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On the qualitative behaviors of a functional differential equation of second order

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Abstract

The aim of this paper is first to investigate the stability of the zero solution to a new Liénard type equation with multiple variable delays by two different methods. The methods to be used in the proofs involve the Lyapunov-Krasovskiĭ functional approach and the fixed point technique under an exponentially weighted metric, respectively. We make a comparison between the applications of these methods with the established conditions on the same stability problems. Then, we obtain three new results for uniformly stability and boundedness/ uniformly boundedness of the solutions to the considered equation by the Lyapunov-Krasovskiĭ functional approach. An example is given to verify the results obtained by the Lyapunov-Krasovskiĭ functional approach. Our results complement and improve some recent ones in the literature.

Keywords: Functional differential equation; second order; multiple variable delays,; Lyapunov-Krasovskiĭ functional; stability; fixed points

2010 Mathematics Subject Classifications: 34K20, 34K40

1. Introduction

In the past years, many researchers claimed that the fixed point theory has an important advantage over the Lyapunov’s direct method. Because, while the Lyapunov ‘s direct method usually requires pointwise conditions, fixed point theory needs average conditions, (see Burton (2005), Burton (2006) and Burton and Furumochi (2001)). In 2001, Burton and Furumochi (2001) observed some difficulties that occur in studying the stability theory of ordinary and functional differential equations by the Lyapunov's second (direct) method. Rather than invent new modifications of the standard Lyapunov function(al) method to overcome the difficulties, the authors demonstrate by various examples that the contraction mapping principle can do the magic in many circumstances. It should be noted that, by using the fixed point theory, Burton (2001) considered the Liénard type equation with constant delay, $L\,(>0)$:

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t)g(x(t-L)) = 0.$$

The author obtained conditions for each solution $x(t)$ to satisfy $(x(t), x'(t)) \to (0,0)$ as $t \to \infty$. After that, Pi (2011) investigated the stability of functional Liénard type equation with variable delay

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t)g(x(t-\tau(t))) = 0.$$

By the fixed point theory, under an exponentially weighted metric, the author gave proper interesting sufficient conditions for the stability and asymptotically stability of the zero solution.

Meanwhile, Tunç and Biçer (2014) considered the Liénard type equation with multiple variable delays

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + \sum_{j=1}^{n} b_j(t) g_j(x(t-\tau_j(t))) = 0.$$

The authors studied the stability of the zero solution of this equation by the fixed point technique under an exponentially weighted metric.

Further, by means of the Lyapunov's function or functional approach, Korkmaz and Tunç (2015), Tunç (2010), Tunç (2011a), Tunç (2011b), Tunç (2013a), Tunç (2013b), Tunç (2014), and Tunç and Yazgan (2013) discussed some problems on the stability, boundedness, uniform-boundedness and existence of periodic solutions of certain nonlinear differential equations of second order without and with delay.

In this paper, we consider the following Liénard type equation with multiple variable delays:

$$\ddot{x} + a_0(t) f(t, x, \dot{x})\dot{x} + a_1(t) f_1(\dot{x}) + a_2(t)g(x) + \sum_{j=1}^{n} b_j(t) g_j(x(t-\tau_j(t))) = p(t), \tag{1}$$

where $t \in \Re^+$, $\Re^+ = [0,\infty)$, $a_0, a_1, a_2, b_j : \Re^+ \to (0,\infty)$ are continuous functions, $f : \Re^+ \times \Re \times \Re \to \Re^+$, f_1, g, $g_j : \Re \to \Re$, $f_1(0) = 0$, $g(0) = 0$, $g_j(0) = 0$, $p : \Re^+ \to \Re$, and $\tau_j : \Re^+ \to \Re^+$ are all continuous functions with $t - \tau_j(t) \geq 0$.

We can write equation (1) in the system form as follows:

$$x' = y,$$

$$y' = -a_0(t) f(t, x, y)y - a_1(t) f_1(y) - a_2(t)g(x)$$

$$- \sum_{j=1}^{n} b_j(t) g_j(x) + \sum_{j=1}^{n} b_j(t) \int_{t-\tau_j(t)}^{t} g_j'(x(s))y(s)ds + p(t). \tag{2}$$

The continuity of $a_0, a_1, a_2, b_j, f, f_1, g, g_j$ and τ_j is a sufficient condition for existence of the solution of equation (1). Further, it is assumed as basic that f, f_1, g and g_j satisfy a Lipschitz condition. Hence, the uniqueness of solutions of equation (1) is guaranteed. Finally, we assume that the derivatives $a_2'(t), b_j'(t), \tau_j'(t), \frac{df_1}{dx'} \equiv f_1'(x')$ and $\frac{dg_j}{dx} \equiv g_j'(x)$ exist and are continuous, and throughout the paper $x(t)$ and $x'(t)$ are abbreviated as x and x', respectively.

Define

$$F_1(y) = \begin{cases} \frac{f_1(y)}{y}, & y \neq 0, \\ f_1'(0), & y = 0. \end{cases}$$

2. Preliminaries

Consider the general non-autonomous delay differential system

$$\dot{x} = G(t, x_t), \ x_t = x(t+\theta), \ -r \le \theta \le 0, \ t \ge 0, \tag{3}$$

where $G:[0,\infty)\times C_H \to \Re^n$ is a continuous mapping, $G(t,0)=0$, and we suppose that G takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi:[-r,0]\to\Re^n$ with supremum norm; $r>0$, C_H is the open H-ball in C; $C_H := \{\phi \in C([-r,0], \Re^n) : \|\phi\| < H\}$. Standard existence theory, see Burton [2], shows that if $\phi \in C_H$ and $t \ge 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0, t_0+\alpha)$ satisfying equation (3) for $t > t_0$, $x_t(t,\phi)=\phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.|$ will denote a convenient norm in $\Re^n$ with $|x| = \max_{1\le i\le n}|x_i|$. Let us assume that $C(t)=\{\phi : [t\text{-}\alpha, t] \to \Re^n \mid \phi$ is continuous$\}$ and ϕ_t denotes the ϕ in the particular $C(t)$, and that $\|\phi_t\| = \max_{t-\alpha\le s\le t}|\phi(t)|$. It can be seen that equation (1) is a particular case of (3).

Definition 1. [Burton (2006)]

1^0) A continuous positive definite function $W:\Re^n \to [0,\infty)$ is called a wedge.

2^0) A continuous function $W:[0,\infty)\to[0,\infty)$ with $W(0)=0$, $W(s)>0$ if $s>0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i is an integer.)

Definition 2. [Burton (2006)]

Let D be an open set in $\Re^n$ with $0 \in D$. A function $V:[0,\infty)\times D \to [0,\infty)$ is called positive definite if $V(t,0)=0$ and if there is a wedge W_1 with $V(t,x) \ge W_1(|x|)$, and is called decrescent if there is a wedge W_2 with $V(t,x) \le W_2(|x|)$.

Theorem 1. [Burton (2006)]

Let $V(t,x_t)$ be a differentiable scalar functional defined when $x:[\alpha,t]\to\Re^n$ is continuous and bounded by some $D \le \infty$. If

$(A1)$ $V(t,0)=0$, $W_1(|x(t)|) \le V(t,x_t)$, (where $W_1(r)$ is a wedge),

$(A2)$ $\dot{V}(t,x_t) \le 0$.

Then, the zero solution of equation (3) is stable.

Theorem 2. [Burton (2006)]

Assume that there exists a Lyapunov-Krasovskiì functional for (3) and wedges satisfying;

$(B1)$ $W_1(|\varphi(0)|) \le V(t,\varphi) \le W_2(\|\varphi\|)$, (where $W_1(r)$ and $W_2(r)$ are wedges),

$(B2)$ $\dot{V}(t,\varphi) \le 0$.

Then, the zero solution of equation (3) is uniformly stable.

Theorem 3. [Yoshizawa (1966)]

Suppose that there exists a continuous Lyapunov-Krasovskiì functional $V(t,\varphi)$ defined for all $t \in \Re^+$ and $\varphi \in S^{\bullet}$, which satisfies the following conditions;

$(C1)$ $a(|\varphi(0)|) \le V(t,\varphi) \le b_1(|\varphi(0)|) + b_2(\|\varphi\|)$,

where $a(r)$, $b_1(r)$, $b_2(r) \in CI$, (CI denotes the families of continuous increasing functions), and are positive for $r > H$ and $a(r) - b_2(r) \to \infty$ as $r \to \infty$,

$(C2)$ $\dot{V}(t,\varphi) \le 0$.

Then, the solutions of equation (3) are uniformly bounded.

3. Stability and boundedness by the Lyapunov-Krasovskiì functional approach

First, we prove three new theorems by the Lyapunov-Krasovskiĭ functional approach (see Krasovskiĭ (1963)).

Let $p(t) \equiv 0$ in equation (1).

The first main result of this paper is the following theorem.

Theorem 4.

We assume that there exist positive constants $L,\ L_j$ and a continuous function $a(t)$ such that the following conditions hold:

$(D1)$ $t-\tau_j(t)$ is strictly increasing, $\lim_{t\to\infty}(t-\tau_j(t)) = \infty$;

$(D2)$ $a_2,\ (a_2 \geq 1)$, and b_j are positive and decreasing functions and a_0 and a_1 are positive and increasing functions such that

$$a_0(t) f(t,x,y) + a_1(t)F_1(y) \geq a(t) \geq \frac{1}{2}\sum_{j=1}^{n} L_j (b_j(t)+1)\tau_j(t) \geq 0,$$

$$b_j(t) \leq 1-\tau_j'(t);$$

$(D3)$ $f_1(0)=0,\ yf_1(y) \geq 0,\ g(0)=0,\ xg(x) > 0,\ (x \neq 0),\ |g'(x)| \geq L,$

$$g_j(0)=0,\ xg_j(x) > 0,\ (x \neq 0),\ |g_j'(x)| \leq L_j,\ (j=1,2,\ldots,n).$$

Then, the zero solution of equation (1) is stable.

***Proof*:**

Define the Lyapunov-Krasovskiì functional $V = V(t)$ by

$$V = \exp(-2\int_0^t |e(s)|ds) \times \{\sum_{j=1}^{n} b_j(t)\int_0^x g_j(s)ds + a_2(t)\int_0^x g(s)ds + \frac{1}{2}y^2 + 1$$

$$+\sum_{j=1}^{n}\lambda_j \int_{-\tau_j(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\}, \qquad (4)$$

where λ_i are positive constants which will be determined later and $e(t)$ is a continuous function on $\Re^+ = [0,\infty)$ and where $|e(.)| \in L^1(0,\infty)$, $L^1(0,\infty)$ is space of Lebesgue integrable functions, that is, $\int_0^\infty |e(s)|ds < \infty$, say $\int_0^\infty |e(s)|ds = K$, $K \in \Re$, $K > 0$. Then, it follows that

$$\exp(-2\int_0^\infty |e(s)|ds) \times \{\sum_{j=1}^n b_j(t)\int_0^x g_j(s)ds + a_2(t)\int_0^x g(s)ds$$

$$+\frac{1}{2}y^2 + \sum_{j=1}^n \lambda_j \int_{-\tau_j(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds\} \le V.$$

Hence, in view of assumptions of Theorem 4, it may be seen that

$$\frac{1}{2}\exp(-2K)(Lx^2 + y^2) \le V(t, x_t, y_t)$$

so that

$$\frac{1}{2}\exp(-2K)L_0(x^2 + y^2) \le V(t, x_t, y_t),$$

where $L_0 = \min\{1, L\}$. Thus, one can easily show that assumption $(A1)$ of Theorem 1 holds, that is, $V(t,0) = 0$, $V(t, x_t) \ge W_1(|x(t)|)$.

Calculating the time derivative of the Lyapunov-Krasovskiì functional V along system (2), we get

$$\dot{V} = -2|e(t)|\exp(-2\int_0^t |e(s)|ds)$$

$$\times\{\sum_{j=1}^n b_j(t)\int_0^x g_j(s)ds + a_2(t)\int_0^x g(s)ds + \frac{1}{2}y^2 + 1 + \sum_{j=1}^n \lambda_j \int_{-\tau_j(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds\}$$

$$-\exp(-2\int_0^t |e(s)|ds) \times \{a_0(t)f(t,x,y) + a_1(t)F_1(y) - \sum_{j=1}^n \lambda_j\tau_j(t)\}y^2$$

$$+\exp(-2\int_0^t |e(s)|ds) \times \{\sum_{j=1}^n b_j'(t)\int_0^x g_j(s)ds + a_2'(t)\int_0^x g(s)ds$$

$$+\exp(-2\int_0^t |e(s)|ds)$$

$$\times\{y\sum_{j=1}^n b_j(t)\int_{t-\tau_j(t)}^t g_j'(x(s))y(s)ds - \sum_{j=1}^n \lambda_j\{1-\tau_j'(t)\}\int_{t-\tau_j(t)}^t y^2(s)ds\}.$$

By the assumptions of Theorem 4 and the estimate $2|ab| \le a^2 + b^2$, it follows that

$$a_2(t)\int_0^x g(s)ds \ge \int_0^x g(s)ds > 0$$

so that

$$-2|e(t)|\exp(-2\int_0^t |e(s)|ds) \times a_2(t)\int_0^x g(s)ds \le \int_0^x g(s)ds < 0,$$

since a_2, $(a_2 \ge 1)$, $g(0)=0$, $xg(x)>0$, $(x \ne 0)$, and

$$\begin{aligned} b_j(t)y\int_{t-\tau_j(t)}^t g'_j(x(s))y(s)ds &\le \frac{b_j(t)}{2}\int_{t-\tau_j(t)}^t |g'_j(x(s))|(y^2(t)+y^2(s))ds \\ &\le \frac{b_j(t)}{2}\int_{t-\tau_j(t)}^t L_j(y^2(t)+y^2(s))ds \\ &= \frac{L_j b_j(t)\tau_j(t)}{2}y^2(t) + \frac{L_j b_j(t)}{2}\int_{t-\tau_j(t)}^t y^2(s)ds. \end{aligned}$$

Hence, in view of the assumptions of Theorem 4, we get

$$\begin{aligned} \dot{V} &\le -\exp(-2\int_0^t |e(s)|ds) \times \{a_0(t)f(t,x,y) + a_1(t)F_1(y) - \sum_{j=1}^n \lambda_j \tau_j(t)\}y^2 \\ &\quad + \exp(-2\int_0^t |e(s)|ds) \times \{y\sum_{j=1}^n b_j(t)\int_{t-\tau_j(t)}^t g'_j(x(s))y(s)ds \\ &\quad - \sum_{j=1}^n \lambda_j\{1-\tau'_j(t)\}\int_{t-\tau_j(t)}^t y^2(s)ds\} \\ &\le -\{a(t) - \sum_{j=1}^n \lambda_j\tau_j(t) - \frac{1}{2}\sum_{j=1}^n L_j b_j(t)\tau_j(t)\}y^2 \times \exp(-2\int_0^t |e(s)|ds) \\ &\quad + \frac{1}{2}\exp(-2\int_0^t |e(s)|ds) \times \sum_{i=1}^n \{[L_j b_j(t) - 2\lambda_j(1-\tau'_j(t))]\int_{t-\tau_j(t)}^t y^2(s)ds\}. \end{aligned}$$

Let $\lambda_j = \frac{L_j}{2}$. Then, we have

$$\begin{aligned} \dot{V} &\le -\{a(t) - \frac{1}{2}\sum_{j=1}^n L_j(b_j(t)+1)\tau_j(t)\} \times \exp(-2\int_0^t |e(s)|ds)y^2 \\ &\quad + \frac{1}{2}\sum_{j=1}^n \{L_j[b_j(t) - (1-\tau'_j(t))]\int_{t-\tau_j(t)}^t y^2(s)ds\} \times \exp(-2\int_0^t |e(s)|ds) \le 0, \end{aligned}$$

by the assumptions of Theorem 4. This estimate completes the proof of Theorem 4 (see Theorem 1).

Let $p(t) \equiv 0$ in equation (1).

The second main result of this paper is the following theorem.

Theorem 5.

We assume that assumptions $(D1)-(D3)$ of Theorem 4 hold. Then, the zero solution of equation (1) is uniformly stable.

***Proof*:**

In the light of the assumptions of Theorem 5, it can be easily completed the proof. Therefore, we omit the details of the proof (see Theorem 2).

The third main result of this paper is the following theorem.

Let $p(t) \neq 0$ in equation (1).

Theorem 6.

We assume that assumptions $(D1)-(D3)$ of Theorem 4 and the following assumption hold:

$(D4)$ $|p(t)| \leq q(t),$

where $q \in L^1(0,\infty)$, $L^1(0,\infty)$ is space of Lebesgue integrable functions. Then there exists a positive constant K such that the solution $x(t)$ of equation (1) defined by the initial function

$$x(t) = \varphi(t), \;\; x'(t) = \varphi'(t), \;\; t_0 - \tau \leq t \leq t_0,$$

satisfies the estimates

$$|x(t)| \leq K, \;\; |x'(t)| \leq K,$$

for all $t \geq t_0$, where $\varphi \in C^1([t_0 - \tau, \; t_0], \Re)$.

***Proof*:**

Since $|e(.)| \in L^1(0,\infty)$, we can assume that

$$\exp(-2\int_0^\infty |e(s)|\, ds) = K_1, \text{ where } K_1 \in \Re, \; K_1 > 0.$$

Then, it is clear to see that

$$V \geq K_1\{\sum_{j=1}^{n} b_j(t)\int_0^x g_j(s)ds + a_2(t)\int_0^x g(s)ds + \frac{1}{2}y^2 + 1 + \sum_{j=1}^{n}\lambda_j \int_{-\tau_j(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\}.$$

In the light of the assumptions of Theorem 6, we can get

$$\dot{V} \leq p(t)\exp(-2\int_0^t |e(s)|ds)y$$

$$\leq |y|\,|q(t)|$$

$$\leq |q(t)| + K_2|q(t)|V,$$

where $K_2 = 2K_1^{-1}$.

Integrating the last estimate from 0 to t, $(t \geq 0)$, and using the Gronwall inequality, we can conclude that all solutions of equation (1) are bounded.

Remark

If the assumptions of Theorem 6 hold, then by Theorem 3, we conclude that all solutions of equation (1) are uniformly bounded (see Theorem 3).

Example

As a special case of equation (1), we consider the following nonlinear differential equation of second order with two variable delays, $\tau_1(t) = \frac{t}{4}$, $\tau_2(t) = \frac{t}{2}$, $t \geq 0$:

$$x'' + (1+\exp(t))(3 + \frac{t}{2} + x^2 + x'^2)x' + (1+t^2)x' + (2+\exp(-t))(x^3 + x)$$

$$+ (1 + \frac{1}{t^2+1})x\left(\frac{3t}{4}\right) + (\frac{1}{2} + \frac{1}{t^2+1})x\left(\frac{t}{2}\right) = \frac{\sin t}{1+t^2}.$$

We write this equation in system form as

$$x' = y,$$

$$y' = -(1+\exp(t))(3 + \frac{t}{2} + x^2 + y^2)y - (1+t^2)y - (2+\exp(-t))(x^3 + x)$$

$$-(1 + \frac{1}{t^2+1})x + (1 + \frac{1}{t^2+1})\int_{\frac{3t}{4}}^{t} y(s)ds$$

$$-(\frac{1}{2}+\frac{1}{t^2+1})x+(\frac{1}{2}+\frac{1}{t^2+1})\int_{\frac{t}{2}}^{t} y(s)ds+\frac{\sin t}{1+t^2}.$$

When we compare this system with system (2), it can be seen the existence of the following relations:

$(E1)$ $t-\tau_1(t)=\frac{3t}{4}$, $t-\tau_2(t)=\frac{t}{2}$, $t>0$,

$t-\tau_1(t)$ and $t-\tau_2(t)$ are strictly increasing functions,

$\lim_{t\to\infty}(t-\tau_1(t))=\infty$, $\lim_{t\to\infty}(t-\tau_2(t))=\infty$;

$(E2)$ $a_0(t)=1+\exp(t)$, $t>0$, a_0 is a positive and increasing function,

$$f(t,x,y)=3+\frac{t}{2}+x^2+y^2,$$

$$a_0(t)f(t,x,y)=(1+\exp(t))(3+\frac{t}{2}+x^2+y^2)$$

$$\geq 2(3+\frac{t}{2}+x^2+y^2)\geq 6+t;$$

$(E3)$ $a_1(t)=1+t^2$, $t>0$, a_1 is a positive and increasing function,

$$f_1(y)=y,\ a_1(t)F_1(y)=(1+t^2)\frac{f_1(y)}{y}=(1+t^2)\frac{y}{y}=1+t^2,$$

$$a_0(t)f(t,x,y)+a_1(t)F_1(y)\geq t^2+t+7=a(t);$$

$(E4)$ $a_2(t)=2+\exp(-t))$, $t>0$, a_2 is a positive and decreasing function,

$b_1(t)=1+\frac{1}{t^2+1}$, b_1 is a positive and decreasing function, and

$b_2(t)=\frac{1}{2}+\frac{1}{t^2+1}$, b_2 is a positive and decreasing function;

$$\frac{1}{2}\sum_{j=1}^{2}L_j(b_j(t)+1)\tau_j(t)=\frac{1}{2}L_1(b_1(t)+1)\tau_1(t)+\frac{1}{2}L_2(b_2(t)+1)\tau_2(t)$$

$$=(1+\frac{1}{2t^2+2})\frac{t}{4}+(\frac{3}{4}+\frac{1}{2t^2+2})\frac{t}{2}$$

$$=\frac{5t}{8}+\frac{3t}{8t^2+8};$$

$(E5)$ $g(x) = x^3 + x,\ g(0) = 0,\ xg(x) = x^4 + x^2 > 0,\ (x \neq 0),$

$$g'(x) = 3x^2 + 1, |g'(x)| \geq 1 = L,$$

$$g_1(x) = g_2(x) = x,\ g_1(0) = g_2(0) = 0,$$

$$xg_1(x) = x^2 > 0,\ xg_2(x) = x^2 > 0,\ (x \neq 0),$$

$$g_1'(x) = g_2'(x) = 1,\ |g_j'(x)| \leq L_j = 1,\ (j = 1,\ 2);$$

$(E6)$ $a_0(t) f(t, x, y) + a_1(t) F_1(y) \geq a(t) \geq \frac{1}{2} \sum_{j=1}^{n} L_j (b_j(t) + 1) \tau_j(t) \geq 0,$

that is,

$$t^2 + t + 7 \geq a(t) \geq \frac{5t}{8} + \frac{3t}{8t^2 + 8} \text{ for all } t \geq 0,$$

where the choice of the function $a(t)$ can be performed easily;

$(E7)$ $p(t) = \frac{\sin t}{1 + t^2},\ |p(t)| = \left| \frac{\sin t}{1 + t^2} \right| \leq \frac{1}{1 + t^2} = q(t),$

$$\int_0^\infty q(s) ds = \int_0^\infty \frac{1}{1 + s^2} ds = \frac{\pi}{2} < \infty.$$

Finally, in view of the former choice of the functions for the special case of equation (1) and a proper and suitable choice of function $e(t)$, we can also reach the results of Theorems 1-3 by means of the Lyapunov-Krasovskiì functional V. We omit here the details of the mathematical operations. Thus, all the assumptions of Theorem 4 and 5 and Theorem 6 hold, when $p(t) = 0$ and $p(t) \neq 0$, respectively. The above discussion implies that the zero solution of the above equation is stable and uniformly stable when $p(t) \equiv 0$ and all solutions of the same equation are bounded and uniformly bounded when $p(t) \neq 0$.

4. Stability by the fixed point theory

Finally, we prove a new result by the fixed point theory. Let $p(t) \equiv 0$ in equation (1). We can write equation (1) in the system form,

$$\dot{x} = y,$$

$$\dot{y} = -f(t, x, y)y - a_1(t) f_1(y) - a_2(t) g(x) - \sum_{j=1}^{n} b_j(t) g_j(x(t - \tau_j(t))). \quad (5)$$

For each $t_0 \geq 0$, we define $m(t_0) = \inf\{s - \tau_1(s), ..., s - \tau_n(s) : s \geq t_0\}$ and $C(t_0)$ = $C([m(t_0), t_0], R)$ with the continuous function norm $\|.\|$, where

$$\|\psi\| = \sup\{|\psi(s)| : m(t_0) \leq s \leq t_0\}.$$

It will cause no confusion even if we use $\|\phi\|$ as the supremum on $[m(t_0), \infty)$. It can be seen from [2] that for a given continuous function ϕ and a number y_0, there exists a solution of system (5) on an interval $[t_0, T)$, and, if the solution remains bounded, then $T = \infty$.

We introduce some basic assumptions:

(A) Let $t - \tau_j(t)$ is strictly increasing and $\lim_{t \to \infty}(t - \tau_j(t)) = \infty$. The inverse of $t - \tau_j(t)$ exists, denoted by $P_j(t)$, $0 \leq a_2(t) \leq M_0$, and $0 \leq b_j(t) \leq M_j$, $j = 1,2,...,n$. Let $M = \max\{M_1, ..., M_n\}$. Hence, $0 \leq b_j(t) \leq M$.

Now, instead of the Lyapunov-Krasovskiĭ functional approach, we use the fixed point technique under an exponentially weighted metric to discuss the stability of zero solution of equation (1).

Before giving our fourth main result, we introduce some auxiliary results.

Lemma 1.

Let $\psi : [m(t_0), t_0] \to \Re$ be a given continuous function. If $(x(t), y(t))$ is the solution of system (5) on $[t_0, T_1)$ satisfying $x(t) = \psi(t)$, $t \in [m(t_0), t_0]$ and $y(t_0) = x'(t_0)$, then $x(t)$ is the solution of the following integral equation

$$x(t) = \psi(t_0) e^{-\int_{t_0}^{t} K(s)ds} + \int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} B(u)du$$

$$- \int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} D(u) g(x(u))du + \int_{t_0}^{t} E(u,s) g(x(s))ds$$

$$- \int_{t_0}^{t} [\int_{t_0}^{u} E(u,s) g(x(s))ds] e^{-\int_{u}^{t} K(s)ds} K(s)du$$

$$+ \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} \hat{D}_j(u)[x(u) - g_j(x(u))]du$$

$$+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\hat{D}_j(s)g_j(x(s))ds-\sum_{j=1}^{n}e^{-\int_{t_0}^{t}K(s)ds}\int_{t_0-\tau_j(t_0)}^{t_0}\hat{D}_j(s)g_j(x(s))ds$$

$$-\sum_{j=1}^{n}\int_{t_0}^{t}[\int_{u-\tau_j(u)}^{u}\hat{D}_j(s)g_j(x(s))ds]e^{-\int_{u}^{t}K(s)ds}K(u)du$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}E_j(t,s)g_j(x(s-\tau_j(s)))ds$$

$$-\sum_{j=1}^{n}\int_{t_0}^{t}[\int_{t_0}^{u}E_j(u,s)g_j(x(s-\tau_j(s)))ds]e^{-\int_{u}^{t}K(s)ds}K(u)du. \qquad (6)$$

Conversely, if the continuous function $x(t)=\psi(t)$, $t\in[m(t_0),t_0]$ is the solution of equation (1) on $[t_0,T_2]$, then $(x(t),y(t))$ is the solution of system (5) on $[t_0,T_2]$.

***Proof*:**

Let $f(t,x(t),y(t))+a_1(t)F_1(y(t))=A(t)$. Since we assume that $p(t)\equiv 0$, then equation (1) can be written as the following system:

$$\dot{x}=y,$$
$$\dot{y}=-A(t)y-a_2(t)g(x)-\sum_{j=1}^{n}b_j(t)g_j(x(t-\tau_j(t))). \qquad (7)$$

Therefore,

$$\dot{y}+A(t)y+a_2(t)g(x)+\sum_{j=1}^{n}b_j(t)g_j(x(t-\tau_j(t)))=0. \qquad (8)$$

Multiplying both sides of Eq. (8) by $e^{\int_{t_0}^{t}A(s)ds}$ and then integrating from t_0 to t, we obtain

$$y(t)=y(t_0)e^{-\int_{t_0}^{t}A(s)ds}-\int_{t_0}^{t}e^{-\int_{u}^{t}A(s)ds}a_2(u)g(x(u))du$$
$$-\int_{t_0}^{t}e^{-\int_{u}^{t}A(s)ds}\sum_{j=1}^{n}b_j(u)g_j(x(u-\tau_j(u)))du.$$

Hence,

$$\dot{x}(t) = \dot{x}(t_0)e^{-\int_{t_0}^{t} A(s)ds} - \int_{t_0}^{t} e^{-\int_u^t A(s)ds} a_2(u)g(x(u))du$$

$$-\int_{t_0}^{t} e^{-\int_u^t A(s)ds} \sum_{j=1}^{n} b_j(u)g_j(x(u-\tau_j(u)))du. \quad (9)$$

If we choose $\dot{x}(t_0)\exp(-\int_{t_0}^{t} A(s)ds) = B(t),$ then,

$$\dot{x}(t) = B(t) - \int_{t_0}^{t} e^{-\int_u^t A(s)ds} a_2(u)g(x(u))du$$

$$-\sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_u^t A(s)ds} b_j(u)g_j(x(u-\tau_j(u)))du, \quad (10)$$

by (9). Let

$$e^{-\int_u^t A(s)ds} a_2(u) = C(t,u),$$

$$\int_{t_0}^{\infty} C(u+t-t_0,t)du = D(t) \geq 0,$$

$$\int_{t_0+t-s}^{\infty} C(u+s-t_0,s)du = E(t,s) \geq 0,$$

$$\sum_{j=1}^{n} e^{-\int_u^t A(s)ds} b_j(u) = \sum_{j=1}^{n} C_j(t,u),$$

$$\sum_{j=1}^{n} \int_{t_0}^{\infty} C_j(u+t-t_0,t)du = \sum_{j=1}^{n} D_j(t),$$

$$\sum_{j=1}^{n} \frac{D_j(t)}{1-\tau'_j(t)} = \sum_{j=1}^{n} \tilde{D}_j(t),$$

$$\sum_{j=1}^{n} \tilde{D}_j(p_j(t)) = \sum_{j=1}^{n} \hat{D}_j(t)$$

and

$$\sum_{j=1}^{n} \int_{t_0+t-s}^{\infty} C_j(u+s-t_0,s)du = \sum_{j=1}^{n} E_j(t,s) \geq 0.$$

In view of the mentioned estimates, it can be written from (10) that

$$\dot{x}(t) = B(t) - g(x(t))\int_{t_0}^{\infty} C(u+t-t_0,t)du + \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s))ds$$

$$- \sum_{j=1}^{n} g_j(x(t-\tau_j(t)))\int_{t_0}^{\infty} C_j(u+t-t_0,t)du$$

$$+ \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)g_j(x(s-\tau_j(s)))ds$$

$$= B(t) - g(x(t))D(t) + \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s))ds$$

$$- \sum_{j=1}^{n} g_j(x(t-\tau_j(t)))D_j(t) + \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)g_j(x(s-\tau_j(s)))ds$$

$$= B(t) - g(x(t))D(t) + \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s))ds$$

$$- \sum_{j=1}^{n} \tilde{D}_j(p_j(t))g_j(x(t)) + \sum_{j=1}^{n}\frac{d}{dt}\int_{t-\tau_j(t)}^{t} \tilde{D}_j(p_j(s))g_j(x(s))ds$$

$$+ \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)g_j(x(s-\tau_j(s))ds$$

$$= B(t) - g(x(t))D(t) + \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s))ds$$

$$- \sum_{j=1}^{n} \hat{D}_j(t)x(t) + \sum_{j=1}^{n} \hat{D}_j(t)[x(t) - g_j(x(t))] + \sum_{j=1}^{n}\frac{d}{dt}\int_{t-\tau_j(t)}^{t} \hat{D}_j(s)g_j(x(s))ds$$

$$+ \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)g_j(x(s-\tau_j(s)))ds.$$

Let $\sum_{j=1}^{n} \hat{D}_j(t) = K(t)$ and $\sup_{t\geq 0} D(t) \leq \sup_{t\geq 0} \hat{D}_j(t)$. Then,

$$\dot{x}(t) + K(t)x(t) = B(t) - g(x(t))D(t) + \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s))ds$$

$$+ \sum_{j=1}^{n} \hat{D}_j(t)[x(t) - g_j(x(t))] + \sum_{j=1}^{n}\frac{d}{dt}\int_{t-\tau_j(t)}^{t} \hat{D}_j(s)g_j(x(s))ds$$

$$+ \sum_{j=1}^{n}\frac{d}{dt}\int_{t_0}^{t} E_j(t,s)g_j(x(s-\tau_j(s)))ds. \qquad (11)$$

Multiplying both sides of (11) by $\exp(\int_{t_0}^{t} K(s)ds)$ and then integrating from t_0 to t, we get

$$x(t)=\psi(t_0)e^{-\int_{t_0}^{t}K(s)ds}+\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}B(u)du-\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}D(u)g(x(u))du$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}\hat{D}_j(u)[x(u)-g_j(x(u))]du$$

$$+\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}[\frac{d}{du}\int_{t_0}^{u}E(u,s)g(x(s))ds]du$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}[\frac{d}{du}\int_{u-\tau_j(u)}^{u}\hat{D}_j(s)g_j(x(s))ds]du$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}[\frac{d}{du}\int_{t_0}^{u}E_j(u,s)g_j(x(s-\tau_j(s)))ds]du.$$

Applying the integration by parts formula for the last three terms, we have

$$x(t)=\psi(t_0)e^{-\int_{t_0}^{t}K(s)ds}+\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}B(u)du-\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}D(u)g(x(u))du$$

$$+\int_{t_0}^{t}E(u,s)g(x(s))ds-\int_{t_0}^{t}[\int_{t_0}^{u}E(u,s)g(x(s))ds]e^{-\int_{u}^{t}K(s)ds}K(s)du$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}\hat{D}_j(u)[x(u)-g_j(x(u))]du$$

$$+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\hat{D}_j(s)g_j(x(s))ds-\sum_{j=1}^{n}e^{-\int_{t_0}^{t}K(s)ds}\int_{t_0-\tau_j(t_0)}^{t_0}\hat{D}_j(s)g_j(x(s))ds$$

$$-\sum_{j=1}^{n}\int_{t_0}^{t}[\int_{u-\tau_j(u)}^{u}\hat{D}_j(s)g_j(x(s))ds]e^{-\int_{u}^{t}K(s)ds}K(u)du$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}E_j(t,s)g_j(x(s-\tau_j(s)))ds$$

$$-\sum_{j=1}^{n}\int_{t_0}^{t}[\int_{t_0}^{u}E_j(u,s)g_j(x(s-\tau_j(s)))ds]e^{-\int_u^t K(s)ds}K(u)du.$$

Conversely, we assume that a continuous function $x(t)=\psi(t)$ for $t\in[m(t_0),t_0]$ and satisfies the integral equation on $t\in[t_0,T_2]$. Then, it is differentiable on $[t_0,T_2]$. Hence, it is only needed to differentiate the integral equation. When we differentiate the integral equation, we can conclude the desired result.

Let $(C,\|\,.\,\|)$ be the Banach space of bounded continuous functions on $[m(t_0),\infty)$ with the supremum norm $\|\phi\|=\sup\{|\phi(t)|:t\in[m(t_0),\infty)\}$ for $\phi\in C$. Let ρ denote the supremum metric and $\rho(\phi_1,\phi_2)=\|\phi_1-\phi_2\|$, where $\phi_1,\phi_2\in C$. Next, let $\psi:[m(t_0),t_0]\to\Re$ be a given continuous initial function.

Define the set $S\subset C$ by:

$$S=\{\phi:[m(t_0),\infty)\to\Re|\phi\in C,\phi(t)=\psi(t),t\in[m(t_0),t_0]\}$$

and its subset

$$S'=\{\phi:[m(t_0),\infty)\to\Re|\phi\in C,\phi(t)=\psi(t),t\in[m(t_0),t_0]\text{ and }|\phi(t)|\le l,t\ge m(t_0)\},$$

where $\psi:[m(t_0),t_0]\to[-l,l]$ is a given initial function, l is a positive constant. Define the mapping $P:S'\to S'$ by

$$(P\phi)(t)=\psi(t),\quad\text{if } t\in[m(t_0),t_0],$$

and if $t>t_0$, then

$$(P\phi)(t)=\psi(t_0)e^{-\int_{t_0}^{t}K(s)ds}+\int_{t_0}^{t}e^{-\int_u^t K(s)ds}B(u)du$$

$$-\int_{t_0}^{t}e^{-\int_u^t K(s)ds}D(u)g(\phi(u))du+\int_{t_0}^{t}E(u,s)g(\phi(s))ds$$

$$-\int_{t_0}^{t}[\int_{t_0}^{u}E(u,s)g(\phi(s))ds]e^{-\int_u^t K(s)ds}K(s)du$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}e^{-\int_u^t K(s)ds}\hat{D}_j(u)[\phi(u)-g_j(\phi(u))]du$$

$$+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\hat{D}_j(s)g_j(\phi(s))ds-\sum_{j=1}^{n}e^{-\int_{t_0}^{t}K(s)ds}\int_{t_0-\tau_j(t_0)}^{t_0}\hat{D}_j(s)g_j(\psi(s))ds$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}E_j(t,s)g_j(\phi(s))ds-\sum_{j=1}^{n}\int_{t_0}^{t}[\int_{u-\tau_j(u)}^{u}\hat{D}_j(s)g_j(\phi(s))ds]e^{-\int_{u}^{t}K(s)ds}K(u)du$$

$$-\sum_{j=1}^{n}\int_{t_0}^{t}[\int_{t_0}^{u}E_j(u,s)g_j(\phi(s-\tau_j(s)))ds]e^{-\int_{u}^{t}K(s)ds}K(u)du.$$

Since $g(x)$ and $g_j(x)$ satisfy the Lipschitz condition, let $L_0, L_1,...,L_n$, $L_0 \le L_j$, denote the common Lipschitz constants for $g(x)$, $g_j(x)$ and $x-g_j(x)$.

It is also clear that

$$\int_{t_0}^{t}e^{-\int_{u}^{t}K(s)ds}K(u)du=e^{-\int_{u}^{t}K(s)ds}\Bigg|_{t_0}^{t}=1-e^{-\int_{t_0}^{t}K(s)ds}\approx 1 \text{ for large } t.$$

But, since $g(x)$ and $g_j(x)$ are non-linear, then L_0 and L_j may not be small enough. Hence, P may not be a contracting mapping. We can solve this problem by giving an exponentially weight metric via the next lemma.

Lemma 2.

We suppose that there exist a constant $l>0$ such that $g(x)$ and $g_j(x)$ satisfy the Lipschitz condition on $[-l,l]$. Then there exists a metric on S' such that

$(F1)$ the metric space (S',d) is complete,

$(F2)$ P is a contraction mapping on (S',d) if P maps S' into itself.

***Proof*:**

$(F1)$ We change the supremum norm to an exponentially weighted norm $|\phi|_h$, which is defined on S'. Let X be the space of all continuous functions $\phi:[m(t_0),\infty)\to\Re$ such that

$$|\phi|_h=\sup_n\{|\phi(t)|e^{-h(t)}:t\in[m(t_0),\infty)\}<\infty,$$

where $h(t) = kL_0 \int_{t_0}^{t} [\hat{D}(s) + D(s)]ds + k \sum_{j=1}^{n} L_j \int_{t_0}^{t} [\hat{D}_j(s) + D_j(s)]ds$, k is a constant, L_0 and L_j are the common Lipschitz constants for $g(x)$, $x - g_j(x)$ and $g_j(x)$. Then $(X, |\,.\,|_h)$ is a Banach space. Thus (X, d) is a complete metric space with $d(\phi, \varphi) = |\phi - \varphi|_h$, where $\phi, \varphi \in S$. Under this metric, the space S' is a closed subset of X. Thus, the metric space (S', d) is complete.

$(F2)$ Let $P: S' \to S'$. It is clear that $D(t) \geq 0$, $E(t,s) \geq 0$, $\sum_{j=1}^{n} \hat{D}_j(t) \geq 0$ and $\sum_{j=1}^{n} E_j(t,s) \geq 0$.

Then, for $\phi, \varphi \in S'$, we can get

$$
\begin{aligned}
|(P\phi)(t) - (P\varphi)(t)|e^{-h(t)} \leq {} & \int_{t_0}^{t} e^{-\int_u^t K(s)ds} D(u)|g(\phi(u)) - g(\varphi(u))|e^{-h(t)} du \\
& + \int_{t_0}^{t} E(u,s)|g(\phi(u)) - g(\varphi(u))|e^{-h(t)} ds \\
& + \int_{t_0}^{t} [\int_{t_0}^{u} E(u,s)|g(\phi(s)) - g(\varphi(s))|e^{-h(t)} ds] e^{-\int_u^t K(s)ds} K(s) du \\
& + \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \hat{D}_j(u) |[\phi(u) - g_j(\phi(u))] - [\varphi(u) - g_j(\varphi(u))]| e^{-h(t)} du \\
& + \sum_{j=1}^{n} \int_{t_0}^{t} E_j(t,s) |g_j(\phi(s)) - g_j(\varphi(s))| e^{-h(t)} ds \\
& + \sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} \hat{D}_j(s) |g_j(\phi(s)) - g_j(\varphi(s))| e^{-h(t)} ds \\
& + \sum_{j=1}^{n} \int_{t_0}^{t} [\int_{u-\tau_j(u)}^{u} \hat{D}_j(s) ds |g_j(\phi(s)) - g_j(\varphi(s))| e^{-h(t)} ds] e^{-\int_u^t K(s)ds} K(u) du \\
+ \sum_{j=1}^{n} \int_{t_0}^{t} [\int_{t_0}^{u} E_j(u,s) & |g_j(\phi(s - \tau_j(s))) - g_j(\varphi(s - \tau_j(s)))| e^{-h(t)} ds] e^{-\int_u^t K(s)ds} K(u) du.
\end{aligned}
$$

For $u \leq t$, since $D(t) \geq 0$ and $D_j(t) \geq 0$, we have

$$
h(u) - h(t) = -kL_0 \int_u^t [\hat{D}(s) + D(s)]ds - k \sum_{j=1}^{n} L_j \int_u^t [\hat{D}_j(s) + D_j(s)]ds
$$

$$\leq -kL_0 \int_u^t D(s)ds,$$

$$h(u) - h(t) = -kL_0 \int_u^t \hat{D}(s)ds,$$

$$h(u) - h(t) \leq -k \sum_{j=1}^{n} L_j \int_u^t D_j(s)ds$$

and

$$h(u) - h(t) = -kL_0 \int_u^t [\hat{D}(s) + D(s)]ds - k \sum_{j=1}^{n} L_j \int_u^t [\hat{D}_j(s) + D_j(s)]ds$$

$$\leq \sum_{j=1}^{n} (-k) L_j \int_u^t \hat{D}_j(s)ds.$$

Further for $s \leq t$, it can be seen that

$$h(s - \tau_j(s)) - h(t) \leq \sum_{j=1}^{n} (-k) L_j \int_s^t D_j(u)du.$$

Since $E_j(t, s) \geq 0$, then we have

$$\sum_{j=1}^{n} E_j(t, s) = \sum_{j=1}^{n} \int_{t_0+t-s}^{\infty} C_j(u + s - t_0, s)du$$

$$\leq \sum_{j=1}^{n} \int_{t_0}^{\infty} C_j(u + s - t_0, s)du = \sum_{j=1}^{n} D_j(s).$$

Hence,

$$|(P\phi)(t) - (P\varphi)(t)| e^{-h(t)} \leq |\phi - \varphi|_h \times \{L_0 \int_{t_0}^{t} e^{-kL_0 \int_u^t D(s)ds} D(u)du$$

$$+ L_0 \int_{t_0}^{t} E(t, s) e^{h(s)-h(t)} ds$$

$$+ L_0 \int_{t_0}^{t} [\int_{t_0}^{u} E(u, s) e^{h(s)-h(t)} ds] e^{-\int_u^t K(s)ds} K(u)du]$$

$$+ \sum_{j=1}^{n} L_j \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \hat{D}_j(u) e^{h(u)-h(t)} du$$

$$+\sum_{j=1}^{n} L_j \int_{t_0}^{t} E_j(t,s) e^{h(s)-h(t)} ds$$

$$+\sum_{j=1}^{n} L_j \int_{t-\tau_j(t)}^{t} \hat{D}_j(s) e^{h(s)-h(t)} ds$$

$$+\sum_{j=1}^{n} L_j \int_{t_0}^{t} [\int_{u-\tau_j(t)}^{u} \hat{D}_j(s) e^{h(s)-h(t)} ds] e^{-\int_u^t K(s)ds} K(u) du$$

$$+\sum_{j=1}^{n} L_j \int_{t_0}^{t} [\int_{t_0}^{u} E_j(u,s) e^{h(s-\tau_j(s))-h(t)} ds] e^{-\int_u^t K(s)ds} K(u) du\}.$$

Therefore, in view of the above discussion, it follows that

$$L_0 \int_{t_0}^{t} E(t,s) e^{h(s)-h(t)} ds \le L_0 \int_{t_0}^{t} D(s) e^{-kL_0 \int_u^t D(s)ds} ds \le \frac{1}{k},$$

$$L_0 \int_{t_0}^{t} e^{-\int_u^t K(s)ds} D(u) e^{h(s)-h(t)} du \le \frac{1}{k},$$

$$L_0 \int_{t_0}^{t} [\int_{t_0}^{u} E(u,s) e^{h(s)-h(t)} ds] e^{-\int_u^t K(s)ds} K(u) du] \le \frac{1}{k},$$

$$\sum_{j=1}^{n} L_j \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \hat{D}_j(u) e^{h(u)-h(t)} du = \sum_{j=1}^{n} L_j \int_{t_0}^{t} e^{-\sum_{j=1}^{n}\int_u^t \hat{D}_j(s)ds} \hat{D}_j(u) e^{h(u)-h(t)} du$$

$$\le \sum_{j=1}^{n} L_j \int_{t_0}^{t} \frac{e^{-\sum_{j=1}^{n}\int_u^t \hat{D}_j(s)ds} \hat{D}_j(u)}{e^{\sum_{j=1}^{n} kL_j \int_u^t \hat{D}_j(s)ds}} du$$

$$= \sum_{j=1}^{n} L_j \int_{t_0}^{t} e^{-\sum_{j=1}^{n}(kL_j+1)\int_u^t \hat{D}_j(s)ds} \hat{D}_j(u) du$$

$$\le \sum_{j=1}^{n} L_j \frac{1}{\sum_{j=1}^{n}(kL_j+1)} e^{-\sum_{j=1}^{n}(kL_j+1)\int_u^t \hat{D}_j(s)ds} \Big|_{t_0}^{t}$$

$$\leq \sum_{j=1}^{n} L_j \frac{1}{\sum_{j=1}^{n} kL_j} \leq \frac{1}{k},$$

$$\sum_{j=1}^{n} L_j \int_{t_0}^{t} E_j(t,s) e^{h(s)-h(t)} ds \leq \sum_{j=1}^{n} L_j \int_{t_0}^{t} D_j(s) e^{\sum_{j=1}^{n}(-k)L_j \int_s^t D_j(s)ds} ds$$

$$\leq \sum_{j=1}^{n} L_j \frac{1}{\sum_{j=1}^{n} kL_j} e^{-\sum_{j=1}^{n} kL_j \int_u^t D_j(s)ds} \Bigg|_{t_0}^{t} \leq \frac{1}{k}.$$

Similarly, it can be easily obtained that

$$\sum_{j=1}^{n} L_j \int_{t-\tau_j(t)}^{t} \hat{D}_j(s) e^{h(s)-h(t)} ds \leq \frac{1}{k},$$

$$\sum_{j=1}^{n} L_j \int_{t_0}^{t} [\int_{u-\tau_j(t)}^{u} \hat{D}_j(s) e^{h(s)-h(t)} ds] e^{-\int_u^t K(s)ds} K(u)du \leq \frac{1}{k}$$

and

$$\sum_{j=1}^{n} L_j \int_{t_0}^{t} [\int_{t_0}^{u} E_j(u,s) e^{h(s-\tau_j(s))-h(t)} ds] e^{-\int_u^t K(s)ds} K(u)du \leq \frac{1}{k}.$$

Thus, we have

$$|(P\phi)(t) - (P\varphi)(t)| e^{-h(t)} \leq \frac{8}{k} |\phi - \varphi|_h, \quad t > t_0.$$

For $t \in [m(t_0), t_0]$, $(P\phi)(t) = (P\varphi)(t) = \theta(t)$. Thus,

$$d(P\phi, P\varphi) \leq \frac{8}{k} d(\phi - \varphi), \ (k > 8).$$

Therefore, P is contraction mapping on (S', d).

The fourth and last main result of this paper is the following theorem.

Theorem 7.

We suppose that the assumption (A) holds. Moreover, we assume the following:

($G1$) There exists a positive constant l such that g and g_j satisfy the Lipschitz condition on $[-l,l]$ and g and g_j are odd and they are strictly increasing on $[-l,l]$, and $x-g(x)$ and $x-g_j(x)$ are non-decreasing on $[-l,l]$;

$(G2)$ There exist an $\alpha_j, \beta \in (0,1)$, and a continuous function $a(t):[0,\infty)\to[0,\infty)$ such that

$$f(t,x(t),y(t))+a_1(t)F_1(y(t)) \ge a(t) \text{ for } t\ge 0,\ x\in\Re,\ y\in\Re,$$

$$\sup_{t\ge 0}\int_{t_0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{u+s}a(v)dv} b_j(s)duds \le \beta$$

and

$$2\sup_{t\ge 0}\int_{t}^{P_j(t)}\int_{0}^{\infty} e^{-\int_s^{w+s}a(v)dv} b_j(s)dwds + 2\sup_{t\ge 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{w+s}a(v)dv} b_j(s)dwds \le \alpha_j;$$

$(G3)$ There exist constants $a_0>0$ and $Q>0$ such that for each $t\ge 0$, if $J\ge Q$, then

$$\int_t^{t+J} a(v)dv \ge a_0 J.$$

Then, there exists $\delta\in(0,l)$ such that for each initial function $\psi:[m(t_0),t_0]\to\Re$ and $\dot{x}(t_0)$ satisfying $|\dot{x}(t_0)|+\|\psi\|\le\delta$, there is a unique continuous function $x:[m(t_0),\infty)\to\Re$ satisfying $x(t)=\psi(t)$, which is a solution of equation (1) on $[t_0,\infty)$. Moreover, the zero solution of equation (1) is stable.

***Proof*:**

Choosing $\psi:[m(t_0),t_0]\to\Re$ and $\dot{x}(t_0)$ such that

$$(Q+\frac{e^{-a_0.Q}}{a_0})|\dot{x}(t_0)|+\delta+\sum_{j=1}^{n} g_j(\delta)\int_{t_0-\tau_j(t_0)}^{t_0}\hat{D}_j(s)ds$$

$$\le[1-(\alpha_1+\alpha_2+\ldots+\alpha_n)]\sum_{j=1}^{n} g_j(l).$$

In view of the assumptions of Theorem 7, $g(0)=0$ and $g_j(0)=0$, it follows that $g(l)\le l$. and $g_j(l)\le l$. Since $g(x)$ and $g_j(x)$ satisfy Lipschitz condition on $[-l,l]$, $g(x)$ and $g_j(x)$

are continuous function on $[-l, l]$. Then, there exists a constant δ such that $\delta < l$. Thus, we can get

$$|(P\phi)(t)| \le \delta + \int_{t_0}^{t} e^{-\int_u^t K(s)ds} |\dot{x}(t_0)| e^{-\int_{t_0}^{u} A(s)ds} du$$

$$+ \int_{t_0}^{t} e^{-\int_u^t K(s)ds} D(u)g(l)du + \int_{t_0}^{t} E(u,s)g(l)ds$$

$$+ \int_{t_0}^{t} [\int_{t_0}^{u} E(u,s)g(l)ds] e^{-\int_u^t K(s)ds} K(s)du$$

$$+ \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_u^t K(s)ds} \hat{D}_j(u)(l - g_j(l))du$$

$$+ \sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} \hat{D}_j(s) g_j(l)ds + \sum_{j=1}^{n} \int_{t_0}^{t} [\int_{u-\tau_j(u)}^{u} \hat{D}_j(s) g_j(l)ds] e^{-\int_u^t K(s)ds} K(u)du$$

$$+ \sum_{j=1}^{n} \int_{t_0-\tau_j(t_0)}^{t_0} \hat{D}_j(s) g_j(\delta)ds + \sum_{j=1}^{n} \int_{t_0}^{t} E_j(t,s) g_j(l)ds$$

$$+ \sum_{j=1}^{n} \int_{t_0}^{t} [\int_{t_0}^{u} E_j(u,s) g_j(l)ds] e^{-\int_u^t K(s)ds} K(u)du.$$

In view of the assumptions, it also follows that

$$\int_{t_0}^{t} e^{-\int_u^t K(s)ds} D(u)g(l)du \le g(l),$$

$$\int_{t_0}^{t} E(t,s)g(l)ds = g(l) \int_{t_0}^{t} \int_{t_0+t-s}^{\infty} C(u+s-t_0, s)duds$$

$$= g(l) \int_{t_0}^{t} \int_{t_0+t-s}^{\infty} e^{-\int_s^{u+s-t_0} A(v)dv} b(s)duds = g(l) \int_{t_0}^{t} \int_{t-s}^{\infty} e^{-\int_s^{u+s} A(v)dv} b(s)duds.$$

$$\le g(l) \sup_{t \ge 0} \int_{t_0}^{t} \int_{t-s}^{\infty} e^{-\int_s^{u+s} a(v)dv} b(s)duds.$$

Similarly, we have

$$\int_{t_0}^{t}\Big[\int_{t_0}^{u}E(u,s)g(l)ds\Big]e^{-\int_u^t K(s)ds}K(s)du \le g(l)\sup_{t\ge 0}\int_{t_0}^{t}\int_{t-s}^{\infty}e^{-\int_s^{u+s}a(v)dv}b(s)duds,$$

$$\int_{t_0}^{t}\Big[\int_{t_0}^{u}E(u,s)g(l)ds\Big]e^{-\int_u^t K(s)ds}K(s)du \le g(l)\int_{t_0}^{t}\Big[\int_{t_0}^{u}E(u,s)ds\Big]du$$

$$\sum_{j=1}^{n}\int_{t_0}^{t}E_j(t,s)ds = \sum_{j=1}^{n}\int_{t_0}^{t}\int_{t_0+t-s}^{\infty}C_j(u+s-t_0,s)duds$$

$$= \sum_{j=1}^{n}\int_{t_0}^{t}\int_{t_0+t-s}^{\infty}e^{-\int_s^{u+s-t_0}A(v)dv}b_j(s)duds = \sum_{j=1}^{n}\int_{t_0}^{t}\int_{t-s}^{\infty}e^{-\int_s^{u+s}A(v)dv}b_j(s)duds$$

$$\le \sup_{t\ge 0}\int_0^t\int_{t-s}^{\infty}e^{-\int_s^{u+s}a(v)dv}b_1(s)duds + \ldots + \sup_{t\ge 0}\int_0^t\int_{t-s}^{\infty}e^{-\int_s^{u+s}a(v)dv}b_n(s)duds,$$

$$\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\hat{D}_j(s)ds = \sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\tilde{D}_j(P_j(s))ds$$

$$= \sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\frac{D_j(P_j(s))}{1-\tau_j'(s)}ds = \sum_{j=1}^{n}\int_{t}^{P_j(t)}D_j(s)ds$$

$$= \sum_{j=1}^{n}\int_{t}^{P_j(t)}\int_{0}^{\infty}e^{-\int_s^{w+s}a(v)dv}b_j(s)dwds$$

$$\le \sup_{t\ge 0}\int_{t}^{P_1(t)}\int_{0}^{\infty}e^{-\int_s^{w+s}a(v)dv}b_1(s)dwds + \ldots + \sup_{t\ge 0}\int_{t}^{P_n(t)}\int_{0}^{\infty}e^{-\int_s^{w+s}a(v)dv}b_n(s)dwds.$$

From assumption $(G2)$, we have

$$\sum_{j=1}^{n}\int_{t_0}^{t}E_j(t,s)g_j(l)ds + \sum_{j=1}^{n}\int_{t_0}^{t}\Big[\int_{u-\tau_j(u)}^{u}\hat{D}_j(s)g_j(l)ds\Big]e^{-\int_u^t K(s)ds}K(u)du$$

$$+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\hat{D}_j(s)g_j(l)ds$$

$$+\sum_{j=1}^{n}\int_{t_0}^{t}\Big[\int_{t_0}^{u}E_j(u,s)g_j(l)ds\Big]e^{-\int_u^t K(s)ds}K(u)du$$

$$\leq \sum_{j=1}^{n} g_j(l)\{2\sup_{t\geq 0}\int_0^t \int_{t-s}^{\infty} e^{-\int_s^{u+s} a(v)dv} b_1(s)duds + ... + 2\sup_{t\geq 0}\int_0^t \int_{t-s}^{\infty} e^{-\int_s^{u+s} a(v)dv} b_n(s)duds$$

$$+ 2\sup_{t\geq 0}\int_t^{P_1(t)} \int_0^{\infty} e^{-\int_s^{w+s} a(v)dv} b_1(s)dwds + ... + 2\sup_{t\geq 0}\int_t^{P_n(t)} \int_0^{\infty} e^{-\int_s^{w+s} a(v)dv} b_n(s)dwds\}$$

$$\leq (\alpha_1 + \alpha_2 + ... + \alpha_n)\sum_{j=1}^{n} g_j(l).$$

Hence,

$$|(P\phi)(t)| \leq \delta + g(l) + \sum_{j=1}^{n} g_j(\delta) \int_{t_0-\tau_j(t_0)}^{t_0} \hat{D}_j(s)ds + \sum_{j=1}^{n}(l - g_j(l))$$

$$+ 2\beta g(l) + (\alpha_1 + \alpha_2 + ... + \alpha_n)\sum_{j=1}^{n} g_j(l)$$

$$+ \int_{t_0}^{t} e^{-\int_u^t K(s)ds} |\dot{x}(t_0)| e^{-\int_{t_0}^{u} A(s)ds} du$$

$$\leq \delta + g(l) + \sum_{j=1}^{n} g_j(\delta) \int_{t_0-\tau_j(t_0)}^{t_0} \hat{D}_j(s)ds + \sum_{j=1}^{n}(l - g_j(l))$$

$$+ 2\beta g(l) + (\alpha_1 + \alpha_2 + ... + \alpha_n)\sum_{j=1}^{n} g_j(l) + \int_{t_0}^{t} |\dot{x}(t_0)| e^{-\int_{t_0}^{u} A(s)ds} du.$$

Using condition $(G3)$ of the theorem, we get

$$\int_{t_0}^{t} e^{-\int_{t_0}^{u} A(s)ds} du = \int_{t_0}^{t_0+Q} e^{-\int_{t_0}^{u} A(s)ds} du + \int_{t_0+Q}^{t} e^{-\int_{t_0}^{u} A(s)ds} du \leq Q + \frac{e^{-a_0.Q}}{a_0}.$$

Thus,

$$|(P\phi)(t)| \leq \delta + g(l) + \sum_{j=1}^{n} g_j(\delta) \int_{t_0-\tau_j(t_0)}^{t_0} \tilde{D}_j(s)ds + \sum_{j=1}^{n}(l - g_j(l))$$

$$+ 2\beta g(l) + (\alpha_1 + \alpha_2 + ... + \alpha_n)\sum_{j=1}^{n} g_j(l) + |\dot{x}(t_0)|\,(Q + \frac{e^{-a_0.Q}}{a_0})$$

and so

$$|(P\phi)(t)| \leq (1 + 2\beta)l + \sum_{j=1}^{n} l.$$

It is obvious that if $t \in [m(t_0), t_0]$, then $(P\phi)(t) = \psi(t)$. Moreover, for $t \in [m(t_0), \infty)$, we get $|(P\phi)(t)| \le (1+2\beta)l + \sum_{j=1}^{n} l$. Therefore, $P\phi : S' \to S'$. Since P is a contraction mapping, then P has unique fixed point $x(t)$ such that $|x(t)| \le (1+2\beta)l + \sum_{j=1}^{n} l$.

From equation (6), we have

$$|y(t)| \le |\dot{x}(t_0)| + \int_{t_0}^{t} e^{-\int_u^t A(s)ds} a_2(u)|g(x(u))|du$$

$$+ \sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_u^t A(s)ds} b_j(u)|g_j(x(u-\tau_j(u)))|du.$$

Since for $t \in [0,\infty)$, $0 \le a_2(t) \le M_0$, $0 \le b_j(t) \le M_j$, then

$$|y(t)| \le |\dot{x}(t_0)| + M_0 \int_{t_0}^{t} e^{-\int_u^t A(s)ds} |x(u)|du + \sum_{j=1}^{n} M_j \int_{t_0}^{t} e^{-\int_u^t A(s)ds} |x(u-\tau_j(u))|du$$

$$\le lM_0 \int_{t_0}^{t} e^{-\int_u^t A(s)ds} du + \sum_{j=1}^{n} l(1 + M_j \int_{t_0}^{t} e^{-\int_u^t A(s)ds} du)$$

$$< M_0 l(Q + \frac{e^{-a_0 Q}}{a_0}) + \sum_{j=1}^{n} l[1 + M_j(Q + \frac{e^{-a_0 Q}}{a_0})].$$

Hence,

$$|x(t)| + |y(t)| < (1+2\beta)l + M_0 l(Q + \frac{e^{-a_0 Q}}{a_0}) + \sum_{j=1}^{n} l[2 + M_j(Q + \frac{e^{-a_0 Q}}{a_0})].$$

If we replace ε by l, then we can conclude that the zero solution of equation (1) is stable.

5. Discussion

A Liénard type equation with multiple variable delays, equation (1), is considered. First, the stability/uniformly stability when $p(t) \equiv 0$ and the boundedness/uniformly boundedness of solutions of this equation, equation (1), when $p(t) \neq 0$, are discussed by the Lyapunov-Krasovskiĭ functional approach. Later, the stability of the solutions of the same equation, when $p(t) \equiv 0$ in equation (1), is investigated by the fixed point technique under an exponentially weighted metric. The claim made by the author is illustrated as the following:

1^0) The obtained results, Theorem 4 and Theorem 7, extend and improve that of Burton (2005), Burton (2006), Burton and Furumochi (2001), Pi (2011) and Tunç and Biçer (2014), and in addition we give additional three new results, Theorem 5, Theorem 6 and Remark to that of Burton (2005), Burton (2006), Burton and Furumochi (2001), Pi (2011) and Tunç and Biçer (2014) by using the Lyapunov-Krasovskiĭ functional approach.

2^o) It is clear that our equation, equation (1), includes the equations investigated by Burton (2005), Burton (2006), Burton and Furumochi (2001), Pi (2011) and Tunç and Biçer (2014). This case is an extension and contribution to the works of Burton (2005), Burton (2006), Burton and Furumochi (2001), Tunç (2010) and Tunç and Biçer (2014).

3^o) It follows that that the assumptions of Theorem 4 and Theorem 7 are completely different from each other except the similarity of the assumption $f(t,x(t),y(t))+a_1(t)F_1(y(t))\geq a(t)$ of Theorem 7 and the assumption $a_0(t)f(t,x,y)+a_1(t)F_1(y)\geq a(t)\geq \frac{1}{2}\sum_{j=1}^{n}L_j(b_j(t)+1)\tau_j(t)\geq 0$ of Theorem 4.

4^o) On the other hand, the assumptions of Theorem 4 are very clear, elegant and comprehensible. That is, the assumptions of Theorem 4 have very simple forms and the applicability and correctness of them can be easily checked and verified. In spite of this fact, to the best of our knowledge, it may be difficult to say the same for the assumptions of Theorem 7. That is, to show the applicability Theorem 7 may be more difficulty. This shows the advantage of the Lyapunov-Krasovskiĭ functional approach over the fixed point technique.

5^o) We assume the existence and continuity of the derivatives $a_2'(t), b_j'(t), \tau_j'(t)$ and $g_j'(x)$ when applying the Lyapunov-Krasovskiĭ functional. However, it is assumed g and g_j are odd and they are strictly increasing on $[-l,l]$, and $x-g(x)$ and $x-g_j(x)$ are non-decreasing on $[-l,l]$ when applying the fixed point technique. It is not needed the differentiability of the mentioned functions when we use the fixed point technique. This is the advantage of the fixed point technique over the Lyapunov-Krasovskiĭ functional approach. Finally, we do not need the restriction of g and g_j are odd and they are strictly increasing on $[-l,l]$, and $x-g(x)$ and $x-g_j(x)$ are non-decreasing on $[-l,l]$ when applying the Lyapunov-Krasovskiĭ functional approach. This case shows that there is no more restriction on the functions g and g_j when applying the Lyapunov-Krasovskiĭ functional approach.

6^o) When we change equation (1) into a more complex form, finding an appropriate Lyapunov-Krasovskiĭ functional, which gives meaningful results, may be very difficult. It should be noted construction or definition of Lyapunov-Krasovskiĭ functionals remain as an open problem in the literature by now. This fact shows that the advantage of the fixed point theory over the Lyapunov's direct method for the special cases. Further, in spite of more effectiveness of the Lyapunov-Krasovskiĭ functionals for ordinary and functional differential equations of higher order, the application of the fixed point theory for those type equations is very difficult because of multiple integrals to be arisen in proofs. By this fact, we mean that the observations of Burton (2005), Burton (2006) and Burton and Furumochi (2001) may not be true in general cases. Depending on the form and order of given functional differential equations, sometimes, the Lyapunov-Krasovskiĭ functional approach has an advantage over the fixed point theory, and sometimes it is in the contrast. However, so far, the most effective method to investigate the qualitative behaviors of non-linear ordinary and functional differential equations of higher order is still the Lyapunov's direct method. At the end, the Lyapunov's direct method is old but it is still more an active method in the scientific literature.

6. Conclusion

A Liénard type differential equation with multiple variable time-lags is considered. The stability of zero solutions of the differential equation considered is investigated by the Lyapunov-Krasovskiĭ functional approach and the fixed point technique under an exponentially weighted metric, respectively. It is done a comparison between the applications of both methods with the established conditions on the same stability problems. A comment is made on the effectiveness of the methods applied. In addition, three new results for uniformly stability and boundedness/ uniformly boundedness of the solutions to the equation considered are obtained by the Lyapunov-Krasovskiĭ functional approach. An example is also given to verify the results obtained by the Lyapunov-Krasovskiĭ functional approach. The results established complement and improve some recent results found in the literature.

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