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Optimal Filtering of an Advertising Production System with Deteriorating Items

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Abstract

In this paper, we consider an integrated stochastic advertising-production system in the case of a duopoly. Two firms spend certain amounts to advertise some product. The expenses processes evolve according to the jumps of two homogeneous, finite-state Markov chains. We assume that the items in stock may be subject to deterioration and the deterioration parameter is assumed to be random.

Keywords: Partially Observed Inventory Systems, Optimal Filtering, Reference Probability Measure, EM algorithm

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1. Introduction

Managers, policy makers, and researchers are interested in understanding the effect of advertising on demand. A large number of response models have been proposed in the literature linking advertising expenditures to sales or market shares. The first of these models seems to be the advertising model of Vidale and Wolfe (1957)

$$\dot{x} = \rho u(1-x) - \delta x, \qquad x(0) = x_0,$$

where x represents the sales rate, u the advertising rate, and ρ and δ are scaling factors. A variety of other models have been built, typically by altering some aspect of the above dynamic equation. For example:

- The advertising model of Ozga (1960)

  $$\dot{x} = ux(1-x) - \delta x.$$

- The advertising model of Nerlove and Arrow (1962)

  $$\dot{x} = u - \delta x.$$

- The advertising model of Bradshaw and Porter (1975)

  $$\dot{x} = \beta(u - x).$$

- The advertising model of Sethi (1983)

  $$\dot{x} = \rho u\sqrt{1-x} - \delta x.$$

- The advertising model of Mahajan and Muller (1986)

  $$\dot{x} = f(u)(1-x) - \delta x.$$

- The advertising model of Feinberg (2001)

  $$\dot{x} = f(u)a(x) - b(x).$$

  where $f(u)$ is S-shaped $a(x)$ and $b(x)$ are "acceleration" and "decay" functions of sales.

Discrete-time analogs of these models have also been considered in the literature, see for example Park and Hahn (1991) and Hahn and Hyun (1991). For a full review of the literature, see Feichtinger et al. (1994). Numerous stochastic models have been proposed too. To review some of them, we start with the stochastic, monopoly advertising model of Sethi (1983) given by the Itô equation

$$dx(t) = \left(\rho u(t)\sqrt{1-x(t)} - \delta x(t)\right) dt + \sigma(x) dw(t),$$

where $w(t)$ represents a standard Wiener process. In the stochastic model of Zhang et al. (2001), the demand rate is a two-state Markov chain. Hitsch (2006) proposes the model

$$x_{t+1} = x_t^{\alpha} e^{v_{t+1}},$$

where v is i.i.d. normal. The model of Gozzi and Marinelli (2006) is given by

$$dx(s) = \left[a_0 x(s) + \int_{-r}^{0} a_1(\xi)x(s+\xi)d\xi + b_0 z(s) + \int_{-r}^{0} b_1(\xi)z(s+\xi)d\xi + b_0 z(s)\right] ds$$
$$+\sigma dW_0(s), \quad 0 \leq s \leq T,$$

$$x(0) = \eta^0, \; x(\xi) = \eta(\xi), z(\xi) = \delta(\xi) \; \forall \xi \in [-r, 0],$$

where W_0 is a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_s)_{s \geq 0}, P)$, with $\mathbb{F}$ being the completion of the filtration generated by W_0. Raman (2006) postulates the following stochastic differential equation:

$$dG = (-\delta G + \beta u)dt + \sigma dW,$$

where $W(t)$ is a standard Brownian motion process.

Another research direction of these advertising models is to generalize them to cater for other system features for a better control. For example, Colombo and Lambertini (2002) investigate an advertising model where product quality is endogenous. Bradshaw and Porter (1975), Zhang et al. (2001), and Bouras et al. (20006) study integrated advertising-production systems. Sethi et al. (2008) propose a model of new-product adoption that incorporates price and advertising effects. Marinelli and Savin (2008) extend the advertising model of Nerlove and Arrow (1962) by adding a spatial dimension while Grosset and Viscolani (2009) extend it by considering the presence of a constant exogenous interference. Bykadorov et al. (2009) take explicitly into account the retailer's sales motivation and performance.

Yet another research direction followed by certain researchers consists in extending the advertising models which deal with a single firm (monopoly) to the case of two firms (duopoly), or even multiple firms (oligopoly). Assuming a saturation market point M, Kim (2001) considers that the market share $z_i = x_i/M$ of firm i follows the dynamics given by

$$\dot{z}_1 = \alpha_1 u_1 (1 - z_1 - z_2) + \gamma_1 u_1 z_2 - (\delta_1 + \gamma_2 u_2) z_1,$$

$$\dot{z}_2 = \alpha_2 u_2 (1 - z_1 - z_2) + \gamma_2 u_2 z_1 - (\delta_2 + \gamma_1 u_1) z_2.$$

Also, in an n-firm oligopoly market, Prasad and Sethi (2003) assume the dynamics of the i^{th} firm are given by

$$\dot{x}_i = \frac{n}{n-1} \rho_i u_i \sqrt{1 - x_i} - \delta \left(x_i - \frac{1}{n} \right) - \frac{1}{n-1} \sum_{j \neq i} \rho_j u_j \sqrt{1 - x_j}.$$

Prasad and Sethi (2004) extend the work of Sethi (1983) by examining a dynamic duopoly with stochastic disturbances. Denoting by $x(t)$ and $y(t)$ the market shares of firms 1 and 2 at time t as, they use the following model dynamics

$$dx = \left[\rho_1 u_1(x, y) \sqrt{1 - x} - \rho_2 u_2(x, y) \sqrt{x} - \delta(x - y) \right] dt + \sigma(x, y) dw,$$

$$dy = \left[\rho_2 u_2(x, y) \sqrt{1 - y} - \rho_1 u_1(x, y) \sqrt{y} - \delta(y - x) \right] dt - \sigma(x, y) dw,$$

where $\sigma(x,y)$ is a white noise term. Another oligopolistic and deterministic extension of the monopoly model of Sethi (1983) is given by Erickson (2009):

$$\dot{x}_i = \beta_i u_i \sqrt{N - \sum_{j=1}^{n} x_j} - \rho_i x_i.$$

In the present paper we combine the two research direction cited above by considering an integrated stochastic advertising-production system in the context of a duopoly. A firm produces a certain product and spends a certain amount on advertisement while a competing company producing the same product also spends another (unknown) amount on advertisement. We assume that the item in stock may deteriorate at some (unknown) rate. Items deterioration is of great importance in inventory theory, as shown by the survey of Goyal and Giri (2001). Examples of deteriorating items include blood, photographic films, certain pharmaceutical, and food stuff.

The problem facing our firm is the estimation of two unknown yet crucial parameters, namely the items deterioration rate and the amount spent by the competing firm. To the best of our knowledge, parameter estimation in this context has not received a large amount of attention so far. Such estimation is crucial though, as it enables decision makers to make educated decisions. Indeed, knowing how much the competing firm is spending on advertisement will help our firm decide how much it wants to spend on advertising itself. Also, knowing the amount of product that deteriorates helps management decide how much to produce. The deterioration parameter is assumed to be random. Concerning expense processes, it is intuitively clear that they are not sequences of independent random variables. The simplest and most common form of dependence used in modeling stochastic phenomena is the Markovian property, which usually is a good approximation of what is happening in reality. It is similar to using Taylor expansions to approximate highly non-linear functions. For this reason, we will assume that the expense processes evolve according to the jumps of a homogeneous, finite-state Markov chain.

We are interested in their conditional probability distributions given past information. This paper gives algorithms to find the maximum likelihood estimators (MLEs) of these unknown probabilities. The identification of MLEs of parameters by the Expectation Maximization (EM) algorithm and its variants appears in many contexts, e.g., voice recognition in artificial intelligence and fitting phase-type distributions in queueing systems. The transformation of probability measures is used in contexts such as importance sampling to estimate the probabilities of rare events. The mathematical procedures as outlined in this paper have been detailed in references (2004) and (1995).

In the next section we formally describe the system. In Section 3 we introduce a 'reference' probability measure under which all calculations are performed. The 'real world' probability measure under which the dynamics of our model are given is then defined via a suitable martingale. In section 4 we estimate recursively the joint conditional probability distribution of the (hidden) Markov chains which represent the expenses of the competing company and the perishability parameter.

2. Model Formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which we develop a parametric, discrete-time multi-period integer-valued inventory model with perishable items.

Definition 1. *A discrete-time stochastic process* $\{\eta_n\}$, *with finite-state space* $S = \{s_1, s_2, \cdots, s_N\}$, *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *is a Markov chain if*

$$P(\eta_{n+1} = s_{i_{n+1}} \mid \eta_0 = s_{i_0}, \cdots, \eta_n = s_{i_n}) = P(\eta_{n+1} = s_{i_{n+1}} \mid \eta_n = s_{i_n}),$$

for all $n \geq 0$ *and all states* $s_{i_0}, \cdots, s_{i_n}, s_{i_{n+1}} \in S$. *The Markov chain* $\{\eta_n\}$ *is homogeneous if*

$$P(\eta_{n+1} = s_j \mid \eta_n = s_i) \triangleq \pi_{ji}$$

is independent of n. *The matrix* $\Pi = \{\pi_{ji}\}$ *is called the transition probability matrix of the homogeneous Markov chain and it satisfies the property* $\sum_{j=1}^{N} \pi_{ji} = 1$.

Note that our transition matrix Π is the transpose of the traditional transition matrix defined elsewhere. The convenience of this choice will be apparent later.

Consider the filtration $\{\mathcal{H}_n\} = \sigma\{\eta_0, \eta_1, \cdots, \eta_n\}$ and write $X_n = \left(I_{(\eta_n = s_1)}, I_{(\eta_n = s_2)}, \cdots, I_{(\eta_n = s_N)}\right)$. Then, $\{X_n\}$ is a discrete-time Markov chain with state space the set of unit vectors $e_1 = (1, 0, \cdots, 0)', \cdots, e_N = (0, 0, \cdots, 1)'$ of $\mathbb{R}^N$, where the 'prime' means transpose. However, the transition probability matrix of X is Π. We can write:

$$E[X_n \mid \mathcal{H}_{n-1}] = E[X_n \mid X_{n-1}] = \Pi X_{n-1},$$

from which we conclude that ΠX_{n-1} is the predictable part of X_n, given the history of X up to time $n-1$ and the non-predictable part of X_n must be $M_n \triangleq X_n - \Pi X_{n-1}$. In fact, it can easily be shown that $M_n \in \mathbb{R}^N$ is a mean 0, $\mathcal{H}_n$-vector martingale and we have the semimartingale (or Doob decomposition) representation of the Markov chain $\{X_n\}$:

$$X_n = \Pi X_{n-1} + M_n. \tag{2.1}$$

Let X^1 and X^2 be such Markov chains with transition probability matrices A and B and state spaces $e_1 = (1, 0, \cdots, 0)', \cdots, e_N = (0, 0, \cdots, 1)'$ of $\mathbb{R}^M$, and $f_1 = (1, 0, \cdots, 0)', \cdots, f_M = (0, 0, \cdots, 1)'$ of $\mathbb{R}^M$, respectively. Then, we can write

$$\begin{aligned} X_n^1 &= AX_{n-1}^1 + W_n, \\ X_n^2 &= BX_{n-1}^2 + V_n, \end{aligned} \tag{2.2}$$

where V_n and W_n are mean 0, $\mathcal{F}_n$-vector martingales and $\{\mathcal{F}_n\}$ is the filtration generated by the Markov chains X^1 and X^2.

Now consider a firm that manufactures a certain product, selling some and stocking the rest in a warehouse. Assume that at epoch n, an amount equal to X_n^1 is spent on advertisement and another (hidden) amount equal to X_n^2 is spent on advertisement by a competing company producing the same item. The dynamics of the expenses processes X^1 and X^2 are regulated by equations (2.2).

We are assuming that items in stock may be subject to deterioration. Suppose $\theta \in (0,1)$ is an unknown parameter representing the proportion of items which did not perish, and let $\{v_\ell\}$ be a sequence of i.i.d. random variables with a suitable probability density function ξ. The interval (0,1) is partitioned into L conveniently chosen from past experience disjoint intervals,

$$I_1 = (0, \alpha_1), I_2 = [\alpha_1, \alpha_2), \cdots, I_{L-1} = [\alpha_{L-2}, \alpha_{L-1}), I_L = [\alpha_{L-1}, 1)$$

Write

$$x_n = \theta + v_n, \tag{2.3}$$

and let Y_n an L-dimensional unit vector such that $Y_n^i = 1$ if $x_n \in I_i$. Then, we see that

$$\begin{aligned} P(Y_\ell^i = 1 \mid \theta) &= P(a_{i-1} \leq x_\ell \leq a_i \mid \theta) \\ &= \int_{a_{i-1}-\theta}^{a_i-\theta} \xi(x)\, dx \\ &:= c_\ell^i(\theta), \qquad 1 \leq i \leq L. \end{aligned}$$

Let $\Im_n$, D_n, and P_n represent the inventory level, the demand rate, and the production rate at the beginning of the period n, respectively. It is clear that $\Im_n$ comprises the proportions of intact inventory which survived from the previous period plus the inventory produced during that same previous period minus the quantity sold. Taking into account equations (2.2) and (2.3), the dynamics of our integrated advertising-production system are as follows:

$$\begin{aligned} \Im_n &= \theta\Im_{n-1} - D_n + P_{n-1}, \\ X_n^2 &= BX_{n-1}^2 + V_n, \\ X_n^1 &= AX_{n-1}^1 + W_n, \\ x_n &= \theta + v_n. \end{aligned} \tag{2.4}$$

We assume here the demand $\{D_n\}$ is a non-negative random variable with a known probability density functions φ_n.

Define the filtrations $\mathcal{G}_n = \sigma(X_\ell^1, X_\ell^2, \Im_\ell, x_\ell, D_\ell, P_\ell,\ \ell \leq n)$ and $\mathcal{Y}_n = \sigma(\Im_\ell, X_\ell^1, D_\ell, P_\ell,\ \ell \leq n)$. We assume that:

(1) Processes X^2 and x are not observed.

(2) Processes $\Im, D, P, X^1$ are either observed or predictable with respect to whatever information is available at each epoch.

We wish to derive recursive conditional probability distributions for X^2 and θ given the filtration $\mathcal{Y}$.

3. Reference Probability

In our context, the objective of the method of reference probability is to choose a measure $\bar{P}$, on the measurable space $(\Omega, \mathcal{F})$, under which

(3) $\{\Im_n\}$ is a random variable with density function φ_n ,

(4) X^2 is a sequence of i.i.d. random variables uniformly distributed on the set of unit vectors $\{f_1, f_2, \cdots, f_N\}$,

(5) Y is a sequence of i.i.d. random variables uniformly distributed on the set of unit vectors $\{h\_1, h\_2, \cdots, h_L\}$.

The probability measure P is referred to as the 'real world' measure, that is, under this measure

$$\begin{aligned} \Im_n &= \theta \Im_{n-1} - D_n + P_{n-1}, \\ X_n^2 &= B X_{n-1}^2 + V_n, \\ X_n^1 &= A X_{n-1}^1 + W_n, \\ x_n &= \theta + v_n, \\ P\left(Y_t^l = 1 \mid \theta\right) &= c_t^l(\theta), \qquad 1 \le l \le L. \end{aligned} \tag{3.1}$$

Denote by $\Lambda = \{\Lambda_n, 0 \le n\}$ the stochastic process whose value at time n is given by

$$\Lambda_n = \prod_{k=0}^{n} \lambda_k, \tag{3.2}$$

where $\lambda_0 = 1$ and

$$\lambda_k = \prod_{i,j=1}^{N} (NB_{ji})^{<X_k^2, e_j><X_{k-1}^2, e_i>} \prod_{l=1}^{L} \left(Lc_k^l(\theta)\right)^{Y_k^l} \times \left\{\frac{\varphi_k(\theta \Im_{k-1} - \Im_k + P_{k-1})}{\varphi_k(\Im_k)}\right\}.$$

It is easily seen that the sequence $\{\Lambda_n\}_{n \in \mathbb{N}}$ given by (3.2) is a $\mathcal{G}_n$-martingale.

Define the 'real world' measure P in terms of $\bar{P}$, by setting

$$\left.\frac{dP}{d\bar{P}}\right|_{\mathcal{G}_n} \triangleq \Lambda_n.$$

The existence of P follows from Kolmogorov Extension Theorem. Under probability measure P, the 'real world' dynamics in (3.1) hold. For proofs and more details on measure change techniques, see Aggoun and Elliott (2004) and Elliott et al. (1995).

***Remark* 1**. The purpose of the change of measure is to work under a "nice" artificial probability measure under which calculation are made easy. Note that, at each time n, the two probability measures are connected via Λ_n which is the projection of the Radon-Nikodym Λ on the information available at time n. This, of course, allows for the results to be expressed under the original `real world' probability measure P.

4. Recursive State Estimation

As mentioned earlier, we are interested in deriving recursive conditional probability distributions for X^2 and θ given the filtration $\mathcal{Y}$. We shall be working under probability measure $\bar{P}$. Write

$$p_n(v,\theta)d\theta \triangleq E[I(\theta \in d\theta) < X_n^2, f_v > \Lambda_n | \mathcal{Y}_n],$$

$$q_n(v,\theta)d\theta \triangleq E[I(\theta \in d\theta) < X_n^2, f_v > | \mathcal{Y}_n].$$

Using a generalized version of Bayes' Theorem; see Aggoun and Elliott (2004) and Elliott et al. (1995)

$$p_n(v,\theta) = \frac{E[I(\theta \in d\theta) < X_n^2, f_v > \Lambda_n | \mathcal{Y}_n]}{E[\Lambda_n | \mathcal{Y}_n]}$$

$$= \frac{q_n(v,\theta)}{\int \sum_{m=1}^{N} q_n(m,u)du}$$

Theorem 1.: *For $n \geq 1$, we have*

$$q_n = \sum_{t=1}^{N} B_{vt} \int \prod_{t=1}^{L} (Lc_n^t(u))^{Y_n^t} \times \varphi_n(u\mathfrak{I}_{n-1} - I + P_{n-1}) q_{n-1}(t,u) dI.$$

***Proof*.**

In view (3.2) and the independence and distribution assumption under $\bar{P}$, for an arbitrary Borel function g, we see that

$$E[g(\theta) < X_n^2, f_v > \Lambda_n | \mathcal{Y}_n] = \int g(u) q_n(v,u) du$$

$$= E\left[g(\theta) < X_n^2, f_v > \lambda_n \Lambda_{n-1} | \mathcal{Y}_n\right]$$

$$= E\left[g(\theta) < X_n^2, f_v > \Lambda_{n-1} \prod_{i,j=1}^{N} (NB_{ji})^{<X_n^2, f_j> <X_{n-1}^2, f_i>} \times \prod_{t=1}^{L} (Lc_n^t(\theta))^{Y_n^t} \times \left\{\frac{\varphi_n(\theta\mathfrak{I}_{n-1} - \mathfrak{I}_n + P_{n-1})}{\varphi_n(\mathfrak{I}_n)}\right\} | \mathcal{Y}_{n-1}\right]$$

Here we replace the product with a summation involving only the index i because index j is equal to v and is fixed. The right-hand side of the above equation becomes

$$= \sum_{i=1}^{N} B_{vi} E\left[g(\theta) \langle X_{n-1}^2, f_i \rangle \Lambda_{n-1} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \times \left\{ \frac{\varphi_n(\theta \Im_{n-1} - \Im_n + P_{n-1})}{\varphi_n(\Im_n)} \right\} \mid \mathcal{Y}_{n-1} \right]$$

$$= \sum_{i=1}^{N} B_{vi} E\Bigg[g(\theta) \langle X_{n-1}^2, f_i \rangle \Lambda_{n-1} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t}$$
$$\times \int \left\{ \frac{\varphi_n(\theta \Im_{n-1} - I + P_{n-1})}{\varphi_n(I)} \right\} \varphi_n(I) dI \mid \mathcal{Y}_{n-1} \Bigg].$$

Here, because of the independence assumptions under $\bar{P}$, we can replace $\mathcal{Y}_n$ with $\mathcal{Y}_{n-1}$ which is in fact the purpose of the change of the probability. Thus,

$$E[g(\theta) \langle X_n^2, f_v \rangle \Lambda_n \mid \mathcal{Y}_n]$$
$$= \sum_{i=1}^{N} B_{vi} \int \int g(u) \prod_{t=1}^{L} \left(Lc_n^t(u)\right)^{Y_n^t} \times \varphi_n(u \Im_{n-1} - I + P_{n-1}) q_{n-1}(i, u) dI du.$$

Since g is arbitrary, the recursion follows.

5. Parameter Updating

Using the EM algorithm, see Baum and Petrie (1966) and Dempster et al. (1977), the transitions probabilities of the Markov chains X^1 and X^2 are updated. Let

$$\eta \triangleq \left(B_{ji}, A_{m\ell}, 1 \le i, j \le N, 1 \le \ell, m \le M\right).$$

Here θ is considered a nuisance parameter and we shall suppose that it is known or that it was estimated via the recursion in Theorem 1.

Suppose our model is determined by such a set η and we wish to determine a new set

$$\hat{\eta} \triangleq \left(\hat{B}_{ji}, \hat{A}_{m\ell}, 1 \le i, j \le N, 1 \le \ell, m \le M\right),$$

which maximizes the conditional pseudo-log-likelihood defined below. To replace, at time n, the parameters η by new ones $\hat{\eta}$, define:

$$\hat{\Lambda}_n = \prod_{k=1}^{n} \prod_{i,j=1}^{N} \left(\frac{\hat{B}_{ji}(n)}{B_{ji}} \right)^{\langle X_k^1, e_j \rangle \langle X_{k-1}^1, e_i \rangle} \prod_{\ell,m=1}^{M} \left(\frac{\hat{A}_{m,\ell}(n)}{A_{m,\ell}} \right)^{\langle X_k^2, f_m \rangle \langle X_{k-1}^2, f_\ell \rangle}.$$

An argument similar to that used earlier shows that we can define a new probability measure $\hat{P}$ by setting

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{G}_n} \triangleq \hat{\Lambda}_n.$$

It is easy to see that under $\hat{P}$, the Markov chains X^2 and X^1 have transition probabilities given by $\hat{B}_{ji}(n)$ and $\hat{A}_{ji}(n)$ respectively. Write

$$\log \hat{\Lambda}_n = \sum_{k=1}^{n} \sum_{i,j=1}^{N} < X_k^2, e_j > < X_{k-1}^2, e_i > \log \hat{B}_{ji}(n)$$
$$+ \sum_{k=1}^{n} \sum_{\ell,m=1}^{M} < X_k^1, f_m > < X_{k-1}^1, f_\ell > \log \hat{A}_{m\ell}(n) + R,$$

where R does not contain $\hat{\eta}$. Recalling that X_n^1 is $\mathcal{Y}_n$-measurable

$$E\left[\log \hat{\Lambda}_n \,|\mathcal{Y}_n, \theta\right]$$
$$= \sum_{k=1}^{n} \sum_{i,j=1}^{N} E\left[< X_k^2, e_j > < X_{k-1}^2, e_i > |\mathcal{Y}_n, \theta\right] \log \hat{B}_{ji}(n) \tag{5.1}$$
$$+ \sum_{k=1}^{n} \sum_{\ell,m=1}^{N} < X_k^1, f_m > < X_{k-1}^1, f_\ell > \log \hat{A}_{m\ell}(n) + \hat{R}.$$

Therefore, to re-estimate parameters η we shall require estimates of

(1) $\mathfrak{N}_n^{(j,i)}$, a discrete time counting process for the state transitions $e_i \to e_j$, where $i \neq j$,

$$\mathfrak{N}_n^{(j,i)} = \sum_{k=1}^{n} < X_{k-1}^2, e_i > < X_k^2, e_j >. \tag{5.2}$$

(2) $\mathfrak{M}_n^{(m,\ell)}$, a discrete time counting process for the state transitions $f_\ell \to f_m$, where $\ell \neq m$,

$$\mathfrak{M}_n^{(m,\ell)} = \sum_{k=1}^{n} < X_{k-1}^1, f_\ell > < X_k^1, f_m >.$$

(3) $\mathcal{J}_n^{2,i}$, the cumulative sojourn time spent by the process X^2 in state e_i,

$$\mathcal{J}_n^{2i} = \sum_{k=1}^{n} < X_{k-1}^2, e_i > .$$

(4) $\mathcal{J}_n^{1\ell}$, the cumulative sojourn time spent by the process X^1 in state f_ℓ,

$$\mathcal{J}_n^{1\ell} = \sum_{k=1}^{n} < X_{k-1}^1, f_\ell > .$$

Rather than directly estimating the quantities $\mathfrak{N}_n^{(j,i)}$, $\mathfrak{M}_n^{(m,\ell)}$, $\mathcal{J}_n^{2i}$ and $\mathcal{J}_n^{1\ell}$, recursive forms can be found to estimate the related product-quantities $\mathfrak{N}_n^{(j,i)} X_n^2 \in \mathbb{R}^N$, $\mathcal{J}_n^{2i} X_n^2 \in \mathbb{R}^N$ etc. The outputs of these filters can then be manipulated to marginalize out the process X^2, resulting in filtered estimates of the quantities of primary interest, namely $\mathfrak{N}_n^{(j,i)}$, $\mathfrak{M}_n^{(m,\ell)}$, $\mathcal{J}_n^{2i}$ and $\mathcal{J}_n^{1\ell}$.

Now the parameters $\hat{\eta}$ must satisfy

$$\textstyle\sum_{j=1}^{N} \mathcal{B}_{ji}(n) = 1, \quad \sum_{m=1}^{M} \mathcal{A}_{\ell m}(n) = 1. \tag{5.3}$$

We wish, therefore, to choose $\hat{\eta}$ to maximize (5.1) subject to the constraint (5.3). The optimum choice of $\hat{\eta}$ is

$$\mathcal{B}_{ji}(n) = \frac{E\left[\mathfrak{N}_n^{(j,i)} | \mathcal{Y}_n, \theta\right]}{E\left[\mathcal{J}_n^{2i} | \mathcal{Y}_n, \theta\right]} = \frac{\mathrm{q}_n\left(\mathfrak{N}_n^{(j,i)}, \theta\right)}{\mathrm{q}_n\left(\mathcal{J}_n^{2i}, \theta\right)}, \forall \text{ pairs } (i,j), i \neq j,$$

$$\mathcal{A}_{m,\ell}(n) = \frac{E\left[\mathfrak{M}_n^{(m,\ell)} | \mathcal{Y}_n, \theta\right]}{E\left[\mathcal{J}_n^{1\ell} | \mathcal{Y}_n, \theta\right]} = \frac{\mathfrak{M}_n^{(m,\ell)}}{\mathcal{J}_n^{1\ell}}, \forall \text{ pairs } (\ell, m), \ell \neq m,$$

Write

$$\mathrm{q}_n\left(\mathfrak{N}_n^{(j,i)} X_n^2, \theta\right) \triangleq E\left[\Lambda_n \mathfrak{N}_n^{(j,i)} X_n^2 \middle| \mathcal{Y}_n, \theta\right].$$

Lemma 1.

Process $\mathrm{q}_n\left(\mathfrak{N}_n^{(j,i)} X_n^2, \theta\right)$ *is computed recursively by the dynamics*

$$
\begin{aligned}
&q_n\left(\mathfrak{N}_n^{(j,i)} X_n^2, \theta\right) \\
&\quad = \sum_{i,j=1}^{N} e_j B_{ji} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \phi_n(\theta\mathfrak{I}_{n-1} - \mathfrak{I}_n + P_{n-1}) \langle q_n\left(\mathfrak{N}_{n-1}^{(j,i)} X_{n-1}^2, \theta\right), e_i\rangle \\
&\quad + e_j B_{ji} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \phi_n(\theta\mathfrak{I}_{n-1} - \mathfrak{I}_n + P_{n-1}) \frac{q_{n-1}(i,\theta)}{\sum_k q_{n-1}(k,\theta)}.
\end{aligned}
$$

Proof.

In view of (3.3) and (5.2):

$$
\begin{aligned}
q_n\left(\mathfrak{N}_n^{(j,i)} X_n^2, \theta\right) &= \overline{E}\left[\left(\mathfrak{N}_{n-1}^{(j,i)} + \langle X_{n-1}^2, e_i\rangle\langle X_n^2, e_j\rangle\right) X_n^2 \Lambda_{n-1}\lambda_n \middle| \mathcal{Y}_n, \theta\right] \\
&= \overline{E}\Bigg[\mathfrak{N}_{n-1}^{(j,i)} X_n^2 \Lambda_{n-1} \prod_{i,j=1}^{N} (NB_{ji})^{\langle X_n^2, e_j\rangle\langle X_{n-1}^2, e_i\rangle} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \\
&\quad \times \left\{\frac{\phi_k(\theta\mathfrak{I}_{n-1} - \mathfrak{I}_k + P_{n-1})}{\phi_n(\mathfrak{I}_n)}\right\}\Bigg] + \overline{E}[\langle X_{n-1}^2, e_i\rangle\langle X_n^2, e_j\rangle X_n^2 \Lambda_{n-1}\lambda_n | \mathcal{Y}_n, \theta].
\end{aligned}
$$

The first expectation is

$$
\begin{aligned}
&= \sum_{i,j=1}^{N} e_j B_{ji} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \phi_n(\theta\mathfrak{I}_{n-1} - \mathfrak{I}_n + P_{n-1}) \times \langle \overline{E}\left[\mathfrak{N}_{n-1}^{(j,i)} X_{n-1}^2 \Lambda_{n-1}\lambda_n \middle| \mathcal{Y}_{n-1}, \theta\right], e_i\rangle \\
&= \sum_{i,j=1}^{N} e_j B_{ji} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \phi_n(\theta\mathfrak{I}_{n-1} - \mathfrak{I}_n + P_{n-1}) \times \langle q_n\left(\mathfrak{N}_{n-1}^{(j,i)} X_{n-1}^2, \theta\right), e_i\rangle.
\end{aligned}
$$

The second expectation yields:

$$
\begin{aligned}
\overline{E}[\langle X_{n-1}^2, e_i\rangle\langle X_n^2, e_j\rangle X_n^2 \Lambda_{n-1}\lambda_n | \mathcal{Y}_n, \theta] &= e_j B_{ji} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \phi_n(\theta\mathfrak{I}_{n-1} - \mathfrak{I}_n + P_{n-1}) \\
&\qquad \overline{E}[\langle X_{n-1}^2, e_i\rangle \Lambda_{n-1} | \mathcal{Y}_{n-1}, \theta] \\
&= e_j B_{ji} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \phi_n(\theta\mathfrak{I}_{n-1} - \mathfrak{I}_n + P_{n-1}) \frac{q_{n-1}(i,\theta)}{\sum_k q_{n-1}(i,\theta)}
\end{aligned}
$$

Write

$$
q_n(\mathcal{J}_n^{2,t} X_n^2, \theta) \triangleq E[\Lambda_n \mathcal{J}_n^{2,t} X_n^2 | \mathcal{Y}_n, \theta].
$$

A similar argument shows that:

Lemma 2. *Process* $q_n(\mathcal{J}_n^{2,t}X_n^2,\theta)$ *is computed recursively by the dynamics*

$$
\begin{aligned}
q_n(\mathcal{J}_n^{2,t}X_n^2,\theta) = {} & \sum_{i,j=1}^{N} e_j B_{jt} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \phi_n(\theta\Im_{n-1} - \Im_n + P_{n-1}) \langle q_{n-1}(\mathcal{J}_{n-1}^{2,t}X_{n-1}^2,\theta), e_i\rangle \\
& + \sum_{j=1}^{N} e_j B_{jt} \prod_{t=1}^{L} \left(Lc_n^t(\theta)\right)^{Y_n^t} \phi_n(\theta\Im_{n-1} - \Im_n + P_{n-1}) \frac{q_{n-1}(t,\theta)}{\sum_k q_{n-1}(k,\theta)}.
\end{aligned}
$$

The filter recursions given by Lemmas 1 and 2 provide updates to estimate product processes, each involving the process X^2. What we would like to do, is manipulate these filters so as to remove the dependence upon the process X^2. Since X^2 takes values on a canonical basis of indicator functions (in fact the standard unit vectors of $\mathbb{R}^N$), we see that, omitting θ,

$$
\begin{aligned}
\langle q_n\left(\mathfrak{M}_n^{(i,t,\theta)} X_n^2\right), \mathbf{1}\rangle &= \langle E\left[\Lambda_n \mathfrak{M}_n^{(i,t,\theta)} X_n^2 \middle| \mathcal{Y}_n\right], \mathbf{1}\rangle \\
&= \overline{E}[\Lambda_n \mathfrak{M}_n^{(i,t,\theta)} \langle X_n^2, \mathbf{1}\rangle | \mathcal{Y}_n] \\
&= q_n\left(\mathfrak{M}_n^{(i,t,\theta)}\right),
\end{aligned}
$$

etc. Here $\mathbf{1} = (1,1,\ldots,1)' \in \mathbb{R}^N$.

6. Concluding Remarks

In this paper an integrated stochastic advertising-production system with deteriorating items is proposed. The firm advertises for its product and is faced with the problem of estimating both the deterioration rate and the amount spent by a competing firm in advertising the same product. Using hidden Markov models techniques, the conditional probability distributions of these parameters given past information are obtained.

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