that amplitude and time-period increases when $q$ increases and it is maximum at $\beta = 0$. $p$ is the factor of velocity dependent terms of P-R drag, when $q$ increases $p$ decreases and hence when P-R decreases then amplitude as well as time-period increases. Figure 3 explains the variation in $A_1$ and time-period $T_1$ respectively for $-90^0 < \beta < 90^0$ and $0 < q < 1$, at resonance $1:1$ with P-R drag. Above graphs show that amplitude and time-period decreases as $\beta$ increases. Figure 4 also explains the amplitude and time period with respect to $\beta$. In this case it can be observed that amplitude becomes very high of greater range of $\beta$ but it is not in the case of velocity dependent terms of P-R drag. Similarly, Figure 5 explains the variation in amplitude for $-90^0 < \beta < 90^0$ and $0 < q < 1$ at resonance $1:2$. Graphs show that amplitude is periodic with respect to $\beta$ and it increases (decreases) when $q$ increases (decreases).

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# Appendix A

Using the following data of a Satellite,

$$
\begin{aligned}
&a = 6921000m,\\
&e = 0.0065,\\
&m_0 = 0.0628766\frac{deg}{sec},\\
&\dot{\beta} = 0.0000114077\frac{deg}{sec},\\
&r_s = 149599 \times 10^6 m,\\
&r_{_E} = 149.6 \times 10^9 m,\\
&c = 3 \times 10^8 \frac{m}{sec}.
\end{aligned}
$$

We make the above quantities dimensionless by taking

$$
\begin{aligned}
M_E + M_s &= 1\text{unit},\\
G &= 1\text{unit},\\
r_s &= \text{distance between the Earth and the Sun}\\
&= 1 \text{ unit}.\\
P_1 &= -\left(\frac{GM_s q}{r_s^3 a(1-e^2)} - \frac{GM_E}{{r_0}^4}\right),\\
P_2 &= -\frac{GM_s q e}{r_s^3 a(1-e^2)},\\
P_3 &= -\frac{r_{_E} e(\dot{\beta}^2 r_s^3 - GM_s q)}{r_s^3 a^2(1-e^2)^2} = P_8,\\
P_4 &= -\frac{r_{_E} e\dot{\beta} GM_s(1-q)}{r_s^2 c_0 a^2(1-e^2)^2} = P_9,\\
P_5 &= -\frac{r_{_E}(\dot{\beta}^2 r_s^3 - GM_s q)}{r_s^3 a^2(1-e^2)^2},\\
P_6 &= -\frac{\left(a^2(1-e^2)^2(m-\dot{\beta}) - r_s\dot{\beta}\right) r_{_E} GM_s(1-q)}{c_0 r_s^3 a^2(1-e^2)^2},\\
P_7 &= -\frac{(m-\dot{\beta}) r_{_E}^2 GM_s(1-q)}{2r_s^3 c_0 a(1-e^2)},\\
P_{10} &= -\frac{r_{_E}^2 e(m-\dot{\beta}) GM_s(1-q)}{4r_s^3 c_0 a(1-e^2)} = P_{11}.
\end{aligned}
$$

# Appendix B

$$A_1 = \frac{2\sqrt{2ca(1-e^2)r_s^3}}{\sqrt{(1-q)GM_s m_0 e r_{_E}^2 (m_0 - \dot{\beta})}} \cos\beta,$$

$$A_2 = \frac{r_s a(1-e^2)\sqrt{2c}}{\sqrt{(1-q)GM_s e m_0 r_{_E} \dot{\beta}}} \cos\beta,$$

$$A_3 = \frac{a(1-e^2)\sqrt{2{r_s}^3}}{\sqrt{m_0 r_{_E} e(\dot{\beta}^2 r_s^3 - GM_s q)}} \cos\beta,$$

$$A_4 = \frac{2\sqrt{ca(1-e^2)r_s^3}}{\sqrt{(1-q)GM_s m_0 r_{_E}^2 (m_0 - \dot{\beta})}} \cos 2\beta,$$

$$A_5 = \frac{r_s a(1-e^2)\sqrt{2c}}{\sqrt{(1-q)GM_s e m_0 r_{_E} \dot{\beta}}} \cos\frac{\beta}{3},$$

$$A_6 = \frac{a(1-e^2)\sqrt{2r_s^3}}{\sqrt{m_0 r_{_E} e(\dot{\beta}^2 r_s^3 - GM_s q)}} \cos\frac{\beta}{3},$$

$$A_7 = \frac{a(1-e^2)\sqrt{2r_s^3}}{\sqrt{m_0 r_{_E} (\dot{\beta}^2 r_s^3 - GM_s q)}} \cos\frac{\beta}{2},$$

$$A_8 = \frac{a(1-e^2)\sqrt{2cr_s^3}\cos\frac{\beta}{2}}{\sqrt{(1-q)GM_s m_0 r_{_E}((m_0 - \dot{\beta})(1-e^2)^2 a^2 - \dot{\beta} r_s)}},$$

$$A_9 = \frac{2\sqrt{2ca(1-e^2)r_s^3}}{\sqrt{(1-q)GM_s e m_0 r_{_E}^2 (m_0 - \dot{\beta})}} \cos\frac{\beta}{2},$$

$$A_{10} = \frac{2\sqrt{ca(1-e^2)r_s^3}}{\sqrt{(1-q)GM_s m_0 r_{_E}^2 (m_0 - \dot{\beta})}} \cos\frac{2\beta}{3}.$$

$$T_1 = \frac{4\pi\sqrt{ca(1-e^2)2r_s^3}}{\sqrt{(1-q)GM_s m_0 e r_{_E}^2 (m_0 - \dot{\beta})}} \cos\beta,$$

$$T_2 = \frac{2\pi r_s a(1-e^2)\sqrt{2c}}{\sqrt{(1-q)GM_s m_0 e r_{_E} \dot{\beta}}} \cos\beta,$$

$$T_3 = \frac{2\pi a(1-e^2)\sqrt{2r_s^3}}{\sqrt{m_0 r_{_E} e(\dot{\beta}^2 r_s^3 - GM_s q)}} \cos\beta,$$

$$T_4 = \frac{4\pi\sqrt{ca(1-e^2)r_s^3}}{\sqrt{(1-q)GM_s m_0 r_{_E}^2 (m_0 - \dot{\beta})}} \cos 2\beta,$$

$$T_5 = \frac{2\pi r_s a(1-e^2)\sqrt{2}c}{\sqrt{(1-q)GM_s e m_0 r_{_E} \dot{\beta}}} \cos\frac{\beta}{3},$$

$$T_6 = \frac{2\pi a(1-e^2)\sqrt{2r_s^3}}{\sqrt{m_0 r_{_E}(\dot{\beta}^2 r_s^3 - GM_s q)}} \cos\frac{\beta}{3},$$

$$T_7 = \frac{2\pi a(1-e^2)\sqrt{2r_s^3}}{\sqrt{m_0 r_{_E}(\dot{\beta}^2 r_s^3 - GM_s q)}} \cos\frac{\beta}{2},$$

$$T_8 = \frac{2\pi a(1-e^2)\sqrt{2cr_s^3}\cos\frac{\beta}{2}}{\sqrt{(1-q)GM_s m_0 r_{_E}((m_0-\dot{\beta})(1-e^2)^2 a^2 - \dot{\beta} r_s)}},$$

$$T_9 = \frac{4\pi\sqrt{ca(1-e^2)2r_s^3}}{\sqrt{(1-q)GM_s m_0 e r_{_E}^2 (m_0-\dot{\beta})}} \cos\frac{\beta}{2},$$

$$T_{10} = \frac{4\pi\sqrt{ca(1-e^2)r_s^3}}{\sqrt{(1-q)GM_s m_0 r_{_E}^2 (m_0-\dot{\beta})}} \cos\frac{2\beta}{3}.$$