Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 4, Issue 2 (December 2009), pp. 343 – 354
(Previously, Vol. 4, No.2)

Applications and Applied Mathematics:
An International Journal
(AAM)

Two-Layered Model of Blood Flow through Composite Stenosed Artery

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Received: May 15, 2009; Accepted: December 15, 2009

Abstract

In this paper a steady, axisymmetric flow, with a constricted tube has been studied. The artery has been represented by a two-layered model consisting of a core layer and a peripheral layer. It has been shown that the resistance to flow and wall shear stress increases as the peripheral layer viscosity increases. The results are compared graphically with those of previous investigators. It has been observed that the existence of peripheral layer is useful in representation of diseased arterial system.

Keywords: Arterial wall, Blood rheology, Peripheral layer viscosity, Stenosis

MSC 2000 #: 76Z05, 92C35

1. Introduction

Stenosis means constriction or narrowing. An artery which is affected by this abnormal growth can lead to serious consequences such as blockage of the artery, stroke and many other arterial diseases. The study of blood flow through stenotic arteries plays an important role in the diagnostic and fundamental understanding of cardiovascular diseases. To understand the effects of mild stenosis, several researchers such as Haldar (1987), Misra and Chakravarty (1986), and Young (1968) investigated the flow of blood through constricted tube treating blood as a Newtonian fluid. The model has been extended by assuming that blood behaves like a non-Newtonian fluid Nakamura and Sawada (1988) and Shukla et al. (1980). In all above models, the blood flow is represented by single-layered model. Bugliarello and Sevilla (1970) have shown experimentally that the blood flowing through narrow tubes can be well represented by a two-layered model instead of one. In this type of models there is a peripheral layer of plasma and a core region of suspension of red blood cells. Shukla et al. (1980) have taken two-layered model to analyze the peripheral layer viscosity. Ponalagusamy (2007) focused on slip velocity, thickness of peripheral layer and core layer viscosity at the vessel wall. Srivastava (2003) studied analytically and numerically effects of mild stenosis on blood flow characteristics in a two-fluid model. In this study we analysed the flow of blood through axisymmetric, mild and composite shaped stenosis. The results obtained are compared with previous investigators Shukla, et al. (1980).

2. Formulation of the Problem

We considered the flow of blood in a tube having axisymmetric mild stenosis. It is assumed that blood is an incompressible fluid which is represented by a two-layered model. The external layer shows peripheral layer of plasma and the internal core layer describes the suspension of red blood cells. The schematic diagram showing the flow is given by the following figure:

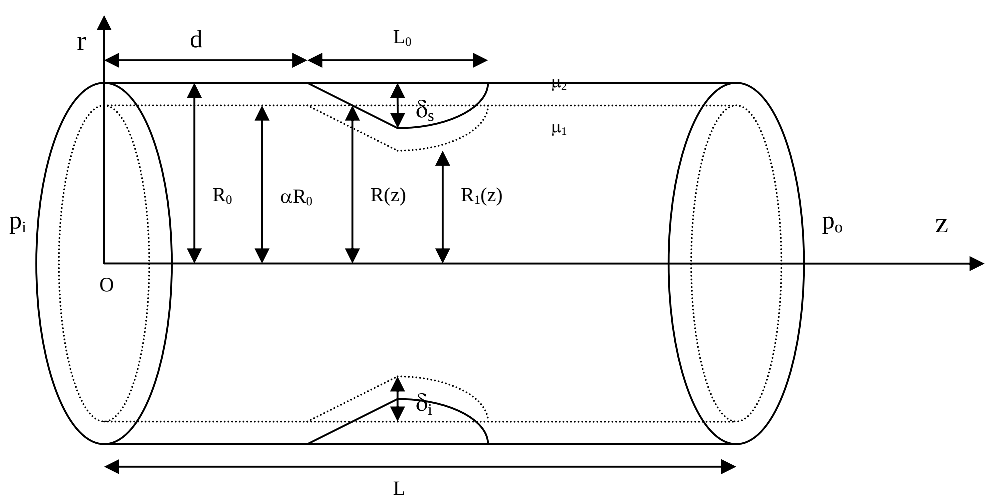

Figure 1: Geometry of stenosed artery,

where the symbols stand for

R_0 : Radius of the non-stenotic region

$R(z)$: Radius of the stenotic region

$R_1(z)$: Radius of the central layer in stenotic region

L: The length of the artery

L_0: The length of the stenosis

d : Location of stenosis
p_i : Inlet fluid pressure
p_0 : Instantaneous outlet fluid pressure
δ_s : Instantaneous maximum height of the stenosis
δ_i : Maximum bulging of interface
μ_1 : Viscosity of fluid in central core layer
μ_2 : Viscosity of fluid in peripheral layer
α : Ratio of central core radius to the tube radius.

The geometry of the stenotic tube without peripheral layer is described as follows:

$$R(z)=\begin{cases} R_0-\dfrac{2\delta_s}{L_0}(z-d), & d\le z\le d+\dfrac{L_0}{2}, \\ R_0-\dfrac{\delta_s}{2}\left\{1+\cos\dfrac{2\pi}{L_0}\left(z-d-\dfrac{L_0}{2}\right)\right\}, & d+\dfrac{L_0}{2}\le z\le d+L_0, \\ R_0, & Otherwise. \end{cases} \tag{1}$$

The governing equation of blood flow is given by Kapur (1985),

$$0=-\frac{dp}{dz}+\frac{1}{r}\frac{\partial}{\partial r}\left\{\mu(r)\,r\frac{\partial w}{\partial r}\right\}\ , \tag{2}$$

where w is axial velocity, p is fluid pressure and $\mu(r)$ is viscosity of fluid.

The boundary conditions are

$$w=0 \text{ at } r=R(z) \tag{3}$$

and

$$\frac{\partial w}{\partial r}=0 \text{ at } r=0\,. \tag{4}$$

Solving equation (2) under boundary conditions (3) and (4), we get

$$w=\left(-\frac{1}{2}\frac{dp}{dz}\right)\int_r^R \frac{r}{\mu(r)}dr \tag{5}$$

The volumetric flow rate is given by

$$Q = \int_0^R 2\pi\, r\, w\, dr\,, \tag{6}$$

which on using equation (5) gives

$$Q = \left(-\frac{\pi}{2}\frac{dp}{dz}\right)\int_0^R \frac{r^3 dr}{\mu(r)}\ . \tag{7}$$

Thus, the pressure gradient can be obtained as

$$\frac{dp}{dz} = -\frac{2Q}{\pi I(z)}\,, \tag{8}$$

where,

$$I(z) = \int_0^R \frac{r^3 dr}{\mu(r)}\,. \tag{9}$$

Integrating equation (8) using conditions $p = p_i \quad at \quad z = 0$ and $p = p_0$ at $z = L$, , we have

$$p_i - p_0 = \frac{2Q}{L}\int_0^L \frac{dz}{I(z)}. \tag{10}$$

The resistance to flow is defined by,

$$\lambda = \frac{p_i - p_0}{Q}\,. \tag{11}$$

From equations (1), (10) and (11), we can find

$$\lambda = \frac{2}{\pi}\left[\frac{L - L_0}{I_0} + \int_d^{d+l_0} \frac{dz}{I(z)}\right], \tag{12}$$

where

$$I_0 = \int_0^{R_0} \frac{r^3}{\mu(r)}\, dr\,. \tag{13}$$

Now, the shear stress at wall is given by,

$$\tau_R = \left[-\mu(r)\frac{\partial w}{\partial r}\right]_{r=R(z)}\,. \tag{14}$$

By using equations (5) and (8) in (14), we can find shear stress at maximum height of stenosis i.e. at $z = d + \frac{L_0}{2}$, which is as follows

$$\tau_s = \left[\frac{R(z)Q}{\pi I(z)} \right]_{z=d+\frac{L_0}{2}} . \tag{15}$$

For finding the effects of peripheral layer viscosity, the viscosity function $\mu(r)$ can be defined as

$$\mu(r) = \begin{cases} \mu_1, & 0 \le r \le R_1(z) \\ \mu_2, & R_1(z) \le r \le R(z), \end{cases} \tag{16}$$

where μ_1 and μ_2 are the viscosities of the central and the peripheral layers respectively. The function $R_1(z)$ represents the shape of the central layer with stenosis. The mathematical representation of this model can be described as

$$R_1(z) = \begin{cases} \alpha R_0 - \frac{2\delta_i}{L_0}(z-d), & d \le z \le d + \frac{L_0}{2} \\ \alpha R_0 - \frac{\delta_i}{2}\left\{1 + \cos\frac{2\pi}{L_0}\left(z - d - \frac{L_0}{2}\right)\right\}, & d + \frac{L_0}{2} \le z \le d + L_0 \\ \alpha R_0, & \text{Otherwise.} \end{cases} \tag{17}$$

By using equation (16) in (5), velocities w_c and w_p can be obtained and then the corresponding volumetric flow rates Q_c and Q_p are obtained as follows

$$Q_c = \int_0^{R_1} 2\pi r w_c \, dr = \left(-\frac{\pi}{8\mu_2} \frac{dp}{dz} \right) 2R_1^2 \left[R^2 - \left(1 - \frac{\bar{\mu}_2}{2}\right) R_1^2 \right] \tag{18}$$

$$Q_p = \int_{R_1}^{R} 2\pi r w_p \, dr = \left(-\frac{\pi}{8\mu_2} \frac{dp}{dz} \right) \left(R^2 - R_1^2 \right)^2 , \tag{19}$$

where $\bar{\mu}_2 = \mu_2 / \mu_1$.

Thus, the total volumetric flow rate Q is defined as

$$Q = Q_c + Q_p = \left(-\frac{\pi}{8\mu_2} \frac{dp}{dz} \right) \left(R^4 - (1 - \overline{\mu_2}) R_1^4 \right) . \tag{20}$$

Equation (20) can also be obtained by equation (7) using (16), which shows that Q is a constant.

Integrating equation (18), (19) and (20) across the length of artery, assuming that pressure drop is same in each case, we obtain

$$Q_c = \frac{(p_i - p_0)\pi R_0^4 M_1}{4\mu_2 L\left(1 - \frac{L_0}{L} + M_1 G_1\right)} , \tag{21}$$

where

$$M_1 = \alpha^2 \left\{1 - \left(1 - \frac{\bar{\mu}_2}{2}\right)\alpha^2\right\}, \tag{22}$$

and

$$G_1 = g_1 + g_2 , \tag{23}$$

where

$$g_1 = \frac{1}{L}\int_{d}^{d+\frac{L_0}{2}} \frac{dz}{\left(\frac{R_1}{R_0}\right)^2 \left\{\left(\frac{R}{R_0}\right)^2 - \left(1 - \frac{\bar{\mu}_2}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right\}}, \tag{24}$$

$$g_2 = \frac{1}{L}\int_{d+\frac{L_0}{2}}^{d+L_0} \frac{dz}{\left(\frac{R_1}{R_0}\right)^2 \left\{\left(\frac{R}{R_0}\right)^2 - \left(1 - \frac{\bar{\mu}_2}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right\}} \tag{25}$$

and

$$Q_p = \frac{(p_i - p_0)\pi R_0^4 M_2}{8\mu_2 L\left(1 - \frac{L_0}{L} + M_2 G_2\right)} , \tag{26}$$

where

$$M_2 = \left(1 - \alpha^2\right)^2 \tag{27}$$

and

$$G_2 = g_3 + g_4 , \tag{28}$$

where,

$$g_3 = \frac{1}{L}\int_{d}^{d+\frac{L_0}{2}} \frac{dz}{\left\{\left(\frac{R}{R_0}\right)^2 - \left(\frac{R_1}{R_0}\right)^2\right\}^2} \tag{29}$$

and

$$g_4 = \frac{1}{L}\int_{d+\frac{L_0}{2}}^{d+L_0} \frac{dz}{\left\{\left(\frac{R}{R_0}\right)^2 - \left(\frac{R_1}{R_0}\right)^2\right\}^2} \quad . \tag{30}$$

Now

$$Q = \frac{(p_i - p_0)\pi R_0^4 M}{8\mu_2 L\left(1 - \frac{L_0}{L} + MG\right)} , \tag{31}$$

where

$$M = 1 - (1 - \bar{\mu}_2)\alpha^4 \tag{32}$$

and

$$G = g_5 + g_6 , \tag{33}$$

where

$$g_5 = \frac{1}{L}\int_{d}^{d+\frac{L_0}{2}} \frac{dz}{\left\{\left(\frac{R}{R_0}\right)^4 - (1-\bar{\mu}_2)\left(\frac{R_1}{R_0}\right)^4\right\}} \tag{34}$$

and

$$g_6 = \frac{1}{L}\int_{d+\frac{L_0}{2}}^{d+L_0} \frac{dz}{\left\{\left(\frac{R}{R_0}\right)^4 - (1-\bar{\mu}_2)\left(\frac{R_1}{R_0}\right)^4\right\}} . \tag{35}$$

From equations (21) through (31) and using $Q = Q_c + Q_p$, we can find

$$\frac{M}{\left(1-\frac{L_0}{L}+MG\right)}=\frac{2M_1}{\left(1-\frac{L_0}{L}+M_1G_1\right)}+\frac{M_2}{\left(1-\frac{L_0}{L}+M_2G_2\right)} \quad . \tag{36}$$

Now using $R_1 = \alpha R$ in equation (17), we get

$$R(z)=\begin{cases} R_0-\dfrac{2\delta i}{\alpha L_0}(z-d), & d\le z\le d+\dfrac{L_0}{2} \\ R_0-\dfrac{\delta_i}{2\alpha}\left\{1+\cos\dfrac{2\pi}{L_0}\left(z-d-\dfrac{L_0}{2}\right)\right\}, & d+\dfrac{L_0}{2}\le z\le d+L_0 \\ R_0, & \textit{Otherwise.} \end{cases} \tag{37}$$

On comparing equations (1) and (37), we can observe that

$$\delta_i = \alpha\delta_s . \tag{38}$$

Now, keeping in mind the equation (16), the dimensionless resistance to flow $\overline{\lambda}$ and the dimensionless shear stress $\overline{\tau_s}$ can be obtained by using equation (31) in equations (11) and (15) as follows

$$\overline{\lambda}=\frac{\lambda}{\lambda_0}=\frac{\overline{\mu_2}}{M}\left(1-\frac{L_0}{L}+MG\right) , \tag{39}$$

where

$$\overline{\mu_2}=\frac{\mu_2}{\mu_1} \text{ and } \lambda_0=\frac{8\mu_1 L}{\pi R_0^4}$$

and

$$\overline{\tau_s}=\frac{\tau_s}{\tau_0}=\frac{\overline{\mu_2}\left(1-\dfrac{\delta_s}{R_0}\right)}{\left[\left(1-\dfrac{\delta_s}{R_0}\right)^4-\left(1-\overline{\mu_2}\right)\left(\alpha-\dfrac{\delta_i}{R_0}\right)^4\right]} , \tag{40}$$

where $\tau_0=\dfrac{4\mu_1 Q}{\pi R_0^3}$, and λ_0 and τ_0 are the resistance to flow and wall shear stress for the case of no stenosis respectively, with $\overline{\mu_2}=1$.

Evaluating the integrals (34) and (35) after using equation (38) and rewriting the expressions for $\overline{\lambda}$ and $\overline{\tau_s}$ as follows

$$\overline{\lambda} = \frac{\overline{\mu_2}}{M}\left[1 - \frac{L_0}{L} + \frac{L_0}{2L}\left\{\left(1 + 2\frac{\delta_s}{R_0} + \frac{10}{3}\left(\frac{\delta_s}{R_0}\right)^2 + \cdots\right) + \left(1 - \frac{\delta_s}{2R_0}\right)\left(1 - \frac{\delta_s}{R_0} + \frac{5}{8}\left(\frac{\delta_s}{R_0}\right)^2\left(1 - \frac{\delta_s}{R_0}\right)^{(-7/2)}\right)\right\}\right] \quad (41)$$

and

$$\overline{\tau_s} = \frac{\overline{\mu_2}}{\left[\left(1 - \frac{\delta_s}{R_0}\right)^3 M\right]} \quad . \quad (42)$$

The $\overline{\tau_s}$ obtained in (42) is the same as in Shukla, et al. (1980).

If $\overline{\mu_2} = 1$ in equations (41) and (42), we get

$$\overline{\lambda} = \left[1 - \frac{L_0}{L} + \frac{L_0}{2L}\left\{\left(1 + 2\frac{\delta_s}{R_0} + \frac{10}{3}\left(\frac{\delta_s}{R_0}\right)^2 + \cdots\right) + \left(1 - \frac{\delta_s}{2R_0}\right)\left(1 - \frac{\delta_s}{R_0} + \frac{5}{8}\left(\frac{\delta_s}{R_0}\right)^2\left(1 - \frac{\delta_s}{R_0}\right)^{(-7/2)}\right)\right\}\right], \quad (43)$$

which is the same as the ratio obtained by Joshi et al. (2009), and

$$\overline{\tau_s} = \left(1 - \frac{\delta_s}{R_0}\right)^{-3},$$

which is the same as the one obtained by Young (1968).

3. Conclusion

A two-layered model of blood flow through a stenosed artery has been considered. The model consists of a central core layer of erythrocytes surrounded by a peripheral plasma layer both with different viscosities. The expressions for $\overline{\lambda}$ and $\overline{\tau_s}$ have been plotted for different values of parameters. Figures 2 and 3 represents the variations of $\overline{\lambda}$ and $\overline{\tau_s}$ with $\frac{\delta_s}{R_0}$ for different values of $\overline{\mu_2}$ and $\frac{L_0}{L}$ respectively. It has been observed that $\overline{\lambda}$ and $\overline{\tau_s}$ increases with the increase in the height of stenosis, this increase have also been noted when $\overline{\mu_2}$ increased. It is further noted that resistance to flow $\bar{\lambda}$ (denoted by broken lines in the fig 2) is lower in the present model as compared to previous investigation of Shukla et al. (1980). Also, using the data $\overline{\mu_2} = 0.3, \frac{L_0}{L} = 1.0, \alpha = 0.95$ and $\frac{\delta_s}{R_0} = 0.1$ in equations (41) and (42), it can be noted that $\bar{\lambda}$ and $\overline{\tau_s}$ are decreased by 27% and 4%, respectively, when compared with the case of no stenosis with $\overline{\mu_2} = 1$. Again, in the absence of peripheral layer these characteristics are increased by 5% and 37% respectively for the same stenosis size and $\overline{\mu_2} = 1$. Thus, it seems that the results of present analysis of two-layered model can better explain the flow behaviour of stenotic arteries.

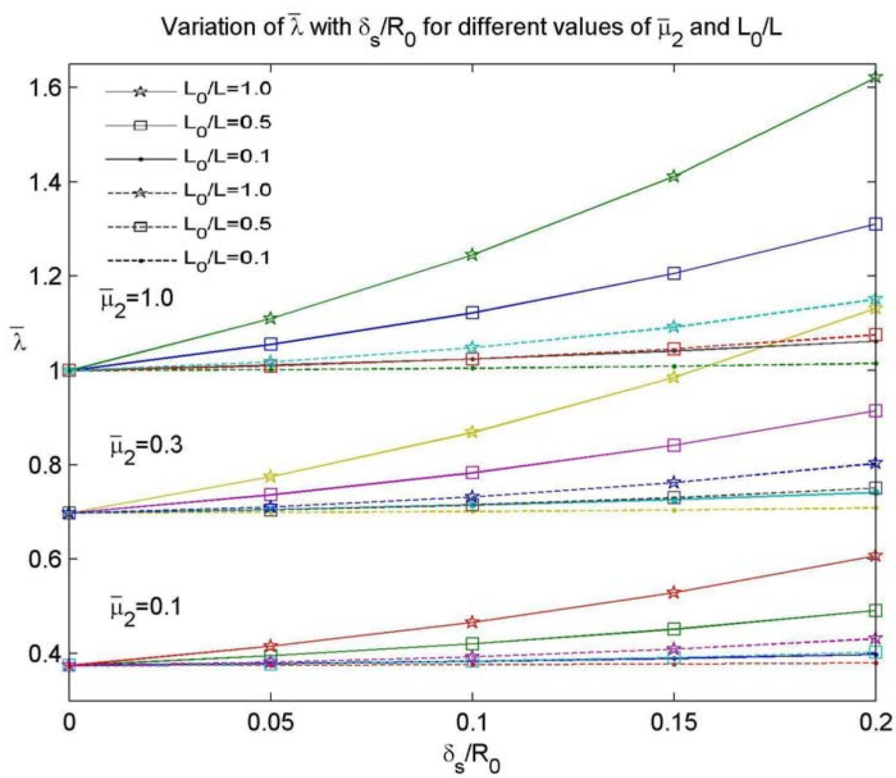

Figure 2.

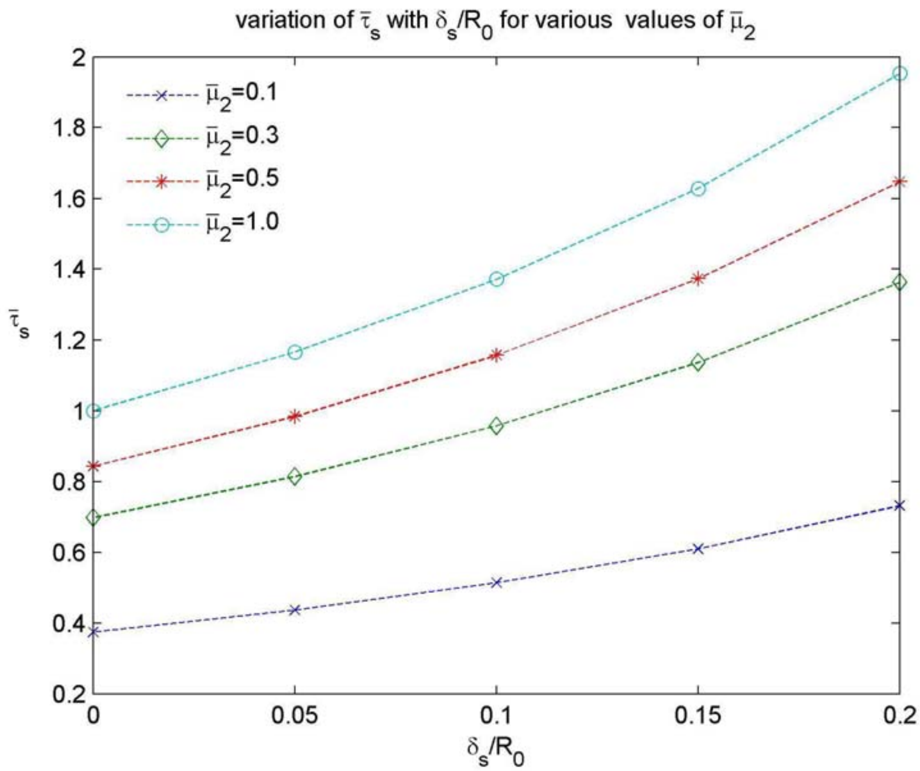

Figure 3.

Acknowledgement:

We are thankful to the anonymous reviewers for useful suggestions.

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