Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied
Mathematics:
An International Journal
(AAM)

Vol. 05, Issue 2 (December 2010), pp. 292 – 302
(Previously, Vol. 05, Issue 10, pp. 1389 – 1399)

An Analytical Technique for Solving Nonlinear Heat Transfer Equations

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Received: November 25, 2009; Accepted: September 25, 2010

Abstract

In this paper, an analytic technique, namely the New Homotopy Perturbation Method (NHPM) is applied for solving the nonlinear differential equations arising in the field of heat transfer. In this method, the solution is considered as an infinite series expansion where converges rapidly to the exact solution. The nonlinear convective–radioactive cooling equation and nonlinear equation of conduction heat transfer with the variable physical properties are chosen as illustrative examples and the exact solutions have been found for each case.

Keywords: NHPM; Nonlinear equations; Conduction and convection heat transfer; Taylor series expansion

MSC (2000) No.: 55P05, 65L05, 65L07

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1. Introduction

Since most of the phenomena in our world are essentially nonlinear and hence described by nonlinear equations, there has developed an ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for solving nonlinear problems. Recently, many new numerical techniques have been widely applied to the nonlinear problems. One of these methods the Homotopy Pertubation Method (HPM) attracted great attention due to its versatility and straightforwardness. HPM was introduced by He (2004, 2005, 2006, 2005, 1999, 2000, 2004, 2003) and has since been used by many mathematicians and engineers to solve various functional equations. This simple method has also been applied to solve linear and nonlinear equations of heat transfer [Rajabi and Ganji (2007), Ganji and Sadighi (2007), Ganji (2006)], fluid mechanics [Abbasbandy (2007)], nonlinear Schrödinger equations [Biazar and Ghazvini (2007)], some boundary value problems and many other topics from a variety of disciplines [Yıldırım and Koçak (2009), Berberler and Yıldırım (2009), Koçak and Yıldırım (2009), Abbasbandy (2006)]. The new homotopy perturbation method (NHPM) was applied to linear and nonlinear ODEs [Aminikhah and Biazar (2009)] and integral equations [Aminikhah and Salahi (2009)]. In this article, the basic idea of the NHPM is introduced and its application in two heat transfer equations is studied. This numerical scheme is based upon the Taylor series expansion and, as we shall soon see, is capable of finding the exact solution of many nonlinear differential equations.

2. Basic ideas of the NHPM

To illustrate the basic ideas of this method, let us consider the following nonlinear differential equation

$$A(u(x)) - f(r(x)) = 0, \quad r(x) \in \Omega, \tag{1}$$

with the following boundary conditions

$$B\left(u(x), \frac{\partial u(x)}{\partial n}\right) = 0, \quad r(x) \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r(x))$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts, L and N, where L is a linear and N is a nonlinear operator. Therefore Eq. (1) can be rewritten as

$$L(u(x)) + N(u(x)) - f(r(x)) = 0 \tag{3}$$

By the homotopy technique, we construct a homotopy $U(r(x), p) : \Omega \times [0,1] \to \mathbb{R}$, which satisfies

$$H(U(x),p) = (1-p)[L(U(x)) - u_0(x)] + p[A(U(x)) - f(r(x))] = 0,\ p \in [0,1],\ r(x) \in \Omega, \tag{4}$$

or equivalently,

$$H(U(x),p) = L(U(x)) - L(u_0(x)) + pL(u_0(x)) + p[N(U(x)) - f(r(x))] = 0, \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, $u_0(x)$ is an initial approximation of solution of Equation (1). Clearly, we have from Equations (4) and (5)

$$H(U(x),0) = L(U(x)) - L(u_0(x)) = 0, \tag{6}$$

$$H(U(x),1) = A(U(x)) - f(r(x)) = 0. \tag{7}$$

According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solutions of Equations (4) and (5) can be represented as a power series in p as

$$U(x) = \sum_{n=0}^{\infty} p^n U_n . \tag{8}$$

Now let us write the Eq. (5) in the following form

$$L(U(x)) = u_0(x) + p[f(r(x)) - u_0(x) - N(U(x))]. \tag{9}$$

By applying the inverse operator, L^{-1} to both sides of Eq. (9), we have

$$U(x) = L^{-1}(u_0(x)) + p[L^{-1}(f(r(x))) - L^{-1}(u_0(x)) - L^{-1}N(U(x))]. \tag{10}$$

Suppose that the initial approximation of Eq. (1) has the form

$$u_0(x) = \sum_{n=0}^{\infty} a_n P_n(x), \tag{11}$$

where $a_0, a_1, a_2, \ldots$ are unknown coefficients and $P_0(x), P_1(x), P_2(x), \ldots$ are specific functions depending on the problem. Now by substituting (8) and (11) into the Equation (10), we get

$$\sum_{n=0}^{\infty} p^n U_n(x) = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right) + p\left[L^{-1}(f(r(x))) - L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right) - L^{-1}N\left(\sum_{n=0}^{\infty} p^n U_n(x)\right)\right]. \tag{12}$$

Comparing coefficients of terms with identical powers of p, leads to

$$
\begin{aligned}
&p^0 : U_0(x) = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right),\\
&p^1 : U_1(x) = L^{-1}\big(f(r(x))\big) - L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right) - L^{-1}N\big(U_0(x)\big),\\
&p^2 : U_2(x) = -L^{-1}N(U_0(x),U_1(x)),\\
&p^3 : U_2(x) = -L^{-1}N(U_0(x),U_1(x),U_2(x)),\\
&\ \vdots\\
&p^j : U_j(x) = -L^{-1}N(U_0(x),U_1(x),U_2(x),\ldots,U_{j-1}(x))\\
&\vdots
\end{aligned}
\tag{13}
$$

Now if we solve these equations in such a way that $U_1(x) = 0$, then Equation (13) results in $U_2(x) = U_3(x) = \cdots = 0$. Therefore the exact solution may be obtained as follows:

$$
u(x) = U_0(x) = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right).
$$

It is worthwhile to mention that if $f(r(x))$ and $u_0(x)$ are analytic at $x = x_0$, then their Taylor series defined as

$$
u_0(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,\ f(r(x)) = \sum_{n=0}^{\infty} a_n^* (x - x_0)^n,
$$

can be used in Equation (12), where $a_0^*, a_1^*, a_2^*, \ldots$ are known coefficients and $a_0, a_1, a_2, \ldots$ are unknown ones, which must be computed. To show the capability of the method, we apply the NHPM to some examples in the next section.

3. Applications

3.1. Cooling of a lumped system by combined convection and radiation

Consider the following problem of the combined convective–radioactive cooling of a lumped system [Aziz and Na (1984)]. Let the system have volume V, surface area A, density ρ, specific heat c, emissivity ε, and the initial temperature T_i. At $t = 0$, the system is exposed to an environment with convective heat transfer with the coefficient of h and the temperature T_a. The system also loses heat through radiation and the effective sink temperature is T_s. The corresponding governing equation of this cooling problem is as follows

$$
\begin{aligned}
&\rho c V \frac{\mathrm{d}T}{\mathrm{d}t} + hA(T - T_a) + \varepsilon \sigma A(T^4 - T_s^4) = 0,\\
&T(0) = T_i.
\end{aligned}
\tag{14}
$$

Under the transformations $\theta = \dfrac{T}{T_i}$, $\theta_a = \dfrac{T_a}{T_i}$, $\theta_s = \dfrac{T_s}{T_i}$, $\tau = \dfrac{hAt}{\rho cV}$ and $\varepsilon = \dfrac{E\sigma T_i^3}{h}$, Equation (14) can be written as

$$\begin{aligned}&\frac{d\theta}{d\tau} + (\theta - \theta_a) + \varepsilon(\theta^4 - \theta_s^4) = 0,\\ &\theta(0) = 1.\end{aligned} \tag{15}$$

For the sake of simplicity, we take $\theta_a = \theta_s = 0$. Therefore, we have

$$\begin{aligned}&\frac{d\theta}{d\tau} + \theta + \varepsilon\theta^4 = 0,\\ &\theta(0) = 1.\end{aligned} \tag{16}$$

The exact solution of above equation was found to be of the form

$$\frac{1}{3}\ln\frac{1+\varepsilon\theta^3}{(1+\varepsilon)\theta^3} = \tau .$$

Expanding $\theta(\tau)$, using Taylor expansion, about $\tau = 0$ gives

$$\begin{aligned}\theta(\tau) = {} & 1 + (-1-\varepsilon)\tau + \left(\frac{5}{2}\varepsilon + \frac{1}{2} + 2\varepsilon^2\right)\tau^2\\ & + \left(-\frac{7}{2}\varepsilon - \frac{1}{6} - 8\varepsilon^2 - \frac{14}{3}\varepsilon^3\right)\tau^3 + \left(\frac{85}{24}\varepsilon + \frac{35}{3}\varepsilon^4 + \frac{77}{3}\varepsilon^3 + \frac{1}{24} + \frac{35}{2}\varepsilon^2\right)\tau^4\\ & + \left(-\frac{341}{120}\varepsilon - \frac{245}{3}\varepsilon^4 - \frac{91}{3}\varepsilon^5 - \frac{1}{120} - \frac{455}{6}\varepsilon^3 - \frac{82}{3}\varepsilon^2\right)\tau^5 + \cdots .\end{aligned} \tag{17}$$

3.1.1. New Homotopy Perturbation Method

To solve Equation (16), by means of NHPM, we construct the following homotopy

$$(1-p)\left[\Theta'(\tau) - \theta_0(\tau)\right] + p\left[\Theta'(\tau) + \Theta(\tau) + \varepsilon\Theta^4(\tau)\right] = 0,$$

or

$$\Theta'(\tau) = \theta_0(\tau) - p\left[\theta_0(\tau) + \Theta(\tau) + \varepsilon\Theta^4(\tau)\right] \tag{18}$$

Applying the inverse operator, $L^{-1} = \int_0^{\tau}(\cdot)\,d\xi$ to the both sides of the above equation, we obtain

$$\Theta(t) = \Theta(0) + \int_0^t q_0(x)dx - p\left[\int_0^t \left[q_0(x) + \Theta(x) + \varepsilon\Theta^4(x)\right]dx\right] \tag{19}$$

Suppose the solution of Equation (19) to have the following form

$$\Theta(t) = \Theta_0(t) + p\Theta_1(t) + p^2\Theta_2(t) + \cdots, \tag{20}$$

where $\Theta_i(t)$ are unknown functions which should be determined. Substituting Equation (20) into Equation (19), collecting the same powers of p, and equating each coefficient of p to zero, results in

$$\begin{aligned}
p^0 &: \Theta_0(t) = \Theta(0) + \int_0^t q_0(x)dx, \\
p^1 &: \Theta_1(t) = -\int_0^t \left[q_0(x) + \Theta_0(x) + \varepsilon\Theta_0^4(x)\right]dx, \\
p^2 &: \Theta_2(t) = -\int_0^t \left[\Theta_1(x) + 4\varepsilon\Theta_0^3(x)\Theta_1(x)\right]dx, \\
&\vdots
\end{aligned} \tag{21}$$

Assuming $q_0(t) = \sum_{n=0}^{\infty} a_n P_n(t), P_k(t) = t^k, \Theta(0) = q(0)$, and solving the above equation for $\Theta_1(t)$ leads to the result

$$\begin{aligned}
\Theta_1(t) = {} & (1 - a_0 - \varepsilon)t + (-2\varepsilon a_0 - \tfrac{1}{2}a_1 - \tfrac{1}{2}a_0)t^2 \\
& + (-\tfrac{1}{3}a_2 - \tfrac{1}{6}a_1 - \tfrac{2}{3}\varepsilon a_1 - 2\varepsilon a_0^2)t^3 + (-\tfrac{1}{3}\varepsilon a_2 - \tfrac{3}{2}\varepsilon a_0 a_1 - \tfrac{1}{12}a_2 - \varepsilon a_0^3 - \tfrac{1}{4}a_3)t^4 \\
& + (-\tfrac{4}{5}\varepsilon a_0 a_2 - \tfrac{1}{5}\varepsilon a_0^4 - \tfrac{6}{5}\varepsilon a_1 a_0^2 - \tfrac{1}{5}a_4 - \tfrac{3}{10}\varepsilon a_1^2 - \tfrac{1}{5}\varepsilon a_3 - \tfrac{1}{20}a_3)t^5 + \cdots
\end{aligned}$$

Vanishing $\Theta_1(t)$ lets the coefficients $a_n (n = 1, 2, 3, \ldots)$ to take the following values

$$\begin{aligned}
a_0 &= -1 - \varepsilon, \\
a_1 &= 4\varepsilon^2 + 5\varepsilon + 1, \\
a_2 &= -24\varepsilon^2 - \frac{21}{2}\varepsilon - \frac{1}{2} - 14\varepsilon^3, \\
a_3 &= \frac{140}{3}\varepsilon^4 + \frac{308}{3}\varepsilon^3 + 70\varepsilon^2 + \frac{85}{6}\varepsilon + \frac{1}{6}, \\
a_4 &= -\frac{341}{24}\varepsilon - \frac{1225}{3}\varepsilon^4 - \frac{455}{3}\varepsilon^5 - \frac{1}{24} - \frac{2275}{6}\varepsilon^3 - \frac{410}{3}\varepsilon^2, \\
&\vdots
\end{aligned}$$

Therefore, we gain the solution of Equation (16) as

$$q(t) = Q_0(t) = 1 + (-1 - e)t + \frac{1}{2}(4e^2 + 5e + 1)t^2$$
$$+ \frac{1}{3}(-24e^2 - \frac{21}{2}e - \frac{1}{2} - 14e^3)t^3 + \frac{1}{4}(\frac{140}{3}e^4 + \frac{308}{3}e^3 + 70e^2 + \frac{85}{6}e + \frac{1}{6})t^4 + L$$

and this in the limit of infinitely many terms, yields the exact solution of (17).

3.2. Cooling of a Lumped System with Variable Specific Heat

Consider the cooling of a lumped system [Y'aziz and Hamad (1977)] exposed to a convective environment at temperature T_a with convective heat transfer coefficient h at time $t = 0$. Let the system have volume V, surface area A, density r, specific heat c and initial temperature T_i. Assume that the specific heat c is a linear function temperature of the form

$$c = c_a[1 + b(T - T_a)], \tag{22}$$

herein c_a is the specific heat, at temperature T_a and b is a constant. The cooling equation corresponding to this problem is

$$\begin{aligned} &rcV\frac{dT}{dt} + hA(T - T_a) = 0, \\ &T(0) = T_i. \end{aligned} \tag{23}$$

Introducing the dimensionless parameters $q = \dfrac{T - T_a}{T_i - T_a}$, $t = \dfrac{hAt}{rcV}$, $e = b(T - T_a)$, Equation (23) can be transformed to the following equation

$$(1 + eq)\frac{dq}{dt} + q = 0, \quad q(0) = 1. \tag{24}$$

The Taylor expansion of the exact solution of Equation (24) about $t = 0$ can be readily obtained using software Maple as

$$\begin{aligned} q(t) = 1 - \frac{t}{1+e} + \frac{t^2}{2(1+e)^3} + \frac{(-1+2e)t^3}{6(1+e)^5} \\ + \frac{(1 - 8e + 6e^2)t^4}{24(1+e)^7} + \frac{(-1 + 22e - 58e^2 + 24e^3)t^5}{120(1+e)^9} + L\ . \end{aligned} \tag{25}$$

3.2.1. New Homotopy Perturbation Method

To solve Eq. (16), by means of NHPM, we construct the following homotopy

$$(1-p)\left[Q'(t)-q_0(t)\right]+p\left[Q'(t)+Q(t)+\varepsilon QQ'(t)\right]=0,$$

or

$$Q'(t)=q_0(t)-p\left[q_0(t)+Q(t)+\varepsilon Q(t)Q'(t)\right] \tag{26}$$

Applying the inverse operator, $L^{-1}=\int_0^t(\cdot)dx$ to the both sides of the above equation, we obtain

$$Q(t)=Q(0)+\int_0^t q_0(x)dx-p\left[\int_0^t\left[q_0(x)+Q(x)+\varepsilon QQ'(z)\right]dx\right] \tag{27}$$

Suppose the solution of Equation (27) to have the following form

$$Q(t)=Q_0(t)+pQ_1(t)+p^2Q_2(t)+\cdots, \tag{28}$$

where $Q_i(t)$ are unknown functions which should be determined. Substituting Equation (28) into Equation (27), collecting the same powers of p, and equating each coefficient of p to zero, results in

$$\begin{aligned}
&p^0: Q_0(t)=Q(0)+\int_0^t q_0(x)dx,\\
&p^1: Q_1(t)=-\int_0^t\left[q_0(x)+Q_0(x)+\varepsilon Q_0Q_0'(x)\right]dx,\\
&p^2: Q_2(t)=-\int_0^t\left[Q_1(x)+\varepsilon Q_0Q_1'(x)+\varepsilon Q_1Q_0'(x)\right]dx,\\
&\vdots
\end{aligned} \tag{29}$$

Assuming $q_0(t)=\sum_{n=0}^{\infty}a_nP_n(t), P_k(t)=t^k, Q(0)=q(0)$, and solving the above equation for $Q_1(t)$ leads to the result

$$\begin{aligned}
Q_1(t)=&(1-a_0-a_0\varepsilon)t-\frac{1}{2}(a_1\varepsilon+a_1+a_0+a_0^2\varepsilon)t^2\\
&-(\frac{1}{3}a_2\varepsilon+\frac{1}{2}a_0a_1\varepsilon+\frac{1}{3}a_2+\frac{1}{6}a_1)t^3-(\frac{1}{4}a_3+\frac{1}{4}a_3\varepsilon+\frac{1}{8}a_1^2\varepsilon+\frac{1}{12}a_2+\frac{1}{3}a_0a_2\varepsilon)t^4\\
&-(\frac{1}{20}a_3+\frac{1}{5}a_4+\frac{1}{4}a_0a_3\varepsilon+\frac{1}{6}a_1a_2\varepsilon+\frac{1}{5}a_4\varepsilon)t^5+\cdots
\end{aligned}$$

Vanishing $Q_1(t)$ lets the coefficients $a_n(n=1,2,3,\ldots)$ to take the following values

$$a_0=-\frac{1}{1+\varepsilon},\ a_1=\frac{1}{(1+\varepsilon)^3},\ a_2=\frac{-1+2\varepsilon}{2(1+\varepsilon)^5},\ a_3=\frac{1-8\varepsilon+6\varepsilon^2}{6(1+\varepsilon)^7},a_4=\frac{-1+22\varepsilon-58\varepsilon^2+24\varepsilon^3}{24(1+\varepsilon)^9},\cdots$$

Therefore, we gain the solution of Equation (24) as

$$q(t) = Q_0(t) = 1 - \frac{t}{1+e} + \frac{t^2}{2(1+e)^3} + \frac{(-1+2e)t^3}{6(1+e)^5} + \frac{(1-8e+6e^2)t^4}{24(1+e)^7} + \frac{(-1+22e-58e^2+24e^3)t^5}{120(1+e)^9} + L,$$

which is exactly the same as the exact solution given by Equation (25). Figure 1 illustrates the variation of the obtained solution of Equation (24) over t for two values of e.

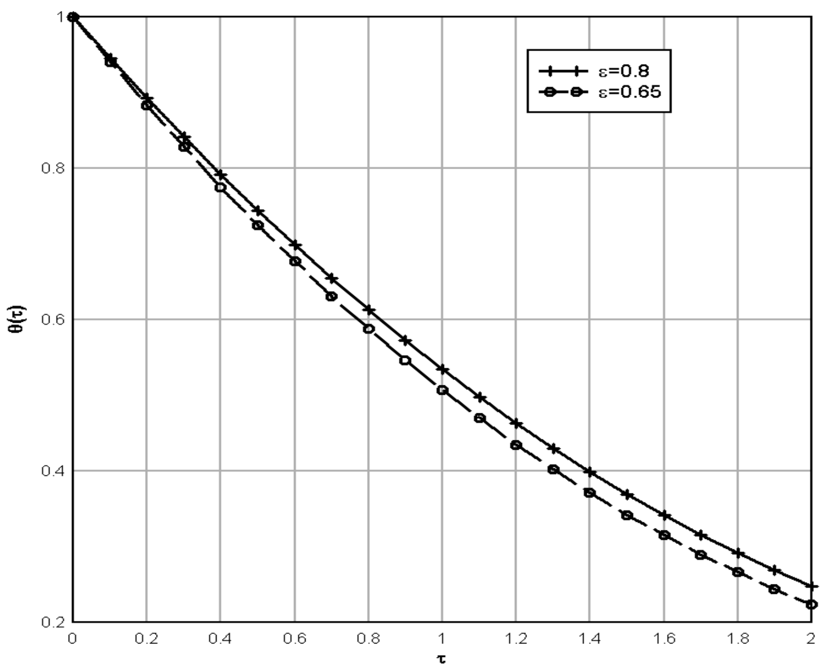

Figure 1. Variation of $q(t)$ over t for the second example.

4. Conclusion

A new homotopy perturbation method (NHPM) is successful in solving two nonlinear differential equations arising in heat transfer problems. In the HPM and VIM [Ganji and Sadighi (2007), Ganji (2006)], we reach a set of recurrent differential equations, which must be solved consecutively to give only an approximate solution of the problem. Further computations may be necessary for higher orders of approximation with a greater degree of accuracy. The new homotopy perturbation method, however, the level of accuracy attained in the first approximate solution $Q_1(t)$ is respectfully high. [The computations corresponding to the examples have been performed using Maple 10.]

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