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Exponentiated Weibull-Exponential Distribution with Applications

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Abstract

In this article, a new four-parameter continuous model, called the exponentiated Weibull exponential distribution, is introduced based on exponentiated Weibull-G family (Hassan and Elgarhy, 2016). The new model contains some new distributions as well as some former distributions. Various mathematical properties of this distribution are studied. General explicit expressions for the quantile function, expansion of distribution and density functions, moments, generating function, Rényi and q – entropies, and order statistics are obtained. The estimation of the model parameters is discussed using maximum likelihood method. The practical importance of the new distribution is demonstrated through real data set where we compare it with several lifetime distributions.

Keywords: Entropy; Exponential distribution; Exponentiated Weibull-G family of distributions; Moments; Order statistics; Maximum likelihood estimation

MSC 2010 No.: 62E15, 62E99

1. Introduction

In the last few years, new generated families of continuous distributions have attracted several statisticians to develop new models. These families are obtained by introducing one or more additional shape parameter(s) to the baseline distribution. Some of the generated families are: the beta-G (Eugene et al., 2002; Jones, 2004), gamma-G (type 1) (Zografos and Balakrishanan, 2009), Kumaraswamy-G (Kw-G; Cordeiro and de Castro, 2011), McDonald-G (Mc-G; Alexander et al., 2012), gamma-G (type 2) (Risti´c and Balakrishanan, 2012), transformed-transformer (T-X; Alzaatreh et al., 2013), Weibull-G (Bourguignon et al. (2014), Kumaraswamy odd log-logistic by Alizadeh et al. (2015), type 1 half-logistic family of distributions by Cordeiro et al. (2016), Kumaraswamy Weibull-G by Hassan and Elgarhy (2016a), additive Weibull-G by Hassan and Saeed (2016) among others. Garhy generated family of distributions introduced by Elgarhy et al. (2016), Hassan et al. (2017) introduced type II half logistic–G. Hassan and Elgarhy (2016b) introduced a new family called exponentiated Weibull-generated (EW-G). The cumulative distribution function (cdf) of exponentiated Weibull-generated family is given by

$$F(x)=\left[1-\exp\left(-\alpha[\frac{G(x)}{1-G(x)}]^{\beta}\right)\right]^{a};x>0\,;\mathrm{a},\alpha,\beta>0, \tag{1}$$

where $a,\beta>0$ are the two shape parameters and $\alpha>0$ is the scale parameter. The cdf (1) provides a wider family of continuous distributions. The probability density function (pdf) corresponding to (1) is given by

$$f(x)=\frac{a\alpha\beta(G(x))^{\beta-1}g(x)}{\left(1-G(x)\right)^{\beta+1}}e^{-\alpha[\frac{G(x)}{1-G(x)}]^{\beta}}\left[1-\exp\left(-\alpha[\frac{G(x)}{1-G(x)}]^{\beta}\right)\right]^{a-1}, \tag{2}$$

where $x>0$, $a,\alpha,\beta>0$.

In this paper, we introduce a new four-parameter model as a competitive extension for the Weibull distribution using the EW-G distribution. The new model extends some recent distributions and provides some new distributions. The rest of the paper is outlined as follows. In Section 2, we define the exponentiated Weibull exponential (EWE) distribution and provide some special models. In Section 3, we derive a very useful representation for the EWE density and distribution functions. In the same section we derive, some general mathematical properties of the proposed distribution. The maximum likelihood method is applied to derive the estimates of the model parameters in Section 4. Section 5 gives a numerical example to explain how a real data set can be modeled by EWE and this paper ends with some conclusions in Section 6.

2. The Exponentiated Weibull Exponential Distribution

In this section, the four-parameter EWE distribution is obtained based on the EW-G family.

Let the random variable X follow the following exponential distribution with scale parameter $\lambda > 0$,

$$g(x;\lambda) = \lambda e^{-\lambda x}; \qquad x, \lambda > 0. \tag{3}$$

The cdf of exponential distribution is given by

$$G(x;\lambda) = 1 - e^{-\lambda x}. \tag{4}$$

Subsituting from pdf (3) and cdf (4) into cdf (1), then the cdf of exponentiated Weibull exponential distribution, $EWE(a, \alpha, \beta, \lambda)$, takes the following form

$$F(x;\Psi) = \left[1 - \exp(-\alpha(e^{\lambda x} - 1)^{\beta})\right]^{a}; \quad a, \alpha, \beta, \lambda > 0, \quad x > 0, \tag{5}$$

where, $\Psi \equiv (a, \alpha, \beta, \lambda)$ is the set of parameters. Inserting the pdf (3) and cdf (4) into (2), then the pdf of EWE takes the following form

$$f(x;\Psi) = a\alpha\beta\lambda[e^{\lambda x} - 1]^{\beta-1} \exp[-\{\alpha(e^{\lambda x} - 1)^{\beta} - \lambda x\}]\left[1 - \exp(-\alpha(e^{\lambda x} - 1)^{\beta})\right]^{a-1}. \tag{6}$$

The pdf (6) contains some new distributions as well as some current distributions. Table 2.1 lists the special sub-models of the EWE distribution (6).

Table 2.1. Special sub models of the exponentiated Weibull exponential distribution

	Model	a	α	β	λ	Distribution function	References
1	Exponentiated Exponential exponential (EEE)	-	-	1	-	$F(x) = \left[1 - \exp(-\alpha(e^{\lambda x} - 1))\right]^{a}$	new
2	Exponentaited Rayligh exponential (ERE)	-	-	2	-	$F(x) = \left[1 - \exp(-\alpha(e^{\lambda x} - 1)^{2})\right]^{a}$	new
3	Weibull exponential (WE)	1	-	-	-	$F(x) = 1 - \exp(-\alpha(e^{\lambda x} - 1)^{\beta})$	Oguntunde et al. (2015)
4	Exponential exponential (EE)	1	-	1	-	$F(x) = 1 - \exp(-\alpha(e^{\lambda x} - 1))$	new
5	Rayleigh Exponential (RE)	1	-	2	-	$F(x) = 1 - \exp(-\alpha(e^{\lambda x} - 1)^{2})$	new

The survival function, hazard rate function, reversed-hazard rate function and cumulative hazard rate function of EWE are, respectively, given by

$$R(x;\Psi)=1-\left[1-\exp(-\alpha(e^{\lambda x}-1)^{\beta}\right]^{a},$$

$$h(x;\Psi)=\frac{a\alpha\beta\lambda[e^{\lambda x}-1]^{\beta-1}\exp[-\{\alpha(e^{\lambda x}-1)^{\beta}-\lambda x\}]\left[1-\exp(-\alpha(e^{\lambda x}-1)^{\beta}\right]^{a-1}}{1-\left[1-\exp(-\alpha(e^{\lambda x}-1)^{\beta}\right]^{a}},$$

$$\tau(x;\Psi)=\frac{a\alpha\beta\lambda[e^{\lambda x}-1]^{\beta-1}\exp[-\{\alpha(e^{\lambda x}-1)^{\beta}-\lambda x\}]}{1-\exp(-\alpha(e^{\lambda x}-1)^{\beta})},$$

and

$$H(x;\Psi)=-\ln\left(R(x;\Psi)\right)=-\ln\left(1-\left(1-\exp(-\alpha\left(e^{\lambda x}-1\right)^{\beta})\right)^{a}\right).$$

Plots of the cdf, pdf, survival function, hazard rate function, and reversed hazard rate function of EWE distribution for some parameter values are displayed in Figures 2.1, 2.2, 2.3, 2.4, and 2.5 respectively.

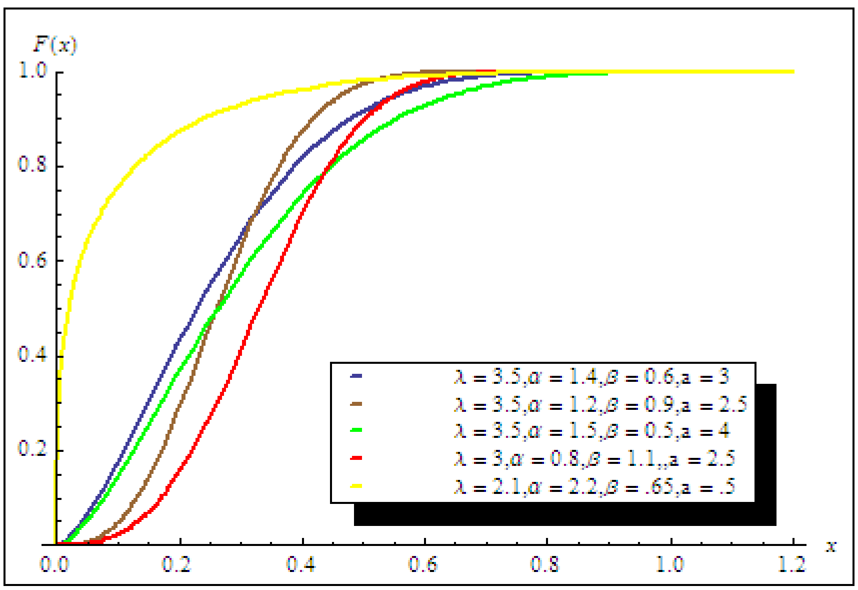

Figure 2.1. Plots of the cdf of the EWE distribution for some parameter values

From Figure 2.2 it appears that the proposed distribution is skewed to the right. Thus, it might be useful to model some life testing data.

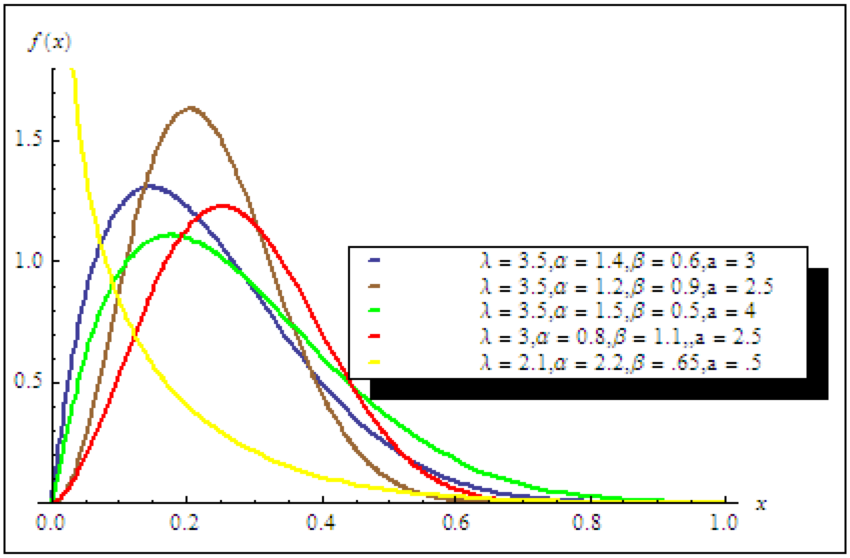

Figure 2.2. Plots of the pdf of the EWE distribution for some parameter values

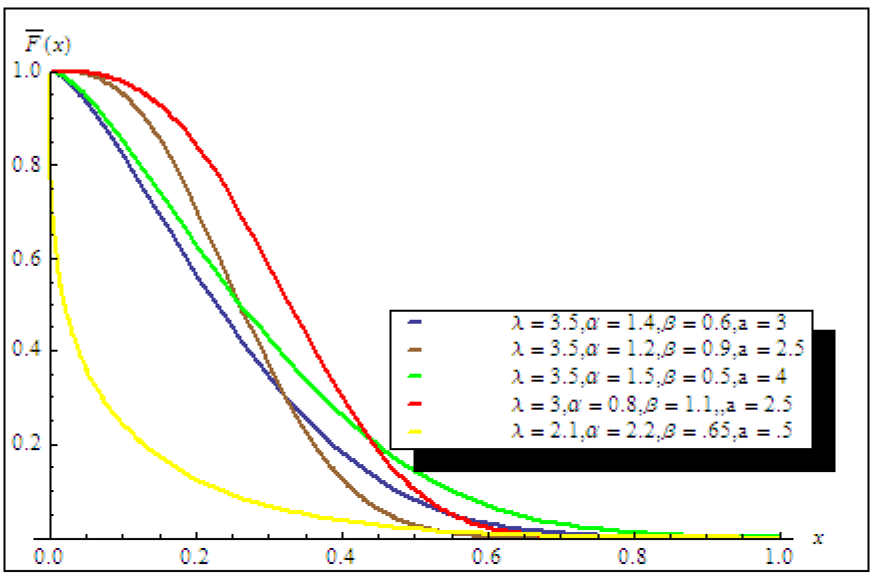

Figure 2.3. Plots of the survival function of the EWE distribution for some parameter values

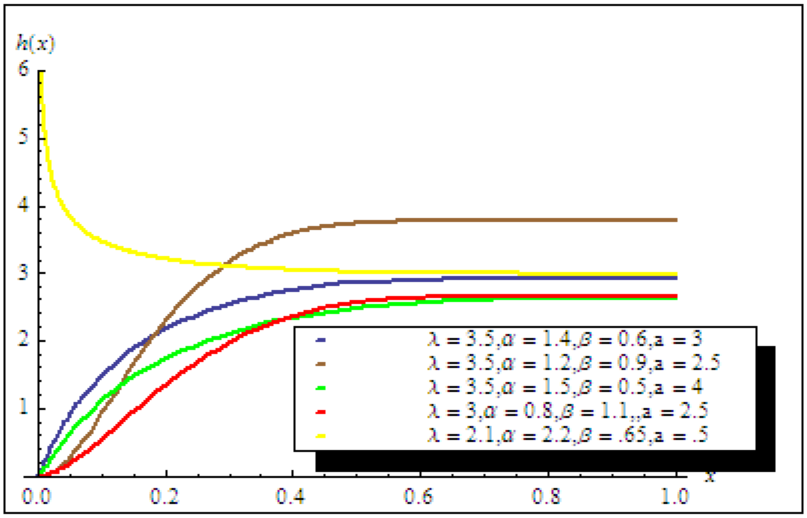

Figure 2.4. Plots of the hazard rate of the EWE distribution for some parameter values

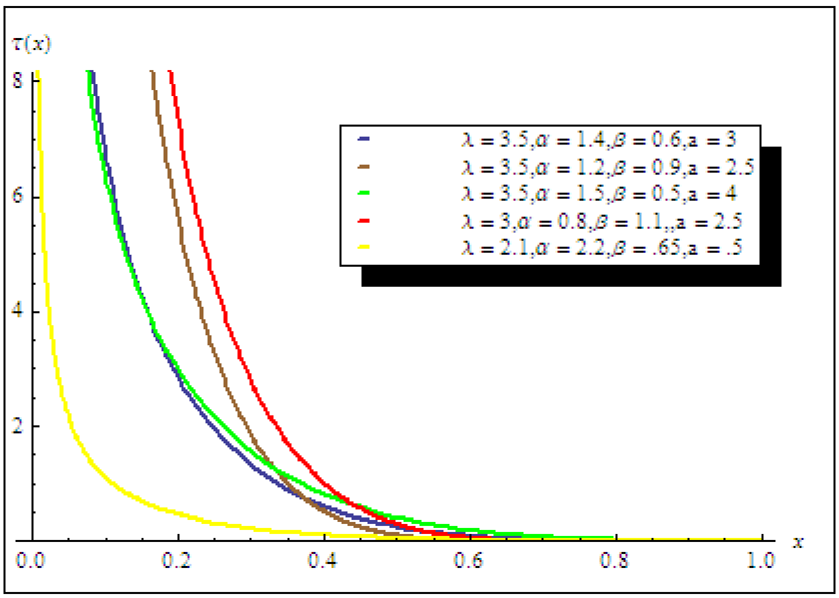

Figure 2.5. Plots of the revised hazard rate function of the EWE distribution for some parameter values

3. Statistical Properties

In this section some statistical properties of the *EWE* distribution are discussed.

3.1 Useful Expansions

In this subsection representations of the pdf and cdf for exponentiated Weibull exponential distribution are derived.

Using the generalized binomial theorem, where $\beta > 0$ is real non integer and $|z| < 1$,

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} z^i. \tag{7}$$

Then, by applying the binomial theorem (7) in (6), the distribution function of EWE distribution where a is real and positive becomes

$$f(x) = a\alpha\beta\lambda \left[\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right]^{\beta-1} e^{\lambda x} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} e^{-\alpha(j+1)\left[\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right]^{\beta}}. \tag{8}$$

By using the power series for the exponential function, we obtain

$$e^{-\alpha(j+1)[\frac{1-e^{-\lambda x}}{e^{-\lambda x}}]^{\beta}} = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k (j+1)^k}{k!} [\frac{1-e^{-\lambda x}}{e^{-\lambda x}}]^{k\beta}. \tag{9}$$

Using Equation (9), Equation (8) becomes,

$$f(x) = a\alpha\beta\lambda \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k} \alpha^k (j+1)^k}{k!} \binom{a-1}{j} \times e^{\lambda x} [\frac{1-e^{-\lambda x}}{e^{-\lambda x}}]^{\beta(k+1)-1},$$

The above equation can also be written as,

$$f(x) = a\alpha\beta\lambda x \sum_{j,k,m=0}^{\infty} \frac{(-1)^{j+k} \alpha^k (j+1)^k e^{-\lambda x}}{k!} \binom{a-1}{j} \binom{\beta(k+1)+m}{m} [1-e^{-\lambda x}]^{m+\beta(k+1)-1}.$$

Using the binomial theorem, the above equation can be expressed as an infinite linear combination of exponential distributin, i.e.,

$$f(x) = \sum_{j,k,m,\ell_1=0}^{\infty} \eta_{j,k,m,\ell_1} e^{-\lambda(\ell_1+1)x}, \tag{10}$$

where

$$\eta_{j,k,m,\ell_1} = \frac{a\beta\lambda\alpha^{k+1}(-1)^{j+k+\ell_1}(j+1)^k}{k!} \binom{a-1}{j} \binom{\beta(k+1)+m}{m} \binom{m+\beta(k+1)-1}{\ell_1}.$$

Now, we express the cumulative density function as an infinite linear combination of exponential distribution.

Since,

$$[F(x)]^h = [1 - e^{-\alpha(\frac{1-e^{-\lambda x}}{e^{-\lambda x}})^\beta}]^{ah},$$

then,

$$[F(x)]^h = \sum_{p=0}^{\infty} (-1)^p \binom{ah}{p} e^{-\alpha p(\frac{1-e^{-\lambda x}}{e^{-\lambda x}})^\beta}.$$

Now, using the power series for the exponential function in the above equation, we obtain

$$[F(x)]^h = \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}(\alpha p)^q}{q!} \binom{ah}{p} [\frac{1-e^{-\lambda x}}{e^{-\lambda x}}]^{\beta q},$$

which can also be written as,

$$[F(x)]^h = \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}(\alpha p)^q}{q!} \binom{ah}{p} \left[\frac{1-e^{-\lambda x}}{1-(1-e^{-\lambda x})}\right]^{\beta q}.$$

By using the binomial expansion, the above equation can be written as

$$[F(x)]^h = \sum_{p,q,t=0}^{\infty} \frac{(-1)^{p+q}(\alpha p)^q}{q!} \binom{ah}{p} \binom{\beta q + t - 1}{t} [1-e^{-\lambda x}]^{\beta q + t}.$$

Then,

$$[F(x)]^h = \sum_{p,q,t,\ell_2=0}^{\infty} \eta_{p,q,t,\ell_2} e^{-\lambda \ell_2 x}, \tag{11}$$

where,

$$\eta_{p,q,t,\ell_2} = \frac{(-1)^{p+q}(\alpha p)^q}{q!} \binom{ah}{p} \binom{\beta q + t - 1}{t} \binom{\beta q + t}{\ell_2}.$$

3.2 Quantile and Median

The quantile function, say $Q(u) = F^{-1}(u)$ of X can be obtained by inverting (5) as follows

$$u = (1 - \exp(-\alpha(e^{\lambda Q(u)} - 1)^\beta))^a.$$

After some simplifications, it reduces to

$$Q(u) = \ln\left\{1 + (\ln(1-(u)^{\frac{1}{a}})^{\frac{-1}{\alpha}})^{\frac{1}{\beta}}\right\}^{\frac{1}{\lambda}}, \tag{12}$$

where, u is a uniform random variable on the unit interval $(0,1)$. In particuler, the median can be derived from (12) by setting $u = 0.5$. That is, the median is given by

$$\text{Median} = Q(u) = \ln\left\{1 + (\ln(1 - (0.5)^{\frac{1}{a}})^{\frac{-1}{\alpha}})^{\frac{1}{\beta}}\right\}^{\frac{1}{\lambda}}.$$

3.3 Moments

This subsection provides the moment and moment generating function of EWE distribution. Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g. tendency, dispersion, skewness and kurtosis).

If X has the pdf (6), then its *rth* moment can be obtained through the following relation

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x;\Psi)dx. \quad (13)$$

Substituting (10) into (13) yields:

$$\mu_r' = E(X^r) = \sum_{j,k,m,l_1}^{\infty} \eta_{j,k,m,l_1} \int_0^{\infty} x^r e^{-\lambda(l_1+1)x} dx.$$

Let $y = \lambda(\ell_1 + 1)x$. Then, μ_r' becomes

$$\mu_r' = \sum_{j,k,m,\ell_1=0}^{\infty} \frac{\eta_{j,k,m,\ell_1}\Gamma(r+1)}{\left[\lambda(\ell_1+1)\right]^{r+1}}, \quad (14)$$

Where

$$\eta_{j,k,m,\ell_1} = \frac{a\beta\lambda\alpha^{k+1}(-1)^{j+k+\ell_1}(j+1)^k}{k!}\binom{a-1}{j}\binom{\beta(k+1)+m}{m}\binom{m+\beta(k+1)-1}{\ell_1}$$

and $\Gamma(.)$ is a gamma function.

Based on the first four moments of the EWE distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the EWE distribution can be obtained as

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^{\frac{3}{2}}},$$

and

$$k(\Phi)=\frac{\mu_4(\theta)-4\mu_1(\theta)\mu_3(\theta)+6\mu_1^2(\theta)\mu_2(\theta)-3\mu_1^4(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^2}.$$

Generally, the moment generating function of EWE distribution is obtained through the following relation

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)=\sum_{r,j,k,m,\ell_1=0}^{\infty}\frac{t^r}{r!}\frac{\eta_{j,k,m,\ell_1}\Gamma(r+1)}{\left[\lambda(\ell_1+1)\right]^{r+1}}. \quad (15)$$

3.4 Rényi and *q* – Entropies

The entropy of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics. According to Rényi (1961), the Rényi entropy is defined by

$$I_\delta(X)=\frac{1}{1-\delta}\log\int_{-\infty}^{\infty}f(x;\Psi)^\delta dx, \quad \delta>0 \text{ and } \delta\neq 1.$$

By applying the binomial theorem (7) in the pdf (6), the pdf $f(x;\Psi)^\delta$ can be expressed as follows

$$f(x;\Psi)^\delta=\sum_{j,k,m,\ell_1=0}^{\infty}W_{j,k,m,\ell_1}e^{-\lambda(\ell_1+1)x},$$

where

$$W_{j,k,m,\ell_1}=\frac{(a\beta\lambda)^\delta\alpha^{k+\delta}(-1)^{j+k+\ell_1}(j+\delta)^k}{k!}\times\binom{\delta(a-1)}{j}\binom{\beta(k+\delta)+m+\delta-1}{m}\binom{\beta(k+\delta)+m-\delta}{\ell_1}.$$

Therefore, the Rényi entropy of EWE distribution is given by

$$I_\delta(X)=\frac{1}{1-\delta}\log\left[\sum_{j,k,m,\ell_1=0}^{\infty}W_{j,k,m,\ell_1}\int_0^{\infty}e^{-\lambda(\ell_1+1)x}dx\right],$$

and then,

$$I_\delta(X)=\frac{1}{1-\delta}\log\left[\sum_{j,k,m,\ell_1=0}^{\infty}\frac{W_{j,k,m,\ell_1}}{\lambda(\ell_1+1)}\right].$$

The *q*- entropy is defined by

$$H_q(X)=\frac{1}{1-q}\log\left(1-\int_{-\infty}^{\infty} f(x;\Psi)^q dx\right), \quad q>0 \text{ and } q\neq 1.$$

Therefore, the q- entropy of EWE distribution is given by

$$H_q(X)=\frac{1}{1-q}\log\left\{1-\left[\sum_{j,k,m,\ell_1=0}^{\infty}\frac{W_{j,k,m,\ell_1}}{\lambda(\ell_1+1)}\right]\right\}. \tag{16}$$

For more on different kinds of entropies we refer our readers to Ahsanuallah et al. (2014) among others.

3.5. Order Statistics

Let $X_{1:n}<X_{2:n}<...<X_{n:n}$ be the order statistics of a random sample of size n following the exponentiated Weibull exponential distribution, with parameters a,α,β and λ. Then the pdf of the kth order statistic (David, (1981)), can be written as follows

$$f_{X_{(k)}}(x_{(k)})=\frac{f(x_{(k)})}{B(k,n-k+1)}\sum_{v=0}^{n-k}(-1)^v\binom{n-k}{v}F(x_{(k)})^{v+k-1}, \tag{17}$$

where, $B(.,.)$ is the beta function. By substituting (10) and (11) in (17), and replacing h with $v+k-1$, leads to

$$f_{X_{(k)}}(x_{(k)})=\frac{1}{B(k,n-k+1)}\sum_{v=0}^{n-k}\sum_{j,k,\ell_1=0}^{\infty}\sum_{q,t,\ell_2=0}^{\infty}\eta^{*}e^{-\lambda[\ell_1+\ell_2-\beta(k+1)-\beta t]x_{(k)}}, \tag{18}$$

where

$$.\eta^{*}=(-1)^v\binom{n-k}{v}\eta_{j,k,\ell_1}\eta_{q,t,\ell_2}$$

Moments of order statistics is given by:

$$E(\mathrm{X}_{(k)}{}^r)=\int_{-\infty}^{\infty}x_{(k)}{}^r f(x_{(k)})dx_{(k)}. \tag{19}$$

Substituting (18) in (19), leads to

$$E(\mathrm{X}_{(k)}{}^r)=\frac{1}{B(k,n-k+1)}\sum_{v=0}^{n-k}\sum_{j,k,\ell_1=0}^{\infty}\sum_{q,t,\ell_2=0}^{\infty}\eta^{*}\int_{0}^{\infty}x_{(k)}{}^r e^{-\lambda[\ell_1+\ell_2-\beta(k+1)-\beta t]x_{(k)}}dx_{(k)}.$$

Then,

$$E(X_{(k)}{}^{r}) = \frac{1}{B(k,n-k+1)} \sum_{v=0}^{n-k} \sum_{j,k,\ell_1=0}^{\infty} \sum_{q,t,\ell_2=0}^{\infty} \frac{\eta^{*}\Gamma(r+1)}{\left[\lambda\left[\ell_1+\ell_2-\beta(k+1)-\beta t\right]\right]^{r+1}}.$$

4. Maximum Likelihood Estimation

The maximum likelihood estimates (MLEs) of the unknown parameters for the exponentiated Weibull exponential distribution are determined based on complete samples. Let $X_1,...,X_n$ be observed values from the EWE distribution with set of parameters $\Psi=(a,\alpha,\beta,\lambda)^T$. The total log-likelihood function for the vector of parameters Ψ can be expressed as

$$\ln L(\Psi) = n\ln a + n\ln\alpha + n\ln\beta + n\ln\lambda + (\beta-1)\sum_{i=1}^{n}\ln\left(e^{\lambda x_i}-1\right) + \lambda\sum_{i=1}^{n}x_i \\ -\alpha\sum_{i=1}^{n}\left[\left(e^{\lambda x_i}-1\right)\right]^{\beta} + (a-1)\sum_{i=1}^{n}\ln\left[1-e^{-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}}\right].$$

The elements of the score function $U(\Psi)=(U_a,U_\alpha,U_\beta,U_\lambda)$ are given by

$$U_a = \frac{n}{a} + \sum_{i=1}^{n}\ln\left[1-e^{-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}}\right], \tag{20}$$

$$U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta} + (a-1)\sum_{i=1}^{n}\frac{\left(e^{\lambda x_i}-1\right)^{\beta}e^{-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}}}{1-e^{-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}}}, \tag{21}$$

$$U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left(e^{\lambda x_i}-1\right)^{\beta} - \alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\ln\left(e^{\lambda x_i}-1\right) + \alpha(a-1) \\ \times\sum_{i=1}^{n}\frac{\left(e^{\lambda x_i}-1\right)^{\beta}e^{-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}}\ln\left(e^{\lambda x_i}-1\right)}{1-e^{-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}}}, \tag{22}$$

and

$$U_\lambda = \frac{n}{\lambda} + (\beta-1)\sum_{i=1}^{n}\frac{x_i e^{\lambda x_i}}{e^{\lambda x_i}-1} - \alpha\beta\sum_{i=1}^{n}x_i\,(e^{\lambda x_i}-1)^{\beta-1}e^{\lambda x_i} \\ + \alpha\beta(a-1)\sum_{i=1}^{n}\frac{x_i e^{\lambda x_i}(e^{\lambda x_i}-1)^{\beta-1}e^{-\alpha(e^{\lambda x_i}-1)^{\beta}}}{1-e^{-\alpha(e^{\lambda x_i}-1)^{\beta}}} + \sum_{i=1}^{n}x_i \tag{23}$$

Then, the maximum likelihood estimates of the parameters a, α, β and λ are obtained by solving the maximum likelihood Equations (20 -23) and applying the Newton-Raphson's iteration method and using the computer package such as Maple or R or other software.

5. Data Analysis

In this section, one real data set are analyzed to illustrate the merit of EWE distribution compare to some sub-models; namely, Weibull exponential (WE), exponential exponential (EE), and Rayleigh exponential (RE) dsitributions.

We obtain the MLE and their corresponding standard errors (in parentheses) of the model parameters. To compare the performance of different distribution models, we consider criteria like minus of log-likelihood function $(-2\ln L)$, Kolmogorov-Smirnov ($K-S$) statistic, Akaike information criterion (AIC), the correct Akaike information criterion ($CAIC$), Bayesian information criterion (BIC) and p-value. However, the better distribution corresponds to the smaller values of $-2\ln L$, AIC, BIC, $CAIC$, $HQIC$, $K-S$ criteria and biggest p-value. Furthermore, we plot the histogram for each data set and the estimated pdf of the EWE, WE, EE and RE models. Moreover, the plots of empirical cdf of the data sets and estimated pdf of EWE, WE, EE and RE models are displayed in Figures 5.1 and 5.2, respectively.

The data have been obtained from Bjerkedal (1960) and represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. The data set are as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 5.1 gives MLEs of parameters and their standard error (S.E) of the EWE distribution. The values of the log-likelihood functions, AIC, **CAIC**, BIC, HQIC, K-S and p-value are presented in Table 5.2

Table 5.1. The MLEs and S.E of the model parameters

Model	**MLEs**				**S. E**			
	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	**S.E($\hat{a}$)**	**S.E($\hat{\alpha}$)**	**S.E($\hat{\beta}$)**	**S.E($\hat{\lambda}$)**
EWE	3.236	15.03	1.00	0.066	0.85278	0.777	0.273	0.025
WE	-	163.73	1.58	0.02	-	0.85849	0.092	0.036
EE	-	73.389	-	0.007	-	0.292	-	0.018
RE	-	85.045	-	0.049	-	0.047	-	0.032

Table 5.2. The values of -2LnL , AIC, BIC, CAIC, HQIC, K-S and p-value for the third data set

Distribution	-2LnL	AIC	CAIC	BIC	HQIC	K-S	p-value
EWE	217.041	225.041	226.641	224.47	228.666	0.091	0.597
WE	292.659	298.659	299.012	298.231	301.378	0.1394	0.122
EE	304.551	308.551	308.725	308.266	310.364	0.286	0.00002
RE	285.026	289.026	289.199	288.74	290.838	0.1343	0.149

We find that the EWE distribution with four parameters provides a better fit than their special sub-models. It has the smallest K-S, AIC, CAIC, BIC and HQIC values among those considered here. Plots of the fitted densities and the histogram are given in Figures 5.1 and 5.2, respectively.

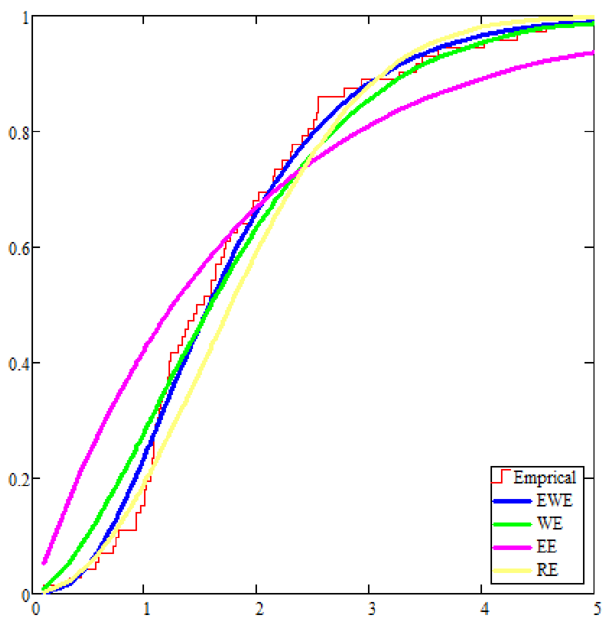

Figure 5.1. Estimated cumulative densities of the data set .

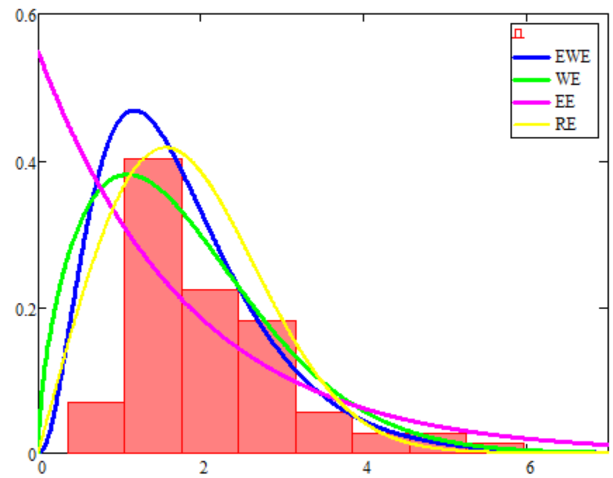

Figure 5.2. Estimated densities of the models models for for data set .

6. Conclusion

In this paper, we have introduced a new four-parameter exponentiated Weibull exponential distribution and studied its different statistical properties. It is noted that the proposed EWE distribution has several desirable properties. The EWE distribution covers some existing distributions and contains some new distributions. The practical importance of the new distribution was demonstrated in application where the EWE distribution provided a better fitting in comparison to several other former lifetime distributions. Applications showed that the EWE models can be used instead of other known distributions.

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