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Numerical solution of fractional integro-differential equations with nonlocal conditions

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Abstract

In this paper, we present a numerical method for solving fractional integro-differential equations with nonlocal boundary conditions using Bernstein polynomials. Some theoretical considerations regarding fractional order derivatives of Bernstein polynomials are discussed. The error analysis is carried out and supported with some numerical examples. It is shown that the method is simple and accurate for the given problem.

Keywords: Bernstein polynomials; Spectral methods; Fractional integro-differential equation

MSC 2010 No.: 34A08; 76M22; 41A10

1. Introduction

Integro-differential equations (IDEs) appear in modeling some phenomena in science and engineering. For example, the kinetic equations, which form the basis in the kinetic theories of rarefied gases, plasma, radiation transfer, coagulation, are expressed by IDEs (Grigoriev et al., 2010). In recent years, various numerical methods have been developed for solving these kind of

equations, for example see (Babolian and Shamloo, 2008; Cuesta and Palencia, 1983; Zhu and Fan, 2012) and the references therein.

This paper is concerned with providing a numerical scheme for the solution of the fractional integro-differential equations of the form (Nazari and Shahmorad, 2010)

$$D^q y(x) = g(x) + \mu_1 \int_a^x k_1(x,t)y(t)dt + \mu_2 \int_a^b k_2(x,t)y(t)dt, \tag{1}$$

for $a \le x \le b$, with integral boundary conditions given by

$$\sum_{j=1}^{m} \left(\alpha_{ij} y^{(j-1)}(a) + \gamma_{ij} y^{(j-1)}(b)\right) + \lambda_i \int_a^b H_i(t)y(t)dt = d_i, \tag{2}$$

for $i = 1, \dots, m$, where, $m = \lceil q \rceil$ $(i.e.,\ m-1 < q \le m;\ m \in \mathbb{N})$, D^q denotes the fractional derivative of order q in the Caputo sense as defined in (11), each $H_i(t)$ is a known continuous function, $g(x)$, $k_1(x,t)$, $k_2(x,t)$ are holomorphic functions, α_{ij}, γ_{ij} , λ_i and d_i are constants and $y(x)$ is the unknown function. For $\mu_1 = 0$ or $\mu_2 = 0$, Equation (1) is reduced to a fractional Fredholm or Volterra IDE, respectively.

In this paper, we use Bernstein polynomials as the basis to approximate the solution of the problem (1)-(2). Properties and applications of these polynomials have been discussed by various authors. For example, Cheng 1983, discussed the rate of convergence of Bernstein polynomials expansion of a certain class of functions. Farouki and Goodman, 1996, proved that a Bernstein polynomial basis on a given interval is an optimally stable basis, in the sense that no other non-negative basis yields systematically smaller condition numbers for the values or roots of arbitrary polynomials on that interval. Applications of Bernstein polynomials in different aspects of computer aided geometric design such as Bezier technique, rational techniques, approximation in spaces of geometric objects and surface construction have been discussed in detail by Farin et al., 2002. Bhatta, 2008, used modified Bernstein polynomials to solve KdV-Burgers equations. The work by Delgado et al., 2009, presents the optimal conditioning of collocation matrices related to Bernstein polynomials known as Bernstein_Vandermonde matrices. Saadatmandi, 2014, derived Bernstein operational matrix of fractional derivatives and he applied it to the collocation method for solving multi-order fractional differential equations. Also, the authors recently derived the exact operational matrices of Bernstein polynomials and applied it to the fractional advection-dispersion equations (Jani et al., 2017).

This paper is organized as follows. In section 2, after providing basic definitions of Bernstein polynomials and fractional calculus, some theoretical properties for fractional order integrals and derivatives involving Bernstein polynomials are derived. Section 3 is devoted to numerical aspect with some applications and a discussion of error estimation of the method.

2. Mathematical Formulation

2.1. Bernstein Polynomial Approximation

Definition 1.

Bernstein polynomials of degree N are defined on the interval $[a, b]$ as follows.

$$B_{i,N}(x) = \frac{\binom{N}{i}(x-a)^i(b-x)^{N-i}}{(b-a)^N}, \quad 0 \le i \le N. \tag{3}$$

Higher-order derivatives of Bernstein polynomials can be obtained using the next theorem (See also Doha et al., 2011).

Proposition 1.

Let p be any positive integer and $0 \le i \le N$. Then,

$$B_{i,N}^{(p)}(x) = c_{p,N} \sum_{k=0}^{p} (-1)^k \binom{p}{k} B_{i-k,N-p}(x), \quad a \le x \le b, \tag{4}$$

with

$$c_{p,N} = \frac{(-1)^p N!}{(b-a)^p (N-p)!}.$$

Proof:

We provide a simple proof based on the Leibniz formula.

$$\begin{aligned} B_{i,N}^{(p)}(x) &= \frac{1}{(b-a)^N}\binom{N}{i}\sum_{k=0}^{p}\binom{p}{k}\frac{d^k}{dx^k}(x-a)^i\frac{d^{p-k}}{dx^{p-k}}(b-x)^{N-i} \\ &= \frac{N!\,(-1)^p}{(b-a)^p(N-p)!}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\binom{N-p}{i-k}\frac{(x-a)^{i-k}(b-x)^{N-i-p+k}}{(b-a)^{N-p}} \\ &= c_{p,N}\sum_{k=0}^{p}(-1)^k\binom{p}{k}B_{i-k,N-p}(x). \end{aligned}$$

Definition 2.

Let f be a continuous function on the interval $[a, b]$. Then,

$$B_N(f)(x) = \sum_{i=0}^{N} f\left(\frac{i}{N}\right) B_{(i,N)}(x), \quad a \le x \le b,$$

is called the Bernstein polynomial of order N with respect to the function f.

Proposition 2.

(Bernstein's Theorem, Voronovskaya, 1932). For any bounded function f on the interval $I = [0,1]$, $\forall x \in I$, $\lim_{n\to\infty} B_n(f)(x) = f(x)$.

Moreover, if f is continuous, then the convergence is uniform. The error bound given by Devore and Lorentz, 1993,

$$|B_N(f)(x) - f(x)| \le \frac{1}{2N} x(1-x)\|f''\|,$$

where, $\|\cdot\|$ is the max norm, shows that the rate of convergence for twice continuously differentiable functions, is at least $\frac{1}{N}$. Also, the following asymptotic formula holds (Voronovskaya, 1932)

$$\lim_{N\to\infty} N\big(B_N(f)(x) - f(x)\big) = \frac{1}{2} x(1-x)\|f''\|.$$

Note that Bernstein polynomials at the endpoints of interval is either zero or one, i.e.,

$$B_{i,N}(a) = \delta_{i,0} = \begin{cases} 1, & i = 0, \\ 0, & i > 0 \end{cases}, \qquad B_{i,N}(b) = \delta_{i,N} = \begin{cases} 1, & i = N, \\ 0, & i < N \end{cases}. \tag{5}$$

This provides a good flexibility to impose boundary conditions. By the beta function, it is easily verified that the integral over $[a,b]$ is independent of i, (Farouki, 1988)

$$\int_a^b B_{i,N}(x)dx = \frac{b-a}{N+1}, \quad 0 \le i \le N. \tag{6}$$

Let $m \ge 0$. Using Definition (3) and taking $y = \frac{t-a}{b-a}$, we obtain ($a \le x \le b$)

$$\int_a^x t^m B_{i,N}(t)dt = \sum_{r=0}^{m} a^{m-r} \binom{m}{r} \sum_{j=i}^{N} (-1)^{i+j} \frac{\binom{N}{i}\binom{N-i}{j-i}}{j+r+1} \frac{(x-a)^{j+r+1}}{(b-a)^j}. \tag{7}$$

For the standard Bernstein polynomials, i.e., $a = 0$, it is reduced to

$$\int_a^x t^m B_{i,N}(t)dt = \binom{N}{i} \sum_{j=i}^{N} \binom{N-i}{j-i} \frac{(-1)^{i+j} x^{j+m+1}}{b^j (j+m+1)}.$$

Specially, for $m = 0$, it reads as

$$\int_a^x B_{i,N}(t)dt = (b-a)\sum_{j=i}^{N} \frac{(-1)^{i+j}}{j+1}\binom{j}{i}\binom{N}{j}\left(\frac{x-a}{b-a}\right)^{j+1} \quad , \quad a \le x \le b. \tag{8}$$

Remark 1.

Substituting $x = b$ and using (6), we have the following combinatorics formula:

$$\sum_{j=i}^{N} (-1)^{i+j}\binom{j}{i}\binom{N+1}{j+1} = 1$$

2.2. Fractional Integral and Derivative

We give some basic definition and properties from the fractional order operators, which are used further in the paper.

Definition 3.

The Riemann-Liouville fractional integral of order q, $(q > 0)$ is defined as

$$\begin{aligned} &J_a^q f(x) = \frac{1}{\Gamma(q)}\int_a^x (x-t)^{q-1} f(t)dt\,, \quad x > a, \\ &J_a^0 f = f. \end{aligned} \tag{9}$$

It is easily proved that for $q \in \mathbb{N}$, the operator J_a^q is the iterated operator of $J_a f = \int_a^x f(t)dt$.

Definition 4.

Riemann-Liouville and Caputo fractional derivative of order $q > 0$ are defined as (see, for instance, Diethelm, 2010)

$$D_a^q f(x) = D^m J_a^{q-m} f(x), \tag{10}$$

$${}_cD_a^q f(x) = J_a^{q-m} D^m f(x), \tag{11}$$

where, $m - 1 < q \le m, m \in \mathbb{Z}$.

Proposition 3.

Let $q \in \mathbb{R}^+$ and $m = \lceil q \rceil$. Then, the Riemann-Liouville fractional integral of Bernstein polynomials of order q is obtained as

$$J_a^q B_{i,N}(x) = (-1)^i (b-a)^q \binom{N}{i} \sum_{k=i}^{N} \binom{N-i}{k-i} \frac{(-1)^k k!}{\Gamma(k+q+1)} z^{k+q}, \tag{12}$$

or equivalently,

$$J_a^q B_{i,N}(x) = (-1)^i (b-a)^q \sum_{k=i}^{N} \binom{N}{k} \binom{k}{i} \frac{(-1)^k k!}{\Gamma(k+q+1)} z^{k+q},$$

where, $z = \frac{x-a}{b-a}$.

Proof.

By relation (9) and taking $y = \frac{t-a}{x-a}$, we have

$$\begin{aligned} J_a^q B_{i,N}(x) &= \frac{\binom{N}{i}(x-a)^{i+q}}{\Gamma(q)(b-a)^i} \int_0^1 y^i (1-y)^{q-1} \left(1 - \frac{x-a}{b-a} y\right)^{N-i} dy \\ &= \frac{\binom{N}{i}(b-a)^q z^{i+q}}{\Gamma(q)} \sum_{k=0}^{N-i} \binom{N-i}{k} (-1)^k z^k \int_0^1 y^{i+k} (1-y)^{q-1} dy \\ &= (-1)^i (b-a)^q \binom{N}{i} \sum_{k=i}^{N} \binom{N-i}{k-i} \frac{(-1)^k k!}{\Gamma(k+q+1)} z^{k+q}. \end{aligned}$$

This completes the proof.

The Caputo fractional derivative of Bernstein polynomials is obtained as follows.

Theorem 1.

Let q be any non-negative real number and $m = \lceil q \rceil$. Then,

$$\begin{aligned} {}_cD_a^q B_{i,N}(x) = c_{m,N} (-1)^i (b-a)^{m-q} \sum_{k=0}^{m} \binom{m}{k} \sum_{r=i-k}^{N-m} \binom{N-m}{r} \binom{r}{i-k} \\ \times \frac{(-1)^r r!}{\Gamma(r+m-q+1)} z^{r+m-q}, \end{aligned} \tag{13}$$

where, as before $z = \frac{x-a}{b-a}$ and $c_{m,N}$ is as defined in Proposition 1.

Proof.

Using (4), we have

$$
\begin{aligned}
{}_cD_a^q B_{i,N}(x) &= J_a^{m-q} D^m B_{i,N}(x) \\
&= c_{m,N} \sum_{k=0}^{m} (-1)^k \binom{m}{k} J_a^{m-q} B_{i-k,N-m}(x) \\
&= c_{m,N} (-1)^i (b-a)^{m-q} \sum_{k=0}^{m} \binom{m}{k} \sum_{r=i-k}^{N-m} \binom{N-m}{r} \binom{r}{i-k} \\
&\quad \times \frac{(-1)^r r!}{\Gamma(r+m-q+1)} z^{r+m-q}.
\end{aligned}
$$

Finally, using (13), it is seen that

$$
\begin{aligned}
\int_a^x {}_cD_a^q B_{i,N}(t)dt &= c_{m,N} (-1)^i (b-a)^{m-q+1} \sum_{k=0}^{m} \binom{m}{k} \sum_{r=i-k}^{N-m} \binom{N-m}{r} \binom{r}{i-k} \\
&\quad \times \frac{(-1)^r r!}{\Gamma(r+m-q+2)} z^{r+m-q+1}.
\end{aligned}
$$

3. Applications to Fractional Integro-differential Equations and Error Analysis

Let $y(x)$ be the solution of (1)-(2) and $y_N(x)$ be the approximate solution written in terms of Bernstein polynomial basis as

$$
y_N(x) = \sum_{j=0}^{N} c_j B_{j,N}(x). \tag{14}
$$

Let $M_i = \sup_{a \le x, s \le b} |k_i(x,t)| < \infty$ for $i = 0,1$ and let $e_N(x) = y(x) - y_N(x)$ be the error function. Let r_N is defined as

$$
D^q y_N(x) - \mu_1 \int_a^x k_1(x,t) y_N(t)dt - \mu_2 \int_a^b k_2(x,t) y_N(t)dt = g(x) + r_N(x).
$$

Then, using equation (1), we have

$$
D^q e_N(x) - \mu_1 \int_a^x k_1(x,t) e_N(t)dt - \mu_2 \int_a^b k_2(x,t) e_N(t)dt = -r_N(x),
$$

and so we obtain the following error bound

$$
|r_N(x) \le |D^q e_N(x)| + (b-a)\|e_N\|(M_1|\mu_1| + M_2|\mu_2|). \tag{15}
$$

On the other hand, for $y \in C^m[a,b]$, it can be shown that $|D^q e_N| \leq \epsilon \frac{x^{m-q}}{\Gamma(m-q+1)}$ (See Theorem 6. in Khosravian and Torres, 2013). So, relation (15) presents the error bound for the perturbation function.

Substituting (14) in equation (1) and using the collocation nodes as $x_i = a + ih,\ (i = 1, \dots, N-1)$ with $h = \frac{b-a}{N}$, the matrix form is obtained as $AX = G$, where, A is a square matrix of order $N+1$ and for $i = m_1, \dots, N - m_2$, (in the case that m is an even number, assume $m_1 = m_2 = \frac{m}{2}$ and when m is odd, $m_1 = m_2 + 1 = \frac{m+1}{2}$)

$$A_{i,j} = \left[C_{i,j} - \mu_1 V_{i,j} - \mu_2 F_{i,j}\right],\ \ X = [c_i],\ \ G = [g(x_i)], \tag{16}$$

where, $C_{i,j} = D^q B_{j,N}(x_i)$ is obtained using the relation (13) and V, F stand for Volterra and Fredholm parts respectively with $V_{i,j} = \int_a^{x_i} k_1(x_i,t) B_{j,N}(t)dt$ and$F_{i,j} = \int_a^b k_2(x_i,t) B_{j,N}(t)dt$. Depending on m, the first and the last row of the matrix A is obtained using the boundary conditions. This is explained in the numerical examples.

Now we consider several examples to illustrate the effectiveness of the proposed method. The computations were performed by Maple 18 on a Levovo Ci3-1.90GH RAM 4 G.

Example 1.

We consider the following fractional differential equation (Nazari and Shahmorad, 2010)

$$D^{\frac{1}{2}} y(x) = \frac{\sqrt{x}}{\Gamma\left(\frac{3}{2}\right)} - \frac{x^2 e^x}{3} y(x) - \frac{1}{2}x^2 + e^x \int_0^x t y(t)dt + \int_0^1 x^2 y(t)dt, \tag{17}$$

with the nonlocal condition

$$y(0) + y(1) - 3\int_0^1 t y(t)dt = 0. \tag{18}$$

Using the boundary values (5) and the relation (6), this condition is simplified as

$$c_0 + c_N - 3\frac{\binom{N}{i}}{\binom{N+1}{i+1}} \frac{1}{N+2} = 0.$$

For the fractional term, using relation (13), we have

$$c_{i,j} = {}_cD_0^{\frac{1}{2}} B_{i,N}(x_j) = (-1)^{i+1} N[\sum_{r=i}^{N-1} \binom{N-1}{r} \binom{r}{i} \frac{(-1)^r r!}{\Gamma\left(r+\frac{3}{2}\right)} x_j^{r+\frac{1}{2}} + \sum_{r=i-1}^{N-1} \binom{N-1}{r} \binom{r}{i-1} \frac{(-1)^r r!}{\Gamma\left(r+\frac{3}{2}\right)} x_j^{r+\frac{1}{2}}].$$

Now substituting the collocation equidistant nodes $x_i = \frac{i}{N}, i = 1, \dots, N$, in (18) with $N = 4$, the approximate solution is as

$$Y(x) = 8 \times 10^{-11} + 0.99999999848x + 4.68 \times 10^{-9}x^2 - 4.72 \times 10^{-9}x^3 + 1.38 \times 10^{-9}x^4. \tag{19}$$

The exact solution of the problem is $y(x) = x$. Table 1. shows the exact, approximate and absolute error with comparison to results of the paper (Nazari and Shahmorad, 2010).

Table 1. Exact solution, approximate solution and absolute error for Example 1

x	Exact	Approximate $Y(x)$	Absolute Error (Present study, N = 4)	Absolute Error (Nazari et al.,2010, N = 20)
0.0	0.0	8.000000000000E-11	8.0000E-11	9.6658E-7
0.2	0.2	1.9999999992768E-1	7.2352E-11	9.2362E-7
0.4	0.4	3.9999999995408E-1	4.5952E-11	7.3108E-7
0.6	0.6	6.0000000001218E-1	1.2128E-11	3.5394E-7
0.8	0.8	8.0000000000788E-1	7.8080E-12	1.9065E-7
1.0	1.0	9.9999999999999E-1	1.0000E-10	8.8091E-7

L_2-norm of the error $e_N(x) = |y(x) - y_N(x)|$ is defined by

$$\|e_N(x)\| = \left(\int_a^b [e_N(x)]^2 dx\right)^{\frac{1}{2}}.$$

In this example, $\|e_N\|_2 = 4.933E - 11$, with$N = 4$.

Example 2.

We consider the following fractional integro-differential equation

$$D^{\frac{5}{4}} y(x) = (\cos x - \sin x) y(x) + f(x) + \int_0^x \sin t \; y(t) \, dt, \tag{20}$$

with the nonlocal conditions

$$y(0)+y(1)+\left(\frac{e+1}{e+2}\right)y'(0)+\frac{1}{2}y'(1)-\int_0^1 ty(t)dt=0, \tag{21}$$
$$2y(0)+2y(1)+\left(\frac{e}{e+1}\right)y'(0)-y'(1)=0,$$

and choose $f(x)$ so that the exact solution is $y(x)=x^2$.

To use the collocation, similar to the previous example, we have

$$C_{i,j}={}_cD_0^{\frac{5}{4}}B_{i,N}(x_j)=N(N-1)(-1)^i\sum_{k=0}^{2}\binom{2}{k}\sum_{r=i-k}^{N-2}\binom{N-2}{r}$$
$$\times\binom{r}{i-k}\frac{(-1)^r r!}{\Gamma\left(r+\frac{7}{4}\right)}x_j^{r+\frac{3}{4}}.$$

The given conditions (21) are simply reduced to the following equations

$$c_0+c_N+\left(\frac{e+1}{e+2}\right)N(c_1-c_0)+\frac{1}{2}N(c_N-c_{N-1})-\frac{\binom{N}{i}}{\binom{N+1}{i+1}}\frac{1}{N+2}=0,$$
$$2c_0+2c_N+\left(\frac{e}{e+1}\right)N(c_1-c_0)-N(c_N-c_{N-1})=0.$$

Using the present method with $N=4$, the approximate solution is

$$Y(x)=2.225\times10^{-8}-5.80\times10^{-9}x+1.00000017630x\text{^}2 \tag{22}$$
$$-3.4660\times10^{-9}x^3+2.1285\times10^{-7}x^4.$$

Table 2. shows the exact, approximate and absolute error with comparison to results of the paper Nazar et al., 2010.

Table 2. Exact solution, approximate solution and absolute error for Example 2

x	Exact	Approximate $Y(x)$	Absolute Error (Present study, N = 4)	Absolute Error (Nazari et al. 2010, N = 30)
0.0	0.00	2.225000000000E-8	2.2250E-8	2.9447E-7
0.2	0.04	4.000002570976E-2	2.5709E-8	1.1055E-7
0.4	0.16	1.6000003140456E-1	3.1404E-8	1.3766E-6
0.6	0.36	3.6000003495776E-1	3.4957E-8	3.5376E-6
0.8	0.64	6.4000004016616E-1	4.0166E-8	6.5280E-6
1.0	1.00	1.000000059	5.9000E-8	1.0149E-5

In this example, $\|e_N\|_2=3.5122E-8$ with $N=4$.

Example 3.

We consider the following problem (Nazari et al., 2010)

$$D^{\frac{1}{3}}y(x) = \frac{3}{2}\frac{x^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)} - 1 + e^{x^2} - x^2e^{x^2} + \int_0^x x^2e^{xt}y(t)dt, \tag{23}$$

with the nonlocal conditions

$$y(0) + 2y(1) + 3\int_0^1 ty(t)dt = 3 \tag{24}$$

The given condition is simplified as

$$c_0 + 2c_N + 3\frac{\binom{N}{i}}{\binom{N+1}{i+1}}\frac{1}{N+2} = 3.$$

For the fractional part we have

$$C_{i,j} = {}_cD_a^qB_{i,N}(x_j) = (-1)^{i+1}N[\sum_{r=i}^{N-1}\binom{N-1}{r}\binom{r}{i-k}\frac{(-1)^r r!}{\Gamma\left(r+\frac{4}{3}\right)}x_j^{r+\frac{2}{3}} + \sum_{r=i-1}^{N-1}\binom{N-1}{r}\binom{r}{i-k}\frac{(-1)^r r!}{\Gamma\left(r+\frac{4}{3}\right)}x_j^{r+\frac{2}{3}}].$$

Using a similar formulation like the previous example with $N = 4$, we obtain the matrix A as follows:

$$A = \begin{bmatrix} 1.1000000 & 0.20000000 & 0.30000000 & 0.40000000 & 2.5000000 \\ 0.056907644 & 0.57628823 & 0.40477857 & 0.10817697 & 0.0099653939 \\ -0.21468847 & 0.032723479 & 0.42701050 & 0.41684130 & 0.12653776 \\ -0.24443625 & -0.30049159 & -0.12979780 & 0.39590585 & 0.52533796 \\ -0.30589790 & 0.38441664 & -0.51693498 & -0.93939718 & 1.1668515 \end{bmatrix}$$

and the approximate solution is

$$Y(x) = 9.8460 \times 10^{-7} + 9.9998465 \times 10^{-1}x + 5.0409000 \times 10^{-5}x^2 - 6.0556800 \times 10^{-5}x^3 + 2.4175400 \times 10^{-5}x^4. \tag{25}$$

The exact solution of the problem is $y(x) = x$. Table 3. shows the exact, approximate and absolute error with comparison to results of the paper Nazari et al. 2010.

Table 3. Exact solution, approximate solution and absolute error for Example 3

x	Exact	Approximate $Y(x)$	Absolute Error (Present study, N = 4)	Absolute Error (Nazari et al. 2010, N = 20)
0.0	0.0	9.829000000000E-7	9.8290E-7	1.4247E-5
0.2	0.2	1.999994825336E-1	5.1746E-7	8.2823E-6
0.4	0.4	3.999996552968E-1	3.4470E-7	2.5271E-6
0.6	0.6	5.999999786872E-1	2.1312E-8	2.9990E-6
0.8	0.8	7.999998654344E-1	1.3456E-7	8.8544E-6
1.0	1.00	9.999996635000E-1	3.3650E-7	1.6811E-5

In this example, $\|e_N\|_2 = 3.3098E-7$ with $N = 4$.

Example 4.

Consider the following problem

$$D^{\frac{1}{2}}y(x) + \int_0^x ty(t)dt + \int_0^1 t^2 y(t)dt = f(x), \quad 0 < x < 1,$$
$$y(0) - \int_0^1 ty(t)dt = 0,$$

where, $f(x) = \left(\operatorname{erf}\sqrt{x} + x - 1\right)e^x + e - 1$ and $\operatorname{erf} x$ is the error function. The exact solution of the problem is $y(x) = \exp(x)$. Table 4. shows the relative error for different values of N. From this table it is seen that the error tends to zero as N increases.

4. Conclusion

In this paper, we have proposed a numerical approach for solving fractional integro-differential equations with nonlocal boundary conditions. After deriving some analytical results involving fractional derivatives of Bernstein basis, we have implemented the collocation method to transform the problem to a system of algebraic equations. We also discuss error estimate of the method. Finally, by using some numerical examples, it has been shown that the method described in the paper is simple and accurate to implement for solving fractional integro-differential equations. Comparison with the method in Nazari et al. 2010, indicate a better performance of the proposed scheme.

Table 4. Exact solution, approximate solution and absolute error for Example 4

x	Exact	$N = 4$	$N = 6$	$N = 12$	$N = 20$
0.0	1	1.82E-05	4.39E-08	5.22E-17	1.50E-18
0.2	1.2214027581601	1.05E-04	1.57E-07	1.78E-16	4.50E-18
0.4	1.4918246976412	2.91E-05	9.36E-08	1.00E-16	2.41E-18
0.6	1.8221188003905	2.85E-05	4.92E-08	6.21E-17	1.48E-18
0.8	2.2255409284924	1.50E-05	4.45E-08	3.69E-17	8.53E-19
1.0	2.7182818284590	4.20E-06	8.85E-09	9.01E-18	3.05E-18

For Example 4, Figure 1 illustrates the spectral accuracy of the method.

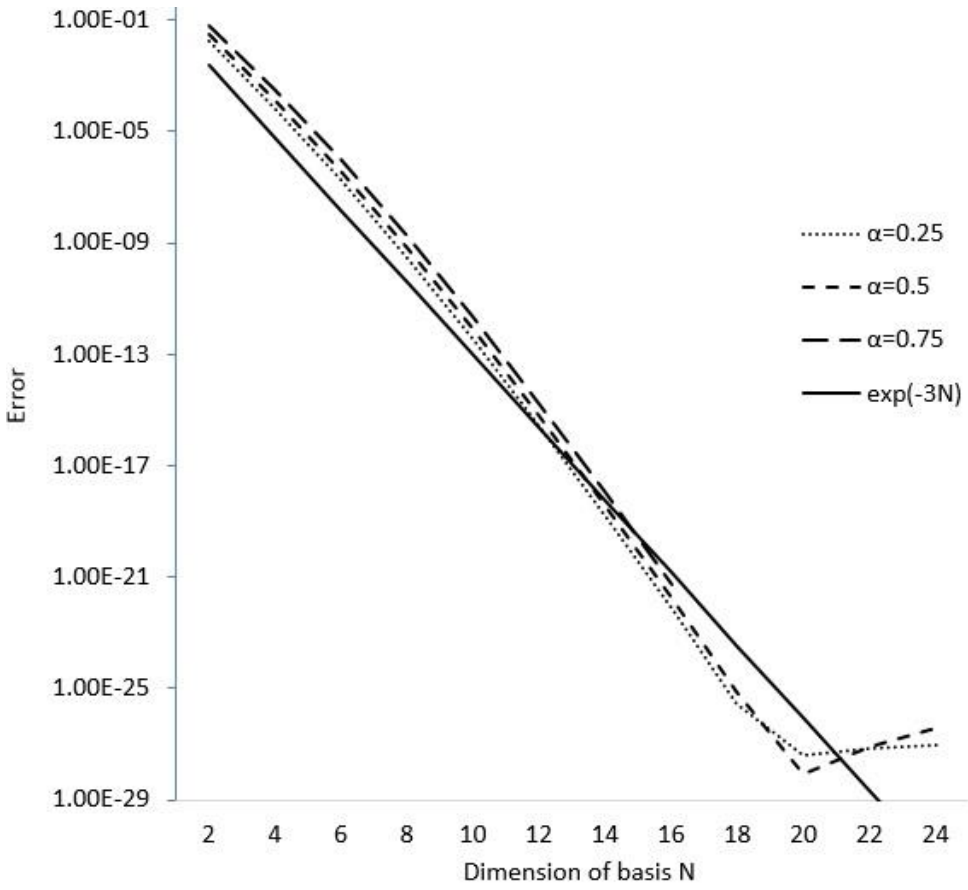

Figure 1. Convergence of the method for some fractional orders

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