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A Generalized Newton-Penalty Algorithm for Large Scale Ill-Conditioned Quadratic Problems

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Abstract

Large scale quadratic problems arise in many real world applications. It is quite often that the coefficient matrices in these problems are ill-conditioned. Thus, if the problem data are available even with small error, then solving them using classical algorithms might result to meaningless solutions. In this short paper, we propose an efficient generalized Newton-penalty algorithm for solving these problems. Our computational results show that our new simple algorithm is much faster and better than the approach of Rojas et al. (2000), which requires parameter tuning for different problems.

Keywords: Large Scale Ill-Quadratic Problems, Penalty Method

MSC (2000) No.: 90C06, 90C20

1. Introduction

Large scale quadratic problems arise in many disciplines like image restoration [Bertero and Bocacci (1998), Rojas and Steihaug (2002)].

$$\min_{x\in R^n}\|Ax-b\|^2\ , A\in R^{m\times n}\ , m\geq n. \qquad (1)$$

This is a well studied problem and efficient algorithms have been developed to solve various forms of (1), for example see Salahi (2009 a, b). It is often the case that the problem is ill-conditioned. Thus, even a small error in problem data might significantly change the solution.

Regularization is a technique to deal with such situation and the well known one is the Tikhonov regularization [Tikhonov (1963)], which considers the following problem instead of (1):

$$\min_{x\in R^n} \|Ax-b\|^2+\rho\|x\|^2, \qquad (2)$$

where ρ is the so called regularization parameter [Hansen (1994)]. However, in applications like image restoration [Bertero and Bocacci (1998)], there are extra nonnegativity constraints on (1) and instead of Tikhonov regularization a bound constraint is added to the problem. Namely, we have the following version of (1):

$$\min_{x\in R^n} \ \|Ax-b\|^2 \ , \ \|x\|^2\leq\beta, \quad x\geq 0. \qquad (3)$$

As we see this formulation requires prior information on the solution norm. Obviously (3) is a convex quadratic optimization problem which can be solved using efficient interior-point software packages like LOQO [Vanderbei (1999)]. In Rojas and Steihaug (2002), the authors have developed a trust region interior-point algorithm to solve (3) which itself uses LSTRS software package [Rojas et al. (2000)]. However it requires tuning several parameters and fails on several problems. In this paper we propose an efficient generalized Newton-penalty algorithm to solve (3). Several well know examples are presented to show the efficiency of the proposed algorithm to the one in Rojas and Steihaug (2002).

2. Generalized Newton-Penalty Algorithm

In this section we present an efficient algorithm for solving (3) based on penalty method. To do so, let us consider the following problem instead of (3)

$$\min_x \|Ax-b\|^2+M\|(-x)_+\|^2+M((\|x\|^2-\beta)_+)^2 \ , \qquad (4)$$

where $M>0$ is a large number called penalty parameter and $x_+=max(x,0)$.

Lemma 2.1.

The objective function in (4) is once differentiable.

Proof:

See Hiriat-Urruty (1984).

Obviously, the objective function of (4) has just the first derivative. However, one can define the generalized Hessian for this function, which has many properties of regular Hessian [Hiriat-Urruty (1984)]. The gradient and generalized Hessian of this function are

$$\nabla f(x) = 2A^T(Ax - b) - 2M(-x)_+ + 4Mx(\|x\|^2 - \beta)_+$$
$$H(x) = 2A^T A + 2MD + 4M(\|x\|^2 - \beta)_+ I_n + 8Mxx^T d,$$

where d is a scalar equal to 1 if $\|x\|^2 - \beta > 0$ and zero else and D is a diagonal matrix with diagonal element at position (i, i) equal to one if $x(i) < 0$ and zero otherwise. Obviously for a given $M > 0$, the generalized Hessian is positive semi-definite, thus the objective function in (4) is convex. At each iteration of the algorithm we solve (4) by moving in the Newton like direction called generalized Newton until certain stopping criterion is met [Mangasarian (2004)]. The detailed algorithm is as follows:

Generalized Newton-Penalty Algorithm

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Step 0: Let $M = 1e6$ and $x_0$ be an initial approximation and $k = 0$.

Step 1: While $\|\nabla f(x_k)\| > 1e-6$ do
            Step 1-1: Solve system $H(x_k)d_k = -\nabla f(x_k)$ .
            Step 1-2: Let $x_{k+1} = x_k + d_k$ .
            Step 1-3: Set $k = k + 1$, and go to step $1$.
    end while

Step 2: The approximate solution is $(x_k)_+$.
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Remark 2.1. In practice we consider $(H(x_k) + \delta I_n)d_k = -\nabla f(x_k)$ instead of $H(x_k)d_k = -\nabla f(x_k)$, where δ is a small constant that guarantees the positive definiteness of generalized Hessian. In our implementation we use $\delta = 10^{-5}$. Moreover, this small perturbation of generalized Hessian for ill-conditioned problem does not allow the algorithm to give meaningless solution.

Remark 2.2. The main advantage of our algorithm to the algorithm of Rojas et al. (2000) is solving an unconstrained convex problem using a Newton like algorithm. It involves two parameters, one is the penalty parameter and the other one is the regularization parameter. However, their algorithm which uses LSTRS software package involves several parameters and they are required to be tuned for different problems.

Table 1: Comparison of GNP and $Rojas's$ Algorithm

$problem$	GNP	$Rojas$
$foxgood(500)(\|x_{exact}\| = 1.2910e+001, \|x_{exact}\|^2+10)$	$1.2910e+001(0.68sec)$	$1.2904+001(1.27sec)$
$shaw(500)(\|x_{exact}\| = 2.2820e+001, \|x_{exact}\|^2+10)$	$2.2806e+001(0.7sec)$	$1.6377e+001(22.9sec)$
$phillips(500)(\|x_{exact}\| = 3.0000e+000, \|x_{exact}\|^2+10)$	$3.0000e+000(11.5sec)$	$failed$
$heat(500)(\|x_{exact}\| = 5.8034e+000, \|x_{exact}\|^2+10)$	$5.8061e+000(21.8sec)$	$failed$

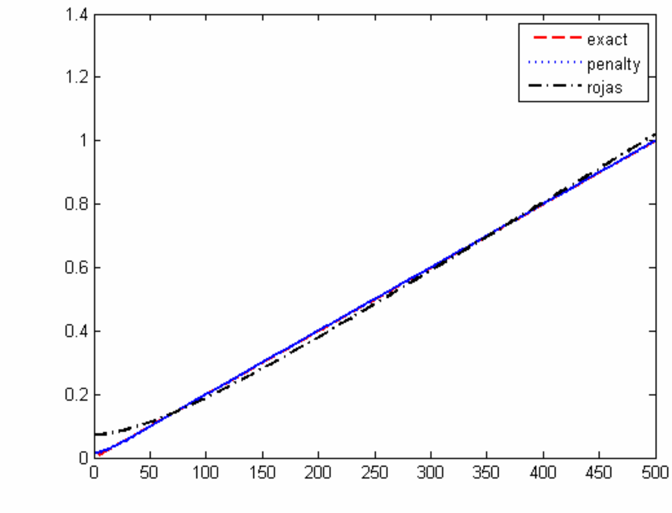

Figure 1: Solution for problem $foxgood$, $n = 500$.

3. Numerical Results

In this section we present several numerical examples showing the practical efficiency of our proposed algorithm to the algorithm of Rojas and Steihaug (2002). Both algorithms are implemented in Matlab 7.2 on a Pentium 4 Laptop with 1 GB of memory. We should note that at each iteration within the algorithm of Rojas and Steihaug (2002) we call LSTRS software package to solve the quadratically constrained quadratic problems. All test problems are taken from Hansen (1994).

As we see in Table 1, our algorithm solves all problems while the algorithm of Rojas and Steihaug (2002) fails on several problems and is much slower than our algorithm. For all test problems we use $\|x_{exact}\|^2 + 10$ as an upper bound for the solution norm, however, Rojas and Steihaug (2002) use $\|x_{exact}\|^2$. Obviously this requires knowing the exact solution norm, while for our algorithm even larger bounds do not affect the solution. For those test problems which the algorithm of Rojas and Steihaug (2002) failed, we even decreased the bound but still it failed.

In Figures $1-4$, we have plotted the solutions norm reported in Table 1. As we see for all four problems the solutions norm obtained by our algorithm almost match the exact solutions. For problem $'foxgood'$ in Figure 1, all cases of both methods are more or less the same, while for problem $'shaw'$ the algorithm of Rojas and Steihaug (2002) significantly differs from our and the exact solution. In the last two figures, we also have just the results of our algorithm with the exact solutions, as the algorithm of Rojas and Steihaug (2002) failed to solve them.

4. Conclusions

In this short paper, using the penalty method, we have considered a large scale quadratic minimization problem as a convex once differentiable unconstrained problem. Then, using the concept of the generalized Hessian, a generalized Newton-penalty algorithm is designed to solve it. Our computational experiments on several well known ill-conditioned test problems show that our algorithm is much faster and reliable than the algorithm of Rojas and Steihaug (2002), which uses LSTRS software package at each iteration to solve a quadratically constrained quadratic problem. Moreover, their algorithm requires tuning several parameters and fails on several problems.

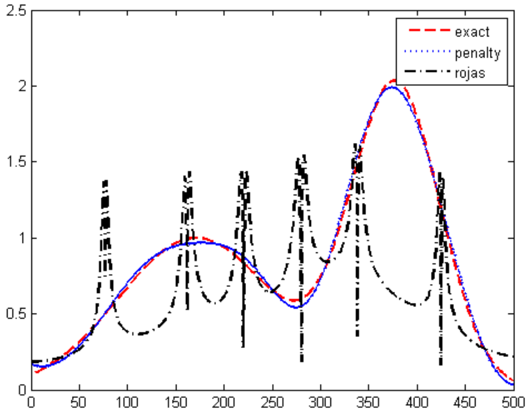

Figure 2. Solution for problem $shaw, n = 500$.

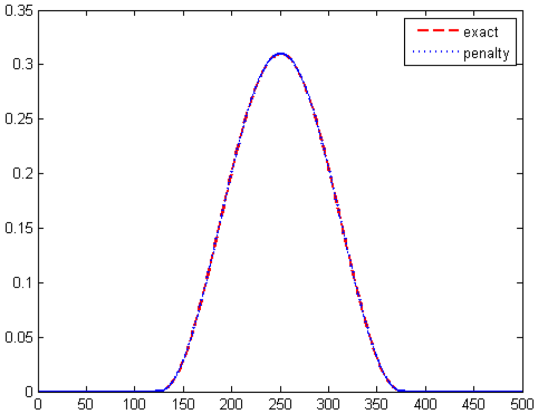

Figure 3. Solution for problem $phillips, n = 500$.

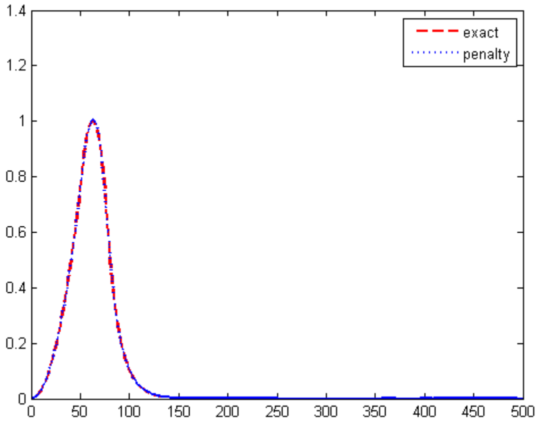

Figure 4. Solution for problem $heat, n = 500$.

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