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Exponential Chain Dual to Ratio cum Dual to Product Estimator for Finite Population Mean in Double Sampling Scheme

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Abstract

This paper considers an exponential chain dual to ratio cum dual to product estimator for estimating finite population mean using two auxiliary variables in double sampling scheme when the information on another additional auxiliary variable is available along with the main auxiliary variable. The expressions for bias and mean square error of the asymptotically optimum estimator are identified in two different cases. The optimum value of the first phase and second phase sample size has been obtained for the fixed cost of survey. To illustrate the results, theoretical and empirical studies have also been carried out to judge the merits of the suggested estimator with respect to strategies which utilized the information on two auxiliary variables.

Key words: Auxiliary variate; bias; mean square error (MSE); efficiency

MSC 2010 No.: 62D05, 94A20

1. Introduction

The use of an auxiliary variable x at the estimation stage improves the precision of an estimate of the population mean of a character y under study. Using the information on the auxiliary variable x, we often use classical ratio and product estimators depending upon the condition $\rho_{yx} > C_x / 2C_y$ and $\rho_{yx} < -C_x / 2C_y$ respectively, where C_y and C_x denote the coefficients of variation of the variable y and x and ρ_{yx} is the correlation coefficient between y and x. However, in many situations of practical importance, the population mean $\bar{X}$ is not known before the start of a survey. In such a situation, the usual thing to do is to estimate it by the sample mean $\bar{x}_1$

based on a preliminary sample of size n_1 of which n is a subsample $(n < n_1)$. If the population mean $\bar{Z}$ of another auxiliary variable z, closely related to x but compared to x remotely related to y is known, it is advisable to estimate $\bar{X}$ by $\bar{X} = \bar{x}_1\bar{Z}/\bar{z}_1$, which would provide better estimate of $\bar{X}$ than $\bar{x}_1$ to the terms of order $o\left(n^{-1}\right)$ if $\rho_{xz} C_x/C_z > 1/2$.

Sukhatme and Chand (1977) proposed a technique of chaining the available information on auxiliary characteristics with the main characteristics. Kiregyera (1984) also proposed some chain type ratio and regression estimators based on two auxiliary variables. Al-Jararha and Ahmed (2002) defined two classes of estimators by using prior information on parameter of one of the two auxiliary variables under double sampling scheme.

Consider a finite population $U = (U_1, U_2, ..., U_N)$ of N units, let y be the study variable, x and z are the two auxiliary variables. Let $\bar{X}$ be unknown, but $\bar{Z}$ the population mean of another cheaper auxiliary variable z closely related to x but compared to x remotely related to y (i.e., $\rho_{yx} > \rho_{yz}$) is known. In this case, Chand (1975) defined the chain ratio estimator

$$\bar{y}_R^{(c)} = \bar{y}\frac{\bar{x}_1\bar{Z}}{\bar{x}\,\bar{z}_1},$$

where $\bar{x}$ and $\bar{y}$ are the sample means of x and y, respectively based on a sample size n out of the population N units and

$$\bar{x}_1 = \left(1/n_1\right)\sum_{i=1}^{n_1} x_i,$$

denotes the sample mean of x based on the first-phase sample of the size n_1 and

$$\bar{z}_1 = \left(1/n_1\right)\sum_{i=1}^{n_1} z_i,$$

denote the sample mean based on n_1 units of the auxiliary variable z.

Using the transformation

$$\bar{x}_i^{\sigma} = \frac{\left(N\bar{X} - nx_i\right)}{\left(N-n\right)}, \left(i = 1, 2, 3, ..., N\right).$$

Srivenkataramana (1980) proposed dual to ratio estimator to estimate population mean as

$$\bar{y}_R = \bar{y}\frac{\bar{x}^{\sigma}}{\bar{X}},$$

where

$$\bar{x}^{\sigma} = \frac{\left(N\bar{X} - nx\right)}{\left(N - n\right)}.$$

Bandyopadhyay (1980) obtained dual to product estimator as

$$\bar{y}_P = \bar{y}\frac{\bar{X}}{\bar{x}^{\sigma}}.$$

Kumar *et al.* (2006) used the transformation

$$\bar{x}_i^{\sigma} = \frac{\left(n_1\bar{X} - nx_i\right)}{\left(n_1 - n\right)}, \left(i = 1, 2, 3, ..., N\right),$$

and proposed a dual to ratio estimator in double sampling as

$$\bar{y}_R^d = \bar{y}\frac{\bar{x}^{\tau}}{\bar{X}},$$

where

$$\bar{x}^{\tau} = \frac{\left(n_1\bar{x}_1 - n\bar{x}\right)}{\left(n_1 - n\right)},$$

and is an unbiased estimator of $\bar{X}$.

Singh and Choudhury (2012) proposed dual to product estimator for estimating population mean in double sampling as

$$\bar{y}_P^d = \bar{y}\frac{\bar{X}}{\bar{x}^{\tau}}.$$

Using an additional auxiliary variable z in dual to ratio and product estimators in double sampling of Kumar *et al.* (2006) and Singh and Choudhury (2012) estimators converts to chain dual to ratio and chain dual to product estimators in double sampling as

$$\bar{y}_{dR}^c = \bar{y}\left[\frac{n_1\frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{\left(n_1 - n\right)\frac{\bar{x}_1}{\bar{z}_1}\bar{Z}}\right]$$

and

$$\bar{y}_{dP}^{c} = \bar{y}\left[\frac{\left(n_1 - n\right)\frac{\bar{x}_1}{\bar{z}_1}\bar{Z}}{n_1\frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}\right],$$

respectively.

The chain linear regression estimator in double sampling suggested by Kiregyera (1984) is given as

$$\bar{y}_{reg.}^{c} = \bar{y} + b_{yx}\left[\bar{x}_1 + b_{xz}\left(\bar{Z} - \bar{z}'\right) - \bar{x}\right],$$

where b_{yx} and b_{xz} are the regression coefficients of y on x and x on z, respectively.

Bahl and Tuteja (1991) suggested the exponential ratio and product type estimators as

$$\bar{y}_{\mathrm{Re}} = \bar{y}\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right),$$

and

$$\bar{y}_{Pe} = \bar{y}\exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right),$$

for the population mean $\bar{Y}$.

Saini and Kumar (2015) also proposed some exponential type product estimator for finite population mean with information on auxiliary attribute.

Singh and Vishwakarma (2007) suggested an exponential ratio and product-type estimators for Y in double sampling as

$$\bar{y}_{\mathrm{Re}}^{d} = \bar{y}\exp\left(\frac{\bar{x}_1 - \bar{x}}{\bar{x}_1 + \bar{x}}\right),$$

and

$$\bar{y}_{\mathrm{Pe}}^{d} = \bar{y}\exp\left(\frac{\bar{x} - \bar{x}_1}{\bar{x} + \bar{x}_1}\right).$$

Singh and Choudhury (2012) suggested the exponential chain ratio and product type estimators under double sampling scheme as

$$\bar{y}_{\mathrm{Re}}^{c} = \bar{y}\exp\left(\frac{\bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1} - \bar{x}}{\bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1} + \bar{x}}\right),$$

and

$$\bar{y}_{P\mathrm{e}}^{c} = \bar{y}\exp\left(\frac{\bar{x} - \bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1}}{\bar{x} + \bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1}}\right).$$

Singh et al. (2013) again generalized the above estimators to a class of exponential chain ratio-product type estimator in double sampling scheme as

$$\bar{y}_{\mathrm{R}P\mathrm{e}}^{c} = \bar{y}\left[\alpha\exp\left(\frac{\bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1} - \bar{x}}{\bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1} + \bar{x}}\right) + \beta\exp\left(\frac{\bar{x} - \bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1}}{\bar{x} + \bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1}}\right)\right],$$

where α and β are unknown constants such that $\alpha + \beta = 1$. Singh et al. (2014) proposed exponential ratio cum exponential dual to ratio estimator in double sampling.

Motivated by Singh et al. (2013), Srivenkataramana (1980) and Bandyopadhyay (1980) and with an aim to provide a more efficient estimator, we have proposed a class of exponential chain dual to ratio cum dual to product estimator in double sampling for estimating population mean $\bar{Y}$ using two auxiliary characters. Throughout the paper simple random sampling without replacement (SRSWOR) scheme has been considered. Theoretical comparisons of the proposed estimator are carried out to demonstrate the performance of the suggested estimator over others and empirical studies have also been carried out in the support of the present study.

2. The Proposed Estimator

Let instead of $\bar{X}$, the population mean $\bar{Z}$ of another auxiliary variable z, which has a positive correlation with x (i.e., $\rho_{xz} > 0$) be known. We further assume $\rho_{yx}, \rho_{yz} > 0$. Let $\bar{x}_1$ and $\bar{z}_1$ be the sample means of x and z respectively based on a preliminary sample of size n_1 drawn from the population of size N with simple random sampling without replacement strategy in order to get an estimate of $\bar{X}$. Then, we suggest a class of estimators for $\bar{Y}$ as

$$\bar{y}^{c}_{d\mathrm{R}Pe} = \bar{y}\left[\alpha \exp\left\{\frac{\left(\frac{n_1 \frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{n_1 - n}\right) - \frac{\bar{x}_1}{\bar{z}_1}\bar{Z}}{\left(\frac{n_1 \frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{n_1 - n}\right) + \frac{\bar{x}_1}{\bar{z}_1}\bar{Z}}\right\} + (1-\alpha)\exp\left\{\frac{\frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - \left(\frac{n_1 \frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{n_1 - n}\right)}{\frac{\bar{x}_1}{\bar{z}_1}\bar{Z} + \left(\frac{n_1 \frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{n_1 - n}\right)}\right\}\right].$$

Let

$$\bar{y}^{c}_{d\mathrm{R}Pe} = \alpha T_1 + (1-\alpha) T_2, \tag{1}$$

where α is unknown constant to be determined and

$$T_1 = \bar{y}^{c}_{dRe} = \bar{y}\exp\left\{\frac{\left(\frac{n_1 \frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{n_1 - n}\right) - \frac{\bar{x}_1}{\bar{z}_1}\bar{Z}}{\left(\frac{n_1 \frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{n_1 - n}\right) + \frac{\bar{x}_1}{\bar{z}_1}\bar{Z}}\right\},$$

and

$$T_2 = \bar{y}^{c}_{dPe} = \bar{y}\exp\left\{\frac{\frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - \left(\frac{n_1 \frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{n_1 - n}\right)}{\frac{\bar{x}_1}{\bar{z}_1}\bar{Z} + \left(\frac{n_1 \frac{\bar{x}_1}{\bar{z}_1}\bar{Z} - n\bar{x}}{n_1 - n}\right)}\right\}.$$

To obtain the bias (B) and MSE of $\bar{y}^{c}_{d\mathrm{R}Pe}$, we write

$$\bar{y} = \bar{Y}\left(1+e_0\right),\ \bar{x} = \bar{X}\left(1+e_1\right),\ \bar{x}_1 = \bar{X}\left(1+e_2\right)$$

and

$$\bar{z}_1 = \bar{Z}\left(1+e_3\right).$$

Expressing $\bar{y}^c_{d\mathrm{RPe}}$ in terms of e's, we have

$$\begin{aligned}\bar{y}^c_{d\mathrm{RPe}} \cong \bar{y}\Bigg[\alpha\Bigg\{&1+\frac{1}{2}g\left(e_2-e_3-e_1-e_2^2+e_2e_3\right)\\ &-\frac{1}{8}g^2\left(e_2^2-2e_2e_3+e_3^2+2e_1e_3-2e_1e_2+e_1^2\right)\Bigg\}\\ &+\left(1-\alpha\right)\Bigg\{1+\frac{4}{8}g\left(e_1-e_2+e_3+e_2^2+e_1e_3-e_1e_2-e_2e_3\right)\\ &+\frac{3}{8}g^2\left(e_1^2+e_2^2+e_3^2+2e_1e_3-2e_1e_2-2e_2e_3\right)\Bigg\}\Bigg],\end{aligned}$$

or

$$\begin{aligned}\bar{y}^c_{d\mathrm{RPe}} &= \bar{y}\left[\alpha(1+i_1)+\left(1-\alpha\right)(1+i_2)\right]\\ &= \bar{Y}(1+e_0)\left[1+i_2+\alpha(i_1-i_2)\right],\end{aligned}$$

or

$$\bar{y}^c_{d\mathrm{RPe}} - \bar{Y} = \bar{Y}\left[e_0+\frac{1}{2}gi_3+\frac{3}{8}g^2i_4+\alpha gi_5+\alpha g^2i_6\right], \tag{2}$$

where

$$i_1 = \frac{1}{8}\left[4g\left(e_2-e_3-e_1-e_2^2+e_2e_3\right)-g^2\left(e_2^2-2e_2e_3+e_3^2+2e_1e_3-2e_1e_2+e_1^2\right)\right],$$

$$i_2 = \frac{1}{8}\left[4g\left(e_1-e_2+e_3+e_2^2+e_1e_3-e_1e_2-e_2e_3\right)+3g^2\left(e_1^2+e_2^2+e_3^2+2e_1e_3-2e_1e_2-2e_2e_3\right)\right],$$

$$i_3 = \left[e_1-e_2+e_3+e_2^2+e_1e_3-e_1e_2-e_2e_3+e_0e_1-e_0e_2+e_0e_3\right],$$

$$i_4 = \left[e_1^2+e_2^2+e_3^2+2e_1e_3-2e_1e_2-2e_2e_3\right],$$

$$i_5 = \left[e_2-e_3-e_1-e_2^2+e_2e_3+\frac{1}{2}e_1e_2-\frac{1}{2}e_1e_3+e_0e_2-e_0e_3-e_0e_1\right],$$

$$i_6 = \left[e_1e_2-e_1e_3+e_2e_3-\frac{1}{2}\left(e_1^2+e_2^2+e_3^2\right)\right],$$

$$g = \frac{n}{n_1-n}$$

and

$$(1+g)=\frac{n_1}{n_1-n}.$$

To obtain the bias (B) and MSE of $\bar{y}^{c}_{d\mathrm{R}Pe}$, let

$$C_y^2=S_y^2/\bar{Y}^2,\ C_x^2=S_x^2/\bar{X}^2,\ C_z^2=S_z^2/\bar{Z}^2,$$
$$C_{yx}=\frac{\rho_{yx}C_y}{C_x},\ \ C_{yz}=\frac{\rho_{yz}C_y}{C_z},\ C_{xz}=\frac{\rho_{xz}C_x}{C_z},$$
$$\rho_{xy}=S_{xy}/S_xS_y,\ \rho_{yz}=S_{yz}/S_yS_z$$

and

$$\rho_{zx}=S_{zx}/S_zS_x,$$

where

$$S_y^2=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(y_i-\bar{Y}\right)^2,\ \ S_x^2=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(x_i-\bar{X}\right)^2,$$
$$S_z^2=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(z_i-\bar{Z}\right)^2,S_{yx}=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(y_i-\bar{Y}\right)\left(x_i-\bar{X}\right),$$
$$S_{yz}=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(y_i-\bar{Y}\right)\left(z_i-\bar{Z}\right)$$

and

$$S_{zx}=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(x_i-\bar{X}\right)\left(z_i-\bar{Z}\right).$$

In this paper the following two cases will be considered separately.

Case I: When the second phase sample of size *n* is a subsample of the first phase sample of size n_1.

Case II: When the second phase sample of size *n* is drawn independently of the first phase sample of size n_1.

3. Case I

3.1. Bias, MSE and Optimum value of α for $\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)_I$

In case I, we get the following results

$$\left.\begin{array}{l}
\mathrm{E}(\mathrm{e}_0)=\mathrm{E}(\mathrm{e}_1)=\mathrm{E}(\mathrm{e}_2)=\mathrm{E}(\mathrm{e}_3)=0, \\
\mathrm{E}(\mathrm{e}_0^2)=\left(\frac{1}{n}-\frac{1}{N}\right)\mathrm{C}_{\mathrm{y}}^2=\mathrm{A}_{0,0}, \quad \mathrm{E}(\mathrm{e}_1^2)=\left(\frac{1}{n}-\frac{1}{N}\right)\mathrm{C}_{\mathrm{x}}^2=A_{1,1}, \\
\mathrm{E}(\mathrm{e}_2^2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)\mathrm{C}_{\mathrm{x}}^2=A_{2,2}, \mathrm{E}(\mathrm{e}_3^2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)\mathrm{C}_{\mathrm{z}}^2=A_{3,3}, \\
\mathrm{E}(\mathrm{e}_0\mathrm{e}_1)=\left(\frac{1}{n}-\frac{1}{N}\right)C_{yx}C_x^2=A_{0,1}, \mathrm{E}(\mathrm{e}_0\mathrm{e}_2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_{yx}C_x^2=A_{0,2}, \\
\mathrm{E}(\mathrm{e}_0\mathrm{e}_3)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_{yz}C_z^2=A_{0,3}, \mathrm{E}(\mathrm{e}_1\mathrm{e}_2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)\mathrm{C}_{\mathrm{x}}^2=A_{1,2}, \\
\mathrm{E}(\mathrm{e}_1\mathrm{e}_3)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_{xz}C_z^2=A_{1,3}, \mathrm{E}(\mathrm{e}_2\mathrm{e}_3)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_{xz}C_z^2=A_{2,3}.
\end{array}\right\} \tag{3}$$

Taking expectations in (2) and using the results of (3), we get the bias of $\left(\bar{y}_{d\mathrm{RPe}}^c\right)_I$ to the first order of approximation as

$$\begin{aligned}
B(\bar{y}_{d\mathrm{RPe}}^c)_I = \bar{Y}g\Bigg[&\frac{1}{2}\left\{A_{1,3}-A_{2,2}-A_{1,3}+A_{2,2}+A_{0,1}-A_{0,2}+A_{0,3}\right\} \\
&+\frac{3}{8}g\left\{A_{1,1}+A_{2,2}+A_{3,3}-2A_{2,2}-2A_{1,3}+2A_{1,3}\right\} \\
&+\alpha\left\{A_{2,3}-A_{2,2}+\frac{1}{2}A_{2,2}-\frac{1}{2}A_{2,3}+A_{0,2}-A_{0,3}-A_{0,1}\right\} \\
&+\alpha g\left\{-\frac{1}{2}A_{2,2}+A_{2,3}-\frac{1}{2}A_{3,3}-A_{1,3}+A_{2,2}-\frac{1}{2}A_{1,1}\right\}\Bigg],
\end{aligned}$$

or

$$B(\bar{y}_{d\mathrm{RPe}}^c)_I = \bar{Y}g\left[\left(\frac{1}{2}-\alpha\right)Q-\frac{3}{8}gP-\frac{\alpha}{2}\left(A_{2,2}+A_{2,3}\right)-\frac{\alpha gP}{2}\right], \tag{4}$$

where

$$P = A_{1,1}-A_{2,2}+A_{3,3}$$

and

$$Q = A_{0,1}-A_{0,2}+A_{0,3}.$$

Again from Equation (2), we have

$$\bar{y}^{c}_{d\mathrm{R}Pe} - \bar{Y} = \bar{Y}\left[e_0 + \frac{1}{2} g\left(e_1 - e_2 + e_3\right) + \alpha g\left(e_2 - e_3 - e_1\right)\right] . \tag{5}$$

Squaring both the sides of Equation (5), taking expectations and using the results of (3), we obtain the MSE of the estimator $\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)_I$ to the first order of approximation as

$$MSE\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)_I = \bar{Y}^2\left[A_{0,0} + g^2\left(\frac{1}{2} - \alpha\right)^2 \left\{A_{1,1} + A_{2,2} + A_{3,3} - 2A_{1,2} - 2A_{2,3} + 2A_{1,3}\right\} \right.$$
$$\left. + g\left\{A_{0,1} - A_{0,2} + A_{0,3}\right\} - 2\alpha g\left\{A_{0,1} - A_{0,2} + A_{0,3}\right\}\right],$$

or

$$MSE\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)_I = \bar{Y}^2\left[A_{0,0} + g^2\left(\frac{1}{2} - \alpha\right)^2 P + gQ - 2\alpha gQ\right], \tag{6}$$

The MSE of $\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)_I$ is minimum when

$$\alpha = \frac{2gQ + g^2 P}{2g^2 P} = \alpha_{Iopt.}(say) . \tag{7}$$

Putting the value of α from (7) in (1) yields the 'asymptotically optimum estimator' (AOE) as

$$\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)_{I(opt)} = \alpha_{I(opt)} T_1 + \left(1 - \alpha_{I(opt)}\right) T_2 ,$$

Thus, the resulting MSE of $\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)_{I(opt)}$ is given by

$$MSE\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)_{I(opt)} = \bar{Y}^2\left[A_{0,0} - \frac{Q^2}{P}\right] . \tag{8}$$

Remarks:

1. When $\alpha = 0$, the estimator $\left(\bar{y}^{c}_{d\mathrm{R}Pe}\right)$ in (1) reduces to the exponential chain dual to product estimator $\left(\bar{y}^{c}_{dPe}\right)_I$ in double sampling. The bias and MSE of $\left(\bar{y}^{c}_{dPe}\right)_I$ is obtained by putting $\alpha = 0$ in (4) and (6), respectively as

$$B(\bar{y}^{c}_{dPe})_I = \bar{Y} g\left[\frac{1}{2} Q - \frac{3}{8} gP\right]$$

and

$$MSE\left(\bar{y}_{dPe}^{c}\right)_{I} = \bar{Y}^{2}\left[A_{0,0} + \frac{g^{2}P}{4} + gQ\right] . \tag{9}$$

2. When $\alpha = 1$, the estimator $\left(\bar{y}_{d\mathrm{R}Pe}^{c}\right)$ in (1) reduces to the exponential chain dual to ratio estimator $\left(\bar{y}_{dRe}^{c}\right)_{I}$ in double sampling. The bias and MSE of $\left(\bar{y}_{dRe}^{c}\right)_{I}$ is obtained by putting $\alpha = 1$ in (4) and (6) respectively as

$$B(\bar{y}_{d\mathrm{Re}}^{c})_{I} = -\bar{Y}g\left[\frac{1}{2}\left(A_{2,2} + A_{2,3} + Q\right) + \frac{7gP}{8}\right]$$

and

$$MSE\left(\bar{y}_{d\mathrm{Re}}^{c}\right)_{I} = \bar{Y}^{2}\left[A_{0,0} + \frac{g^{2}P}{4} - gQ\right] . \tag{10}$$

4. Efficiency Comparisons in Case I

4.1. Comparison with sample mean per unit estimator $\bar{y}$

The variance of usual unbiased estimator $\bar{y}$ is given by

$$V\left(\bar{y}\right) = \bar{Y}^{2}\left(\frac{1}{n} - \frac{1}{N}\right)C_{y}^{2} . \tag{11}$$

From (8) and (11), we have

$$V\left(\bar{y}\right) - MSE\left(\bar{y}_{d\mathrm{R}Pe}^{c}\right)_{I(opt)} = \bar{Y}^{2}\frac{Q^{2}}{P} > 0, \tag{12}$$

if

$$A_{1,1} + A_{3,3} > A_{2,2},$$

where

$$P = A_{1,1} - A_{2,2} + A_{3,3}$$

and

$$Q = A_{0,1} - A_{0,2} + A_{0,3} .$$

4.2. Comparison with the chain dual to ratio estimator $\left(\bar{y}_{dR}^{c}\right)_I$

The MSE of the chain dual to ratio estimator $\left(\bar{y}_{dR}^{c}\right)_I$ is given by

$$MSE\left(\bar{y}_{dR}^{c}\right)_I = \bar{Y}^2\left[A_{0,0} - 2gQ + g^2P\right]. \tag{13}$$

From (8) and (13), we have

$$MSE\left(\bar{y}_{dR}^{c}\right)_I - MSE\left(\bar{y}_{d\mathrm{R}P\mathrm{e}}^{c}\right)_{I(opt)} = \frac{\bar{Y}^2}{P}\left(Q - gP\right)^2 > 0. \tag{14}$$

4.3. Comparison with the chain dual to product estimator $\left(\bar{y}_{dP}^{c}\right)_I$

The MSE of the chain dual to product estimator $\left(\bar{y}_{dP}^{c}\right)_I$ is given by

$$MSE\left(\bar{y}_{dP}^{c}\right)_I = \bar{Y}^2\left[A_{0,0} + 2gQ + g^2P\right]. \tag{15}$$

From (8) and (15), we have

$$MSE\left(\bar{y}_{dP}^{c}\right)_I - MSE\left(\bar{y}_{d\mathrm{R}P\mathrm{e}}^{c}\right)_{I(opt)} = \frac{\bar{Y}^2}{P}\left(Q + gP\right)^2 > 0. \tag{16}$$

4.4. Comparison with the exponential chain dual to ratio estimator $\left(\bar{y}_{d\mathrm{Re}}^{c}\right)_I$

From (8) and (10), we have

$$MSE\left(\bar{y}_{d\mathrm{Re}}^{c}\right)_I - MSE\left(\bar{y}_{d\mathrm{R}P\mathrm{e}}^{c}\right)_{I(opt)} = \bar{Y}^2\frac{\left(gP - 2Q\right)^2}{4P} > 0. \tag{17}$$

4.5. Comparison with the exponential chain dual to product estimator $\left(\bar{y}_{dPe}^{c}\right)_I$

From (8) and (9), we have

$$MSE\left(\bar{y}_{dPe}^{c}\right)_I - MSE\left(\bar{y}_{d\mathrm{R}P\mathrm{e}}^{c}\right)_{I(opt)} = \bar{Y}^2\frac{\left(gP + 2Q\right)^2}{4P} > 0. \tag{18}$$

4.6. Comparison with the chain linear regression estimator $\left(\bar{y}_{reg.}^{c}\right)_I$

The MSE of the chain linear regression estimator $\left(\bar{y}^{c}_{reg.}\right)_I$ is given by

$$MSE\left(\bar{y}^{c}_{reg.}\right)_I = \bar{Y}^2\left[\left(\frac{1}{n}-\frac{1}{N}\right)\left(C_y^2 - C_{yx}^2 C_x^2\right)+\left(\frac{1}{n_1}-\frac{1}{N}\right)\left\{C_{yx}^2 C_x^2 + C_{yx}C_{xz}C_z^2\left(C_{yx}C_{xz}-2C_{yz}\right)\right\}\right]. \tag{19}$$

From (8) and (19), we have

$$MSE\left(\bar{y}^{c}_{reg.}\right)_I - MSE\left(\bar{y}^{c}_{d\mathrm{R}P\mathrm{e}}\right)_{I(opt)} = \bar{Y}^2\left[-\left(\frac{1}{n}-\frac{1}{n_1}\right)C_{yx}^2 C_x^2 + \left(\frac{1}{n_1}-\frac{1}{N}\right) C_{yx}C_{xz}C_z^2\left(C_{yx}C_{xz}-2C_{yz}\right)\right\}+\frac{Q^2}{P}\right] > 0, \tag{20}$$

if

$$\left(\frac{1}{n}-\frac{1}{n_1}\right)C_{yx}^2 C_x^2 < \left(\frac{1}{n_1}-\frac{1}{N}\right)\left\{C_{yx}C_{xz}C_z^2\left(C_{yx}C_{xz}-2C_{yz}\right)\right\}.$$

Now, we state the following theorem:

Theorem.

To the first order of approximation, the proposed strategy under optimality condition (7) is always more efficient than $V(\bar{y})$, $MSE\left(\bar{y}^{c}_{dR}\right)_I$, $MSE\left(\bar{y}^{c}_{dP}\right)_I$, $MSE\left(\bar{y}^{c}_{d\mathrm{R}e}\right)_I$, $MSE\left(\bar{y}^{c}_{dPe}\right)_I$ and $MSE\left(\bar{y}^{c}_{reg.}\right)_I$.

5. Case II

5.1. Bias, MSE and Optimum Value of α for $\left(\bar{y}^{c}_{d\mathrm{R}P\mathrm{e}}\right)_{II}$

In case II, we have

$$\left.\begin{array}{l}
E(e_0)=E(e_1)=E(e_2)=E(e_3)=0\ , \\
E(e_0^2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_y^2=A_{0,0}, \quad E(e_1^2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_x^2=A_{1,1}, \\
E(e_2^2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_x^2=A_{2,2},\ E(e_3^2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_z^2=A_{3,3}, \\
E(e_0e_1)=\left(\frac{1}{n}-\frac{1}{N}\right)C_{yx}C_x^2=A_{0,1}, E(e_2e_3)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_{xz}C_z^2=A_{2,3}, \\
E(e_0e_2)=E(e_0e_3)=E(e_1e_2)=E(e_1e_3)=0\ .
\end{array}\right\} \tag{21}$$

Taking expectations in (2) and using the results of (21), we obtain the bias of $\left(\bar{y}_{d\mathrm{RPe}}^{c}\right)_{II}$ up to the terms of order n^{-1} as

$$\begin{aligned}
B(\bar{y}_{d\mathrm{RPe}}^{c})_{II} = \bar{Y}g\Bigg[&\frac{1}{2}\left\{A_{0,1}-A_{0,2}+A_{0,3}\right\}+\frac{3}{8}g\left\{A_{1,1}-A_{2,2}+A_{3,3}\right\} \\
&+\alpha\left\{\frac{1}{2}A_{2,3}-\frac{1}{2}A_{2,2}+A_{0,2}-A_{0,3}-A_{0,1}\right\} \\
&+\alpha g\left\{\frac{1}{2}A_{2,2}-\frac{1}{2}A_{3,3}-\frac{1}{2}A_{1,1}\right\}\Bigg],
\end{aligned}$$

or

$$B\left(\bar{y}_{d\mathrm{RPe}}^{c}\right)_{II} = \bar{Y}g\left[\frac{A_{0,1}}{2}+\frac{3gP}{8}+\frac{\alpha}{2}\left(2A_{0,1}-A_{2,2}+A_{2,3}\right)-\frac{\alpha gP}{2}\right]. \tag{22}$$

Since the population size N is large as compared to the sample sizes n and n_1 so the finite population correction (FPC) terms 1/N and 2/N are ignored.

Ignoring the FPC in (22), the bias of $\left(\bar{y}_{d\mathrm{RPe}}^{c}\right)_{II}$ is given by

$$B\left(\bar{y}_{d\mathrm{RPe}}^{c}\right)_{II} = \bar{Y}g\left[\frac{A'_{0,1}}{2}+\frac{3gP'}{8}+\frac{\alpha}{2}\left(2A'_{0,1}-A'_{2,2}+A'_{2,3}\right)-\frac{\alpha gP'}{2}\right], \tag{23}$$

where

$$P' = A'_{1,1}-A'_{2,2}+A'_{3,3}$$

and

$$\left.\begin{aligned} &A'_{0,0}=\left(\frac{1}{n}\right)C_y^2,\ A'_{1,1}=\left(\frac{1}{n}\right)C_x^2,\ A'_{2,2}=\left(\frac{1}{n_1}\right)C_x^2,\ A'_{3,3}=\left(\frac{1}{n_1}\right)C_z^2,\\ &A'_{0,1}=\left(\frac{1}{n}\right)C_{yx}C_x^2,\ A'_{2,3}=\left(\frac{1}{n_1}\right)C_{xz}C_z^2.\end{aligned}\right\} \tag{24}$$

Squaring both the sides in (5), taking expectations and using the results from (24), we obtain the MSE of the estimator $\left(\bar{y}^c_{d\mathrm{R}Pe}\right)_{II}$ to the first order of approximation as

$$MSE\left(\bar{y}^c_{d\mathrm{R}Pe}\right)_{II}=\bar{Y}^2\left[A'_{0,0}+g^2\left(\frac{1}{2}-\alpha\right)^2\left(A'_{1,1}-A'_{2,2}+A'_{3,3}\right)+gA'_{0,1}-2\alpha gA'_{0,1}\right],$$

or

$$MSE\left(\bar{y}^c_{d\mathrm{R}Pe}\right)_{II}=\bar{Y}^2\left[A'_{0,0}+g^2\left(\frac{1}{2}-\alpha\right)^2P'+gA'_{0,1}-2\alpha gA'_{0,1}\right]. \tag{25}$$

Differentiation of (25) with respect to α yields its optimum value as

$$\alpha=\frac{A'_{0,1}}{gP'}=\alpha_{IIopt.}(say). \tag{26}$$

Thus, the resulting optimum MSE of $\left(\bar{y}^c_{d\mathrm{R}Pe}\right)_{II(opt)}$ is given by

$$MSE\left(\bar{y}^c_{d\mathrm{R}Pe}\right)_{II(opt)}=\bar{Y}^2\left[A'_{0,0}-\frac{\left(A'_{0,1}\right)^2}{P'}\right]. \tag{27}$$

Remarks:

1. When $\alpha=0$, the estimator $\left(\bar{y}^c_{d\mathrm{R}Pe}\right)$ in (1) reduces to the exponential chain dual to product estimator $\left(\bar{y}^c_{dPe}\right)_{II}$ in double sampling. The MSE of $\left(\bar{y}^c_{dPe}\right)_{II}$ is obtained by putting $\alpha=0$ in (25) as

$$MSE\left(\bar{y}^c_{dPe}\right)_{II}=\bar{Y}^2\left[A'_{0,0}+\frac{g^2P'}{4}+gA'_{0,1}\right]. \tag{28}$$

2. When $\alpha=1$, the estimator $\left(\bar{y}^c_{d\mathrm{R}Pe}\right)$ in (1) reduces to the exponential chain dual to ratio estimator $\left(\bar{y}^c_{d\mathrm{R}e}\right)_{II}$ in double sampling. Thus by putting $\alpha=1$ in (25), we obtain the MSE of $\left(\bar{y}^c_{d\mathrm{R}e}\right)_{II}$ to the first order of approximation as

$$MSE\left(\bar{y}_{d\mathrm{Re}}^{c}\right)_{II} = \bar{Y}^{2}\left[A'_{0,0} + \frac{g^{2}P'}{4} - gA'_{0,1}\right]. \tag{29}$$

6. Efficiency Comparisons in Case II

6.1. Comparison with sample mean per unit estimator $\bar{y}$

The variance of usual unbiased estimator $\bar{y}$ is given by

$$V\left(\bar{y}\right) = \bar{Y}^{2}A'_{0,0}. \tag{30}$$

From (27) and (30), we have

$$V\left(\bar{y}\right) - MSE\left(\bar{y}_{d\mathrm{R}Pe}^{c}\right)_{II(opt)} = \bar{Y}^{2}\frac{\left(A'_{0,1}\right)^{2}}{P'} > 0 \ , \tag{31}$$

if

$$A'_{1,1} + A'_{3,3} > A'_{2,2}\,,$$

where

$$P' = A'_{1,1} - A'_{2,2} + A'_{3,3}\,.$$

6.2. Comparison with the chain dual to ratio estimator $\left(\bar{y}_{dR}^{c}\right)_{II}$

The MSE of the chain dual to ratio estimator $\left(\bar{y}_{dR}^{c}\right)_{II}$ is given by

$$MSE\left(\bar{y}_{dR}^{c}\right)_{II} = \bar{Y}^{2}\left[A'_{0,0} - 2gA'_{0,1} + g^{2}P'\right]. \tag{32}$$

From (27) and (32), we have

$$MSE\left(\bar{y}_{dR}^{c}\right)_{II} - MSE\left(\bar{y}_{d\mathrm{R}Pe}^{c}\right)_{II(opt)} = \frac{\bar{Y}^{2}}{P'}\left(A'_{0,1} - gP'\right)^{2} > 0\,. \tag{33}$$

6.3. Comparison with the chain dual to product estimator $\left(\bar{y}_{dP}^{c}\right)_{II}$

The MSE of the chain dual to product estimator $\left(\bar{y}_{dP}^{c}\right)_{II}$ is given by

$$MSE\left(\bar{y}_{dP}^{c}\right)_{II} = \bar{Y}^{2}\left[A'_{0,0} + 2gA'_{0,1} + g^{2}P'\right]. \tag{34}$$

From (27) and (34), we have

$$MSE\left(\bar{y}_{dP}^{c}\right)_{II} - MSE\left(\bar{y}_{d\mathrm{R}P\mathrm{e}}^{c}\right)_{II(opt)} = \frac{\bar{Y}^{2}}{P'}\left(A'_{0,1} + gP'\right)^{2} > 0. \tag{35}$$

6.4. Comparison with the exponential chain dual to ratio estimator $\left(\bar{y}_{d\mathrm{Re}}^{c}\right)_{II}$

From (27) and (29), we have

$$MSE\left(\bar{y}_{d\mathrm{Re}}^{c}\right)_{II} - MSE\left(\bar{y}_{d\mathrm{R}P\mathrm{e}}^{c}\right)_{II(opt)} = \bar{Y}^{2}\frac{\left(gP' - 2A'_{0,1}\right)^{2}}{4P'} > 0\ . \tag{36}$$

6.5. Comparison with the exponential chain dual to product estimator $\left(\bar{y}_{dPe}^{c}\right)_{II}$

From (27) and (28), we have

$$MSE\left(\bar{y}_{dPe}^{c}\right)_{II} - MSE\left(\bar{y}_{d\mathrm{R}P\mathrm{e}}^{c}\right)_{II(opt)} = \bar{Y}^{2}\frac{\left(gP' + 2A'_{0,1}\right)^{2}}{4P'} > 0. \tag{37}$$

6.6. Comparison with the chain linear regression estimator $\left(\bar{y}_{reg.}^{c}\right)_{II}$

The MSE of the chain linear regression estimator $\left(\bar{y}_{reg.}^{c}\right)_{II}$ is given by

$$MSE\left(\bar{y}_{reg.}^{c}\right)_{II} = \bar{Y}^{2}\left[A'_{0,0} - \left(\frac{1}{n} - \frac{1}{n_1}\right)C_{yx}^{2}C_{x}^{2} - \left(\frac{1}{n_1}\right)C_{yx}^{2}C_{xz}^{2}C_{z}^{2}\right], \tag{38}$$

From (27) and (38), we have

$$MSE\left(\bar{y}_{reg.}^{c}\right)_{II} - MSE\left(\bar{y}_{d\mathrm{R}P\mathrm{e}}^{c}\right)_{II(opt)} = \bar{Y}^{2}\left[\frac{\left(A'_{0,1}\right)^{2}}{P'} - \left(\frac{1}{n} - \frac{1}{n_1}\right)C_{yx}^{2}C_{x}^{2} - \left(\frac{1}{n_1}\right)C_{yx}^{2}C_{xz}^{2}C_{z}^{2}\right] > 0,$$

if

$$\frac{\left(A'_{0,1}\right)^2}{P'} > \left(\frac{1}{n} - \frac{1}{n_1}\right) C_{yx}^2 C_x^2 + \left(\frac{1}{n_1}\right) C_{yx}^2 C_{xz}^2 C_z^2 \ . \tag{39}$$

Theorem.

To the first order of approximation, the proposed strategy under optimality condition (27) is always more efficient than $V(\bar{y})$, $MSE\left(\bar{y}_{dR}^c\right)_I$, $MSE\left(\bar{y}_{dP}^c\right)_I$, $MSE\left(\bar{y}_{d\mathrm{Re}}^c\right)_I$, $MSE\left(\bar{y}_{dPe}^c\right)_I$ and $MSE\left(\bar{y}_{reg.}^c\right)_I$.

7. Cost Aspect

The different estimators reported in this paper have so far been compared with respect to their MSE. However, in practical applications the cost aspect should also be taken into account. In the literature, therefore, convention is to fix the total cost of the survey and then have to find optimum sizes of preliminary and final samples so that the variance of the estimator is minimized. In most of the practical situations, total cost is a linear function of samples selected at first and second phases.

In this section, we shall consider the cost of the survey and find the optimum sizes of the preliminary and second-phase samples in **Case I** and **Case II** separately.

Case I: When one auxiliary variable *x* is used then the cost function is given by

$$C = nC_1 + n_1 C_2 ,$$

where

C = the total cost,

C_1 = the cost per unit of collecting information on the study variable *y*, and

C_2 = the cost per unit of collecting information on the auxiliary variable *x*.

When we use additional auxiliary variable z to estimate $\left(\bar{y}_{d\mathrm{R}Pe}^c\right)_I$, then the cost function is given by

$$C = nC_1 + n_1(C_2 + C_3) , \tag{40}$$

where C_3 is the cost per unit of collecting information on the auxiliary variable z.

Ignoring FPC, the MSE of $\left(\bar{y}_{d\mathrm{R}Pe}^c\right)_I$ in (6) can be expressed as

$$MSE\left(\bar{y}_{d\mathrm{R}Pe}^c\right)_I = \frac{1}{n} V_1 + \frac{1}{n_1} V_2 \ ,$$

where

$$V_1 = \left\{ A'_{0,0} + g^2 \left(\frac{1}{2} - \alpha \right)^2 A'_{1,1} + (1-2\alpha) A'_{1,0} \right\},$$

$$V_2 = \left\{ g^2 \left(\frac{1}{2} - \alpha \right)^2 \left(A'_{3,3} - A'_{2,2} \right) + (1-2\alpha) g \left(A'_{0,3} - A'_{0,2} \right) \right\}.$$

It is assumed that $C_1 > C_2 > C_3$. The optimum values of n and n_1 for fixed cost $C = C_0$, which minimizes the variance of $\left(\bar{y}^c_{d\mathrm{RPe}}\right)_I$ in (6) under cost function are given by

$$n_{opt.} = \frac{C_0 \sqrt{V_1 / C_1}}{\sqrt{V_1 C_1} + \sqrt{V_2 \left(C_2 + C_3 \right)}}$$

and

$$n_{1opt.} = \frac{C_0 \sqrt{V_2 / \left(C_2 + C_3 \right)}}{\sqrt{V_1 C_1} + \sqrt{V_2 \left(C_2 + C_3 \right)}}.$$

Hence, the resulting MSE of $\left(\bar{y}^c_{d\mathrm{RPe}}\right)_I$ is given by

$$MSE\left(\bar{y}^c_{d\mathrm{RPe}}\right)_{I(opt)} = \frac{1}{C_0} \left\{ \sqrt{V_1 C_1} + \sqrt{V_2 \left(C_2 + C_3 \right)} \right\}^2 . \tag{41}$$

If all the resources were diverted towards the study variable y only, then we would have optimum sample size as below

$$n^{**} = \frac{C}{C_1}.$$

Thus, the variance of sample mean $\bar{y}$ for a given fixed cost $C = C_0$ in case of large population is given by

$$V\left(\bar{y}\right)_{(opt)} = \frac{C_1}{C_0} S_y^2 . \tag{42}$$

Now, from (41) and (42), the proposed sampling strategy would be profitable if

$$MSE\left(\bar{y}^c_{d\mathrm{RPe}}\right)_{I(opt)} < V\left(\bar{y}\right)_{(opt)},$$

or equivalently

$$\frac{C_2 + C_3}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_2}} \right]^2 .$$

Case II: We assume that y is measured on n units; x and z are measured on n_1 units. We consider a simple cost function

$$C = nC_1 + n_1(C_2' + C_3') \ , \tag{43}$$

where C_2' and C_3' denotes costs per unit of observing x and z values respectively.

The MSE of $\left(\bar{y}^c_{d\mathrm{RPe}}\right)_{II}$ in (25) can be written as

$$M\left(\bar{y}^c_{d\mathrm{RPe}}\right)_{II} = \frac{1}{n}V_1 + \frac{1}{n_1}V_3 , \tag{44}$$

where

$$V_3 = \left(\frac{1}{2} - \alpha\right)^2 g^2 \left(A'_{3,3} - A'_{2,2}\right).$$

To obtain the optimum allocation of sample between phases for a fixed cost $C = C_0$, we minimize (25) with condition (43). It is easily found that this minimum is attained for

$$n_{opt.} = \frac{C_0\sqrt{V_1/C_1}}{\sqrt{V_1C_1} + \sqrt{V_3\left(C_2' + C_3'\right)}} \ ,$$

and

$$n_{1opt.} = \frac{C_0\sqrt{V_3/\left(C_2' + C_3'\right)}}{\sqrt{V_1C_1} + \sqrt{V_3\left(C_2' + C_3'\right)}} .$$

Thus, the optimum MSE corresponding to these optimum values of n and n_1 are given by

$$MSE\left(\bar{y}^c_{d\mathrm{RPe}}\right)_{II(opt)} = \frac{1}{C_0}\left\{\sqrt{V_1C_1} + \sqrt{V_3\left(C_2' + C_3'\right)}\right\}^2 . \tag{45}$$

From (42) and (45), it is obtained that the proposed estimator $\left(\bar{y}^c_{d\mathrm{RPe}}\right)_{II}$ yields less MSE than that of sample mean $\bar{y}$ for the same fixed cost if

$$\frac{C_2' + C_3'}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_3}} \right]^2 .$$

8. Empirical Study

To examine the merits of the proposed estimator, we have considered the three natural population data sets. The descriptions of the population are given as follows:

Population I (Source: Cochran (1977))
Y: Number of 'placebo' children
X: Number of paralytic polio cases in the placebo group
Z: Number of paralytic polio cases in the 'not inoculated' group
N=34, $n = 10$, $n_1 = 15, \overline{Y} = 4.92, \overline{X} = 2.59, \overline{Z} = 2.91, \rho_{yx} = 0.7326,$
$\rho_{yz} = 0.6430, \rho_{xz} = 0.6837, C_y^2 = 1.0248, C_x^2 = 1.5175, C_z^2 = 1.1492.$

Population II (Source: Srivastava *et al.* (1989, Page 3922))
Y: The measurement of weight of children
X: Mid arm circumference of children
Z: Skull circumference of children
$N = 55, n = 18, n_1 = 30, \overline{Y} = 17.08\,\text{kg}, \overline{X} = 16.92\ \text{cm}, \overline{Z} = 50.44\,\text{cm},$
$\rho_{yx} = 0.54, \rho_{yz} = 0.51, \rho_{xz} = -0.08, C_y^2 = 0.0161, C_x^2 = 0.0049, C_z^2 = 0.0007$.

Population III (Source: Srivastava *et al.* (1989, Page 3922))
Y: The measurement of weight of children
X: Mid arm circumference of children
Z: Skull circumference of children
$N = 82, n = 25, n_1 = 43, \overline{Y} = 5.60$ Kg, $\overline{X} = 11.90$ cm, $\overline{Z} = 39.80$ cm,
$\rho_{yx} = 0.09, \rho_{yz} = 0.12, \rho_{xz} = 0.86, C_y^2 = 0.0107, C_x^2 = 0.0052$ and $C_z^2 = 0.0008.$

To see the efficiency of the proposed estimator, the Percent Relative Efficiencies (PREs) of the proposed estimator along with other estimators under considerations are computed with respect to the usual unbiased estimator $\overline{y}$ using the given formula below in case I and case II.

$$PRE(\overline{y}, *) = \frac{V(\overline{y})}{MSE(*)} \times 100\,,$$

where

$$* = \overline{y}\,, \left(\overline{y}_{dR}^{c}\right)_I, \left(\overline{y}_{dP}^{c}\right)_I, \left(\overline{y}_{d\mathrm{Re}}^{c}\right)_I, \left(\overline{y}_{dPe}^{c}\right)_I, \left(\overline{y}_{reg.}^{c}\right)_I, \left(\overline{y}_{d\mathrm{RPe}}^{c}\right)_{I(opt)}$$

and

$$\left(\overline{y}_{dR}^{c}\right)_{II}, \left(\overline{y}_{dP}^{c}\right)_{II}, \left(\overline{y}_{d\mathrm{Re}}^{c}\right)_{II}, \left(\overline{y}_{dPe}^{c}\right)_{II}, \left(\overline{y}_{reg.}^{c}\right)_{II}, \left(\overline{y}_{d\mathrm{RPe}}^{c}\right)_{II(opt)}\ .$$

They are presented in Tables 1 and 2.

Table 1. Percentage relative efficiency of different estimators with respect to $\bar{y}$ in Case I

Estimators	$\bar{y}$	$\left(\bar{y}^c_{dR}\right)_I$	$\left(\bar{y}^c_{dP}\right)_I$	$\left(\bar{y}^c_{d\mathrm{Re}}\right)_I$	$\left(\bar{y}^c_{dPe}\right)_I$	$\left(\bar{y}^c_{reg.}\right)_I$	$\left(\bar{y}^c_{d\mathrm{R}Pe}\right)_{I(opt)}$
Population I	100	136.91	*	135.99	*	185.53	188
Population II	100	131.91	*	128.21	*	123.53	132.45
Population III	100	*	*	*	*	100.99	148.5

Table 2. Percentage relative efficiency of different estimators with respect to $\bar{y}$ in Case II

Estimators	$\bar{y}$	$\left(\bar{y}^c_{dR}\right)_{II}$	$\left(\bar{y}^c_{dP}\right)_{II}$	$\left(\bar{y}^c_{d\mathrm{Re}}\right)_{II}$	$\left(\bar{y}^c_{dPe}\right)_{II}$	$\left(\bar{y}^c_{reg.}\right)_{II}$	$\left(\bar{y}^c_{d\mathrm{R}Pe}\right)_{II(opt)}$
Population I	100	*	*	123.4	*	162.83	277.87
Population II	100	116.68	*	157	*	121.08	248.8
Population III	100	*	*	*	*	100.73	101.66

*Percent relative efficiency is less than 100%

9. Conclusions

The use of auxiliary information to increase the precision of the estimate has received numerous attentions from several authors. In this paper, we continued this research by developing a new estimator under SRSWOR. The proposed estimator has been analyzed and its bias and MSE equations have been obtained in two different cases. The MSE of the proposed estimator has been compared with the MSEs of the usual unbiased estimator $\bar{y}$, chain dual ratio estimator $\bar{y}^c_{dR}$, chain dual to product estimator $\bar{y}^c_{dP}$, exponential chain dual to ratio estimator $\bar{y}^c_{d\mathrm{Re}}$, exponential chain dual to product estimator $\bar{y}^c_{dP\mathrm{e}}$ and regression estimator $\bar{y}^c_{reg.}$ on a theoretical basis and also conditions for obtaining minimum MSE has been derived. The estimator in its optimality is compared theoretically and numerically with other estimators under considerations. The percentage relative efficiencies of different estimators with respect to $\bar{y}$ have been computed and is shown in Tables 1 and 2. Theoretically, the proposed estimator is found to be more efficient than the other estimators under certain conditions. Three population data sets are taken to check the efficiency of the proposed estimator over others estimators. Numerically in all population sets, the proposed estimator is found to be more efficient than other estimators viz., usual unbiased estimator, chain dual to ratio and product estimators, exponential chain dual to ratio and exponential chain dual to product estimators and regression estimator in both the cases of second phase sample selection. Thus, it is preferred to use the proposed estimator

$$\bar{y}^c_{d\mathrm{R}P\mathrm{e}} \text{ or } \left[\left(\bar{y}^c_{d\mathrm{R}P\mathrm{e}}\right)_{I(opt)} \text{ or } \left(\bar{y}^c_{d\mathrm{R}P\mathrm{e}}\right)_{II(opt)}\right],$$

in practice.

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