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Non Markovian Queue with Two Types service Optional Re-service and General Vacation Distribution

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Abstract

We consider a single server batch arrival queueing system, where the server provides two types of heterogeneous service. A customer has the option of choosing either type 1 service with probability p_1 or type 2 service with probability p_2 with the service times follow general distribution. After the completion of either type 1 or type 2 service a customer has the option to repeat or not to repeat the type 1 or type 2 service. As soon as the customer service is completed, the server will take a vacation with probability θ or may continue staying in the system with probability $1 - \theta$. The re-service periods and vacation periods are assumed to be general. Using supplementary variable technique, the Laplace transforms of time dependent probabilities of system state are derived and thus we deduce the steady state results. We obtain the average queue size and average waiting time. Some system performance measures and numerical illustrations are discussed.

Keywords: Two type of service; Time dependent solution; Optional re-service; Average queue size; Average waiting time

MSC 2010 No.: 60K25, 68M30, 90B22

1. Introduction

Queueing models are essential for designing and monitoring of several communication systems. In queueing theory, customers arrive randomly according to a Poisson process and form a queue. In some practical situations like stored and forward communication networks the arrivals cannot be characterized by a Poisson process the arriving messages are converted into a random number of packets depending upon the size of the message. Similarly, in railway yards and ports the cargo handling is done in batches of random size. For analyzing these sort of situations the queueing models with bulk arrivals are developed. The bulk arrivals can be well characterised by a compound Poisson process. In a typical manufacturing situation, the work-pieces arrive at a machine center in batches and they leave in batches. A batch consists of identical work-pieces that are processed and then transported in batches for further processing. Such a situation can be modeled as queues with bulk arrivals. There is a discipline within the mathematical theory of probability, called a bulk queue (also called batch queue) where customers are served in groups of random size. The purpose of present paper is to provide overview of queueing models with phase service and its applications in real life queueing problems. The motivation for studying the queueing systems with phase service comes from numerous versatile applications in the performance evaluation and dimensioning of production and manufacturing systems, computer and communication networks, inventory and distribution systems, and so forth. During the last few decades, attention has been paid increasingly by many researchers in studying the phase service queueing models. In recent years, computer networks and data communication systems are the fastest growing technologies, which have led to significant development in applications such as swift advance in internet, audio data traffic, video data traffic, etc.

In a vacation queueing system, the server may not be available for a period of time due to many reasons like, being checked for maintenance, working at other queues, scanning for new work (a typical aspect of many communication systems) or simply taking break. This period of time, when the server is unavailable for primary customers is referred as a vacation. A wide class of vacation policies like N-policy, T-policy and D-policy, for governing the vacation mechanisms like single vacation, multiple vacation, Bernoulli vacation and modified vacation, etc., have been discussed in the literature. Single server queues with vacations have been studied extensively by Doshi (1986). Keilson and Servi (1986) were introduced the concept of single server queueing system with Bernoulli vacation, where the server takes a vacation after each service completion with probability p or starts a new service with probability $1-p$. Takagi (1991) examined an $M/G/1$ queueing model with exhaustive service by using supplementary variable technique. He obtained explicit expression for time dependent solutions in terms of their Laplace transforms. These models arise naturally in call centers with multi-task employees, customised manufacturing, telecommunication and computer networks, maintenance activities, production and quality control problem, etc. For a review of main results and methods, the reader is referred to the survey papers by Madan and Anabosi (2003), Kumar and Arumuganathan (2008) and Gharbi and Ioualalen (2010), Gross and Harris (2011) and Zhang and Hou (2012), Haghighi and Mishev $(2016a)$ and Haghighi and Mishev $(2016b)$. Xu et al. (2009) studied the concept of bulk input queue with working vacation.

Madan and Ayman Baklizi (2002) have studied an $M/G/1$ queue with additional second stage service and optional re-service. Madan et al. (2004) considered a bulk arrival queue with optional re-service. In their system, before a service starts customer has the option to choose either type of service, after completion of which the customer may leave the system or may opt for re-service of the service taken by him. Recently Tadj and Ke (2008) analysed a bulk arrival two phase bulk service queueing system with optional re-service facility. Badamchi Zadeh (2009) and Jain and Upadhyaya (2010) discussed about second optional service. Later Ke et al. (2010) studied the operating characteristics of an $M/G/1$ queueing system. Baruah et al. (2013) aimed at studying a queuing model with two stage heterogeneous service where customer arrival in batches and has a single server providing service in two stages, one after the other in succession.

Artalejo and Choudhury (2004) studied an $M/G/1$ queue with repeated attempts and two-phase service. A numbers of papers by Choudhury and Paul (2004), Madan and Choudhury (2004), Choudhury and Madan (2005). Thangaraj and Vanitha (2010)analyzed a single server $M/G/1$ feedback queue with two types of service having general distribution. Tadj (2013) discussed Bernoulli vacation schedule under T-Policy and Rajadurai et al. (2015) have recently appeared in queueing literature in which the server provides each unit two phases of heterogeneous service in succession with Bernoulli schedule vacation.

The motivation for such type of models comes from some computer and communication networks where messages are processed in two stages by a single server. The case where both phases of service are exponentially distributed is the so called Coxian distribution C_2 distribution. Queueing models wherein the server provide two phases of essential service to each customer are known as two-phase essential service queueing model. Such types of queueing situations naturally arise in many real time system namely in manufacturing system wherein the machine producing certain items may require two phases of service in succession. For completing the processing of raw materials, the periodic checking (first phase of service) followed by usual processing (second phase of service) of raw material is required. Kumar and Arumuganathan (2008) considered a single server retrial queue with batch arrivals under the assumption that the server provides preliminary first essential service and second essential service to all arriving calls.

Though a lot of work has been done in queueing systems, there have not been many significant studies on single server batch arrival queue with general service time, two types of service, optional re-service and Bernoulli vacation. To the best of our knowledge, the current work that explores batch arrival, two type of service queueing system with Bernoulli vacation. In this paper, we consider a single server vacation queueing model, in which the server provides two types of service and each arriving customer has the option of choosing either type of service. After completion of type 1 or type 2 service a customer has the option to repeat or not to repeat the type 1 or type 2 service. The re-service periods and vacation periods are assumed to be general.

The outline of the paper is as follows. In Section 2, the detailed description of the mathematical model and practical justification of the model are given. Definitions are given in Section 3. Equations governing the system and the time dependent solution have been obtained in Section 4 and corresponding steady state results have been derived explicitly in Section 5. Average queue size and average waiting time are computed in Section 6. Particular cases and the effects of various parameters on the system performance are analyzed numerically in Sections 7 and 8, respectively. The conclusion is given in Section 9.

2. Mathematical description of the model

We assume the following to describe the queueing model of our study.

- Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ ($i \geq 1$) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.
- There is a single server who provides either type 1 service or type 2 service for all customers, as soon as the service of a customer is completed, he may opt to repeat the type 1 service with probability r_1 or may not repeat with probability $(1 - r_1)$. Similarly after taking the type 2 service he may opt to repeat the type 2 with probability r_2 or may not repeat with probability $(1 - r_2)$. Further, we assume that this option of repeating the type 1 or the type 2 service can be availed once.
- The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x+dx]$, given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2,$$

and, therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2.$$

- The server's vacation time follows a general (arbitrary) distribution with distribution function $V(t)$ and density function $v(t)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)},$$

and, therefore,

$$v(t) = \gamma(t) e^{-\int_0^t \gamma(x)dx}.$$

- Various stochastic processes involved in the system are assumed to be independent of each other.

2.1. Practical justification of the suggested model

An application of our model is in the following automobile repair garage for cars. Consider one mechanic of automobile repair garage, the arrival of cars forms a random process. The mechanic is responsible for routine maintenance of one car at one time (such as the routine maintenance every 500 km, 1000km, etc.) Some vehicles may need tire, windshield wiper, or battery replacement. The mechanic needs to prepare tools before the starting of busy period (i.e., continuous service period of server) and places the tools back at the end of busy period. In this scenario, mechanic, routine maintenance, tire (windshield wiper or battery) replacement and preparing tools are corresponding to the server, type 1 or type 2 service, re-service and vacation, respectively, in the queueing terminology.

Another practical application is that such system can be modeled in a coffee shop. In a one-person coffee shop, the arrival of customers follows a random process. Customers buy various coffee beans and some of them buy various coffee grinders as well. The shopkeeper has to prepare scale and other equipments in advance and put them back after the end of the each busy period. In this scenario, shopkeeper, coffee beans, coffee grinders and preparing scale and other equipments are corresponding to the server, essential service type 1 or type 2 service, re-service and vacation, respectively, in the queueing terminology.

3. Definitions

We define

(1) $P_n^{(i)}(x,t)$ = Pr{at time t, the server is active providing i^{th} type service and there are n $(n \geq 0)$ customers in the queue excluding one customer in the service being served and the elapsed service time for this customer is x}. $P_n^{(i)}(t) = \int\limits_0^\infty P_n^{(i)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the i^{th} type service irrespective of the value of x for $i = 1$, 2.

(2) $R_n^{(i)}(x,t)$ = Probability that at time t, the server is active providing i^{th} type re-service and there are n $(n \geq 0)$ customers in the queue excluding one customer who is repeating i^{th} type service and the elapsed re-service time for this customer is x. Consequently $R_n^{(i)}(t) = \int\limits_0^\infty R_n^{(i)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer who is repeating i^{th} type service irrespective of the value of x for $i = 1, 2$.

(3) $V_n(x,t)$ = Probability that at time t, the server is under vacation with elapsed vacation time x and there are n $(n \geq 0)$ customers in the queue. Accordingly $V_n(t)$=$\int\limits_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under vacation irrespective of the value of x.

(4) $Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system.

4. Equations governing the system

The model is then, governed by the following set of differential - difference equations:

$$\tfrac{\partial}{\partial x}P_0^{(1)}(x,t)+\tfrac{\partial}{\partial t}P_0^{(1)}(x,t)+(\lambda+\mu_1(x))P_0^{(1)}(x,t)=0, \tag{1}$$

$$\begin{aligned}\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+(\lambda+\mu_1(x))P_n^{(1)}(x,t)\\ =\lambda\sum_{k=1}^{n}c_kP_{n-k}^{(1)}(x,t), n\geq 1,\end{aligned} \tag{2}$$

$$\tfrac{\partial}{\partial x}P_0^{(2)}(x,t)+\tfrac{\partial}{\partial t}P_0^{(2)}(x,t)+[\lambda+\mu_2(x)]P_0^{(2)}(x,t)=0, \tag{3}$$

$$\begin{aligned}\tfrac{\partial}{\partial x}P_n^{(2)}(x,t)+\tfrac{\partial}{\partial t}P_n^{(2)}(x,t)+[\lambda+\mu_2(x)]P_n^{(2)}(x,t)\\ =\lambda\textstyle\sum_{k=1}^{n}c_kP_{n-k}^{(2)}(x,t), n\geq 1,\end{aligned} \tag{4}$$

$$\tfrac{\partial}{\partial x}R_0^{(1)}(x,t)+\tfrac{\partial}{\partial t}R_0^{(1)}(x,t)+[\lambda+\mu_1(x)]R_0^{(1)}(x,t)=0, \tag{5}$$

$$\begin{aligned}\tfrac{\partial}{\partial x}R_n^{(1)}(x,t)+\tfrac{\partial}{\partial t}R_n^{(1)}(x,t)+[\lambda+\mu_1(x)]R_n^{(1)}(x,t)\\ =\lambda\textstyle\sum_{k=1}^{n}c_kR_{n-k}^{(1)}(x,t), n\geq 1,\end{aligned} \tag{6}$$

$$\tfrac{\partial}{\partial x}R_0^{(2)}(x,t)+\tfrac{\partial}{\partial t}R_0^{(2)}(x,t)+[\lambda+\mu_2(x)]R_0^{(2)}(x,t)=0, \tag{7}$$

$$\begin{aligned}\tfrac{\partial}{\partial x}R_n^{(2)}(x,t)+\tfrac{\partial}{\partial t}R_n^{(2)}(x,t)+[\lambda+\mu_2(x)]R_n^{(2)}(x,t)\\ =\lambda\textstyle\sum_{k=1}^{n}c_kR_{n-k}^{(2)}(x,t), n\geq 1,\end{aligned} \tag{8}$$

$$\tfrac{\partial}{\partial x}V_0(x,t)+\tfrac{\partial}{\partial t}V_0(x,t)+[\lambda+\gamma(x)]V_0(x,t)=0, \tag{9}$$

$$\begin{aligned}\tfrac{\partial}{\partial x}V_n(x,t)+\tfrac{\partial}{\partial t}V_n(x,t)+[\lambda+\gamma(x)]V_n(x,t)\\ =\lambda\textstyle\sum_{k=1}^{n}c_kV_{n-k}(x,t), \quad n\geq 1,\end{aligned} \tag{10}$$

$$\begin{aligned}\tfrac{d}{dt}Q(t)=\lambda Q(t)+(1-r_1)(1-\theta)\textstyle\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx\\ +(1-r_2)(1-\theta)\textstyle\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx+(1-\theta)\\ \textstyle\int_0^\infty R_0^{(1)}(x,t)\mu_1(x)dx+(1-\theta)\int_0^\infty R_0^{(2)}(x,t)\mu_2(x)dx\\ +\textstyle\int_0^\infty V_0(x,t)\gamma(x)dx,\end{aligned} \tag{11}$$

$$\begin{aligned}P_n^{(1)}(0,t)=p_1\lambda c_{n+1}Q(t)+p_1(1-r_1)(1-\theta)\textstyle\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx\\ +p_1(1-\theta)\Big\{(1-r_2)\textstyle\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx\\ +\textstyle\int_0^\infty R_{n+1}^{(1)}(x,t)\mu_1(x)dx\Big\}+p_1(1-\theta)\\ \times\textstyle\int_0^\infty R_{n+1}^{(2)}(x,t)\mu_2(x)dx+p_1\int_0^\infty V_{n+1}(x,t)\gamma(x)dx, \quad n\geq 0,\end{aligned} \tag{12}$$

$$
\begin{aligned}
P_n^{(2)}(0,t) &= p_2\lambda c_{n+1}Q(t) + p_2(1-r_1)(1-\theta)\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx \\
&+p_2(1-\theta)\Bigg\{(1-r_2)\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx \\
&+\int_0^\infty R_{n+1}^{(1)}(x,t)\mu_1(x)dx\Bigg\} + p_2(1-\theta) \\
&\times\int_0^\infty R_{n+1}^{(2)}(x,t)\mu_2(x)dx + p_2\int_0^\infty V_{n+1}(x,t)\gamma(x)dx, \quad n\geq 0, \qquad (13)
\end{aligned}
$$

$$R_n^{(1)}(0,t) = r_1\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx, \quad n\geq 0, \qquad (14)$$

$$R_n^{(2)}(0,t) = r_2\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx, \quad n\geq 0, \qquad (15)$$

$$
\begin{aligned}
V_n(0,t) &= \theta(1-r_2)\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx \\
&+\theta(1-r_1)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx + \theta\int_0^\infty R_n^{(1)}(x,t)\mu_1(x)dx \\
&+\theta\int_0^\infty R_n^{(2)}(x,t)\mu_2(x)dx, \quad n\geq 0. \qquad (16)
\end{aligned}
$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_n^{(i)}(0) = R_n^{(i)}(0) = V_n(0) = 0 \ \text{ for } \ i=1,2, \quad n\geq 0 \text{ and } Q(0)=1. \qquad (17)$$

We define the probability generating functions, for i =1, 2.

$$P^{(i)}(x,z,t) = \sum_{n=0}^\infty z^nP_n^{(i)}(x,t);\ \ P^{(i)}(z,t) = \sum_{n=0}^\infty z^nP_n^{(i)}(t);$$

$$C(z) = \sum_{n=1}^\infty c_nz^n;\ \ R^{(i)}(x,z,t) = \sum_{n=0}^\infty z^nR_n^{(i)}(x,t);$$

$$R^{(i)}(z,t) = \sum_{n=0}^\infty z^nR_n^{(i)}(t);\ \ V(x,z,t) = \sum_{n=0}^\infty z^nV_n(x,t);$$

$$V(z,t) = \sum_{n=0}^\infty z^nV_n(t), \qquad (18)$$

which are convergent inside the circle given by $\mid z\mid\leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^\infty e^{-st}f(t)dt, \quad \Re(s)>0. \qquad (19)$$

We take the Laplace transform of equation (1) and using (17) and (19), we get

$$\frac{\partial}{\partial x}\int_0^\infty e^{-st}P_0^{(1)}(x,t)dt+\int_0^\infty e^{-st}\frac{\partial}{\partial t}P_0^{(1)}(x,t)dt$$
$$+(\lambda+\mu_1(x))\int_0^\infty e^{-st}P_0^{(1)}(x,t)dt=0,$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s)+s\int_0^\infty e^{-st}P_0^{(1)}(x,t)dt-P_0^{(1)}(0)+(\lambda+\mu_1(x))\bar{P}_0^{(1)}(x,s)=0,$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s)+s\bar{P}_0^{(1)}(x,s)+(\lambda+\mu_1(x))\bar{P}_0^{(1)}(x,s)=0,$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s)+[s+\lambda+\mu_1(x)]\bar{P}_0^{(1)}(x,s)=0. \tag{20}$$

By similar performance on equations (2) - (16), we get

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s)+[s+\lambda+\mu_1(x)]\bar{P}_n^{(1)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{P}_{n-k}^{(1)}(x,s),\quad n\geq 1, \tag{21}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,t)+[s+\lambda+\mu_2(x)]\bar{P}_0^{(2)}(x,s)=0, \tag{22}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s)+[s+\lambda+\mu_2(x)]\bar{P}_n^{(2)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{P}_{n-k}^{(2)}(x,s),\quad n\geq 1, \tag{23}$$

$$\frac{\partial}{\partial x}\bar{R}_0^{(1)}(x,s)+[s+\lambda+\mu_1(x)]\bar{R}_0^{(1)}(x,s)=0, \tag{24}$$

$$\frac{\partial}{\partial x}\bar{R}_n^{(1)}(x,s)+[s+\lambda+\mu_1(x)]\bar{R}_n^{(1)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{R}_{n-k}^{(1)}(x,s),\quad n\geq 1, \tag{25}$$

$$\frac{\partial}{\partial x}\bar{R}_0^{(2)}(x,s)+[s+\lambda+\mu_2(x)]\bar{R}_0^{(2)}(x,s)=0, \tag{26}$$

$$\frac{\partial}{\partial x}\bar{R}_n^{(2)}(x,s)+[s+\lambda+\mu_2(x)]\bar{R}_n^{(2)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{R}_{n-k}^{(2)}(x,s),\quad n\geq 1, \tag{27}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s)+[s+\lambda+\gamma(x)]\bar{V}_0(x,s)=0, \tag{28}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s)+[s+\lambda+\gamma(x)]\bar{V}_n(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{V}_{n-k}(x,s),\ n\geq 1, \tag{29}$$

$$(s+\lambda)\bar{Q}(s) = 1 + (1-r_1)(1-\theta)\int_0^\infty \bar{P}_0^{(1)}(x,s)\mu_1(x)dx$$
$$+(1-r_2)(1-\theta)\int_0^\infty \bar{P}_0^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty V_0(x,s)\gamma(x)dx$$
$$+(1-\theta)\left\{\int_0^\infty \bar{R}_0^{(1)}(x,s)\mu_1(x)dx + \int_0^\infty \bar{R}_0^{(2)}(x,s)\mu_2(x)dx\right\}, \tag{30}$$

$$\bar{P}_n^{(1)}(0,s) = p_1\lambda c_{n+1}\bar{Q}(s) + p_1(1-r_1)(1-\theta)\int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_1(x)dx$$
$$+p_1(1-r_2)(1-\theta)\int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx + p_1\int_0^\infty V_{n+1}(x,s)\gamma(x)dx$$
$$+p_1(1-\theta)\left\{\int_0^\infty \bar{R}_{n+1}^{(1)}(x,s)\mu_1(x)dx + \int_0^\infty \bar{R}_{n+1}^{(2)}(x,s)\mu_2(x)dx\right\},$$
$$n \geq 0, \tag{31}$$

$$\bar{P}_n^{(2)}(0,s) = p_2\lambda c_{n+1}\bar{Q}(s) + p_2(1-r_1)(1-\theta)\int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_1(x)dx$$
$$+p_2(1-r_2)(1-\theta)\int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx + p_2$$
$$\int_0^\infty V_{n+1}(x,s)\gamma(x)dx + p_2(1-\theta)\left\{\int_0^\infty \bar{R}_{n+1}^{(1)}(x,s)\mu_1(x)dx\right.$$
$$\left.+\int_0^\infty \bar{R}_{n+1}^{(2)}(x,s)\mu_2(x)dx\right\}, \quad n \geq 0, \tag{32}$$

$$\bar{R}_n^{(1)}(0,s) = r_1\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx, \tag{33}$$

$$\bar{R}_n^{(2)}(0,s) = r_2\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx, \tag{34}$$

$$\bar{V}_n(0,s) = \theta(1-r_2)\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx + \theta(1-r_1)\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx$$
$$+\theta\int_0^\infty \bar{R}_n^{(1)}(x,s)\mu_1(x)dx + \theta\int_0^\infty \bar{R}_n^{(2)}(x,s)\mu_2(x)dx, \quad n \geq 0. \tag{35}$$

Now multiplying equation (21) by z^n summing over n from 0 to ∞, adding (20) and using (18), we get

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\bar{P}_n^{(1)}(x,s)z^n + [s+\lambda+\mu_1(x)]\sum_{n=0}^{\infty}\bar{P}_n^{(1)}(x,s)z^n = \lambda\sum_{n=0}^{\infty}\sum_{k=1}^{n}c_k\bar{P}_{n-k}^{(1)}(x,s)z^n,$$

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\mu_1(x)]\bar{P}^{(1)}(x,z,s) = 0. \tag{36}$$

Similarly for equations (23) - (29), we get

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s) + [s+\lambda-\lambda C(z)+\mu_2(x)]\bar{P}^{(2)}(x,z,s) = 0, \tag{37}$$

$$\frac{\partial}{\partial x}\bar{R}^{(1)}(x,z,s) + [s+\lambda-\lambda C(z)+\mu_1(x)]\bar{R}^{(1)}(x,z,s) = 0, \tag{38}$$

$$\frac{\partial}{\partial x}\bar{R}^{(2)}(x,z,s) + [s+\lambda-\lambda C(z)+\mu_2(x)]\bar{R}^{(2)}(x,z,s) = 0, \tag{39}$$

$$\frac{\partial}{\partial x}\bar{V}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma(x)]\bar{V}(x,z,s)=0. \tag{40}$$

For the boundary conditions, we multiply both sides of equation (31) by z^n summing over n from 0 to ∞, and use equation (30), we get

$$\begin{aligned} z\bar{P}^{(1)}(0,z,s) &= p_1[1-(s+\lambda)\bar{Q}(s)]+p_1(1-r_1)(1-\theta) \\ &\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx+p_1\lambda C(z)\bar{Q}(s)+p_1(1-r_2)(1-\theta) \\ &\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx+p_1(1-\theta)\left\{\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx\right. \\ &\left.+\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx\right\}+p_1\int_0^\infty V(x,z,s)\gamma(x)dx, \quad n\geq 0. \end{aligned} \tag{41}$$

Performing similar operation on equations (32) to (35), we get

$$\begin{aligned} z\bar{P}^{(2)}(0,z,s) &= p_2[1-(s+\lambda)\bar{Q}(s)]+p_2(1-r_1)(1-\theta) \\ &\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx+p_2\lambda C(z)\bar{Q}(s)+p_2(1-r_2)(1-\theta) \\ &\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx+p_2(1-\theta)\left\{\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx\right. \\ &\left.+\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx\right\}+p_2\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx, \quad n\geq 0, \end{aligned} \tag{42}$$

$$\bar{R}^{(1)}(0,z,s)=r_1\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx, \tag{43}$$

$$\bar{R}^{(2)}(0,z,s)=r_2\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx, \tag{44}$$

$$\begin{aligned} \bar{V}_n(0,z,s) =&\theta(1-r_1)\int_0^\infty \bar{P}_n^{(1)}(x,z,s)\mu_1(x)dx \\ &+\theta\int_0^\infty \bar{R}_n^{(1)}(x,z,s)\mu_1(x)dx+\theta\int_0^\infty \bar{R}_n^{(2)}(x,z,s)\mu_2(x)dx \\ &+\theta(1-r_2)\int_0^\infty \bar{P}_n^{(2)}(x,z,s)\mu_2(x)dx, \quad n\geq 0. \end{aligned} \tag{45}$$

Integrating equations (36) to (40) between 0 and x, we obtain

$$\bar{P}^{(1)}(x,z,s)=\bar{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_1(t)dt}, \tag{46}$$

$$\bar{P}^{(2)}(x,z,s)=\bar{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_2(t)dt}, \tag{47}$$

$$\bar{R}^{(1)}(x,z,s)=\bar{R}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_1(t)dt}, \tag{48}$$

$$\bar{R}^{(2)}(x,z,s) = \bar{R}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \mu_2(t)dt}, \tag{49}$$

$$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int\limits_0^x \gamma(t)dt}. \tag{50}$$

Again integrating equations (46) to (50) with respect to x, we have

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right], \tag{51}$$

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right], \tag{52}$$

$$\bar{R}^{(1)}(z,s) = \bar{R}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right], \tag{53}$$

$$\bar{R}^{(2)}(z,s) = \bar{R}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right], \tag{54}$$

$$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right], \tag{55}$$

where

$$\bar{B}_1(s+\lambda-\lambda C(z)) = \int\limits_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_1(x),$$

$$\bar{B}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_2(x),$$

$$\bar{V}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV(x),$$

are the Laplace-Stieltjes transform of the type 1 service time $B_1(x)$, type 2 service time $B_2(x)$ and vacation time $V(x)$. Now multiplying both sides of equations (46) to (50) by $\mu_1(x)$, $\mu_2(x)$, $\mu_1(x)$, $\mu_2(x)$ and $\gamma(x)$, respectively, and integrating over x, we obtain

$$\int\limits_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda-\lambda C(z)], \tag{56}$$

$$\int\limits_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)], \tag{57}$$

$$\int_0^\infty \bar{R}^{(1)}(x,z,s)\mu_1(x)dx = \bar{R}^{(1)}(0,z,s)\bar{B}_1[s+\lambda-\lambda C(z)], \tag{58}$$

$$\int_0^\infty \bar{R}^{(2)}(x,z,s)\mu_2(x)dx = \bar{R}^{(2)}(0,z,s)\bar{B}_2[s+\lambda-\lambda C(z)], \tag{59}$$

$$\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx = \bar{V}(0,z,s)\bar{V}[s+\lambda-\lambda C(z)]. \tag{60}$$

Using equations (56) and (57) in (43) and (44), we get

$$\bar{R}^{(1)}(0,z,s) = \ r_1\bar{B}_1(a)\bar{P}^{(1)}(0,z,s), \tag{61}$$

$$\bar{R}^{(2)}(0,z,s) = \ r_2\bar{B}_2(a)\bar{P}^{(2)}(0,z,s), \tag{62}$$

where $a = s+\lambda-\lambda C(z)$.

Using equations (56) to (59) in (45), we get

$$\begin{aligned}\bar{V}(0,z,s) =&\theta(1-r_1)\bar{B}_1(a)\bar{P}^{(1)}(0,z,s) + \theta\bar{B}_1(a)\bar{R}^{(1)}(0,z,s)\\ &+\theta(1-r_2)\bar{B}_2(a)\bar{P}^{(2)}(0,z,s) + \theta\bar{B}_2(a)\bar{R}^{(2)}(0,z,s).\end{aligned} \tag{63}$$

Using equations (61) and (62) in the above equation, we have

$$\begin{aligned}\bar{V}(0,z,s) =&\theta\bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a))\bar{P}^{(1)}(0,z,s)\\ &+\theta\bar{B}_2(a)(1-r_2+r_2\bar{B}_2(a))\bar{P}^{(2)}(0,z,s).\end{aligned} \tag{64}$$

Using equations (56) to (64) in (41) and (42), we get

$$\begin{aligned}\bar{P}^{(1)}(0,z,s) =&p_1[1-(s+\lambda)\bar{Q}(s)] + \lambda p_1 C(z)\bar{Q}(s)\\ &+p_1\bar{B}_2(a)B\bar{P}^{(2)}(0,z,s),\end{aligned} \tag{65}$$

$$\begin{aligned}\bar{P}^{(2)}(0,z,s) =&p_2[1-(s+\lambda)\bar{Q}(s)] + \lambda p_2 C(z)\bar{Q}(s)\\ &+p_2\bar{B}_1(a)A\bar{P}^{(1)}(0,z,s),\end{aligned} \tag{66}$$

where

$A = (1-r_1+r_1\bar{B}_1(a))(1-\theta+\theta\bar{V}(a))$ and

$B = (1-r_2+r_2\bar{B}_2(a))(1-\theta+\theta\bar{V}(a))$.

From equations (65) and (66), we get

$$\bar{P}^{(1)}(0,z,s) = \frac{p_1[1 - s\bar{Q}(s)] + p_1\lambda(C(z)-1)\bar{Q}(s)}{Dr}, \tag{67}$$

where

$$\begin{aligned} Dr = z - (1-\theta+\theta\bar{V}(a))[p_1\bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a)) \\ +p_2\bar{B}_2(a)(1-r_2+r_2\bar{B}_2(a))], \end{aligned} \tag{68}$$

$$\bar{P}^{(2)}(0,z,s) = \frac{p_2[1 - s\bar{Q}(s)] + \lambda p_2(C(z)-1)\bar{Q}(s)}{Dr}. \tag{69}$$

By substituting equations (67) and (69) in (61), (62) and (64), we get

$$\bar{R}^{(1)}(0,z,s) = \frac{r_1\bar{B}_1(a)[p_1(1 - s\bar{Q}(s)) + p_1\lambda(C(z)-1)\bar{Q}(s)]}{Dr}, \tag{70}$$

$$\bar{R}^{(2)}(0,z,s) = \frac{r_2\bar{B}_2(a)[p_2(1 - s\bar{Q}(s)) + p_2\lambda(C(z)-1)\bar{Q}(s)]}{Dr}, \tag{71}$$

$$\begin{aligned} \bar{V}(0,z,s) = \tfrac{\theta}{Dr}[p_1\bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a)) + p_2\bar{B}_2(a) \\ \times(1-r_2+r_2\bar{B}_2(a))][\lambda(C(z)-1)\bar{Q}(s) + (1-s\bar{Q}(s))]. \end{aligned} \tag{72}$$

By substituting equations (67), (69) to (72) in (51) to (55), we have

$$\bar{P}^{(1)}(z,s) = \frac{[\lambda p_1(C(z)-1)\bar{Q}(s) + p_1(1-s\bar{Q}(s))]}{Dr}\left[\frac{1-\bar{B}_1(a)}{a}\right], \tag{73}$$

$$\bar{P}^{(2)}(z,s) = \frac{[\lambda p_2(C(z)-1)\bar{Q}(s) + p_2(1-s\bar{Q}(s))]}{Dr}\left[\frac{1-\bar{B}_2(a)}{a}\right], \tag{74}$$

$$\bar{R}^{(1)}(z,s) = \frac{r_1\bar{B}_1(a)[\lambda p_1(C(z)-1)\bar{Q}(s) + p_1(1-s\bar{Q}(s))]}{Dr}\left[\frac{1-\bar{B}_1(a)}{a}\right], \tag{75}$$

$$\bar{R}^{(2)}(z,s) = \frac{r_2\bar{B}_2(a)[\lambda p_2(C(z)-1)\bar{Q}(s) + p_2(1-s\bar{Q}(s))]}{Dr}\left[\frac{1-\bar{B}_2(a)}{a}\right], \tag{76}$$

$$\begin{aligned} \bar{V}(z,s) &= \tfrac{\theta}{Dr}[p_1\bar{B}_1(a)(1-r_1+r_1\bar{B}_1(a)) + p_2\bar{B}_2(a)(1-r_2+r_2\bar{B}_2(a))] \\ &\quad\times[\lambda(C(z)-1)\bar{Q}(s) + (1-s\bar{Q}(s))]\left[\tfrac{1-\bar{V}(a)}{a}\right]. \end{aligned} \tag{77}$$

Thus, $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{R}^{(1)}(z,s)$, $\bar{R}^{(2)}(z,s)$ and $\bar{V}(z,s)$ are completely determined from equations (73) to (77).

5. The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t). \tag{78}$$

In order to determine $\bar{P}^{(1)}(z,s)$, $\bar{P}^{(2)}(z,s)$, $\bar{R}^{(1)}(z,s)$, $\bar{R}^{(2)}(z,s)$ and $\bar{V}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (73) to (77). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + R^{(1)}(1) + R^{(2)}(1) + V(1) + Q = 1.$$

The steady state probabilities for $M^{[X]}/G/1$ queue with two types of service, optional re-service and Bernoulli vacation are given by

$$P^{(1)}(1) = \frac{\lambda p_1 E(I)E(B_1)Q}{dr},$$

$$P^{(2)}(1) = \frac{\lambda p_2 E(I)E(B_2)Q}{dr},$$

$$R^{(1)}(1) = \frac{\lambda r_1 p_1 E(I)E(B_1)Q}{dr},$$

$$R^{(2)}(1) = \frac{\lambda r_2 p_2 E(I)E(B_2)Q}{dr},$$

$$V(1) = \frac{\lambda\theta E(I)E(V)Q}{dr},$$

where

$$dr = 1 - \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)], \tag{79}$$

$P^{(1)}(1)$, $P^{(2)}(1)$, $R^{(1)}(1)$, $R^{(2)}(1)$, $V(1)$ and Q are the steady state probabilities that the server is providing type 1 service, type2 service, type 1 re-optional service, type 2 re-optional service, server under vacation and idle respectively without regard to the number of customers in the queue. Thus multiplying both sides of equations (73) to (77) by s, taking limit as $s \to 0$, applying property (78) and simplifying, we obtain

$$P^{(1)}(z) = \frac{p_1[\bar{B}_1(b) - 1]Q}{D(z)}, \tag{80}$$

$$P^{(2)}(z) = \frac{p_2[\bar{B}_2(b) - 1]Q}{D(z)}, \tag{81}$$

$$R^{(1)}(z) = \frac{p_1 r_1 \bar{B}_1(b)[\bar{B}_1(b) - 1]Q}{D(z)}, \tag{82}$$

$$R^{(2)}(z) = \frac{p_2 r_2 \bar{B}_2(b)[\bar{B}_2(b) - 1]Q}{D(z)}, \tag{83}$$

$$V(z) = \frac{\theta[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) + p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))][\bar{V}(b) - 1]}{D(z)}, \tag{84}$$

where

$$\begin{aligned} D(z) = z - (1 - \theta + \theta\bar{V}(b))[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) \\ +p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))], \end{aligned} \tag{85}$$

and $b = \lambda - \lambda C(z)$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (80) to (84), we obtain

$$\begin{aligned} W_q(z) \quad &= P^{(1)}(z) + P^{(2)}(z) + R^{(1)}(z) + R^{(2)}(z) + V(z) \\ &= \frac{p_1[\bar{B}_1(b) - 1]Q}{D(z)} + \frac{p_2[\bar{B}_2(b) - 1]Q}{D(z)} + \frac{p_1 r_1 \bar{B}_1(b)[\bar{B}_1(b) - 1]Q}{D(z)} \\ &+ \frac{\theta[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) + p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))][\bar{V}(b) - 1]Q}{D(z)} \\ &+ \frac{p_2 r_2 \bar{B}_2(b)[\bar{B}_2(b) - 1]Q}{D(z)}. \end{aligned} \tag{86}$$

In order to find Q, we use the normalization condition $W_q(1)$ + Q =1. We see that for z=1, $W_q(1)$ is indeterminate of the form $0/0$. Therefore, we apply L'Hospital's rule and on simplifying, we get

$$W_q(1) = \frac{\lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)]Q}{dr}, \tag{87}$$

where C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $E(V) = -\bar{V}'(0)$, $E(B_i) = -\bar{B}_i'(0)$ for i = $1, 2$.

Therefore, adding Q to equation (87), equating to 1 and simplifying, we get

$$Q = 1 - \rho, \tag{88}$$

and hence the utilization factor ρ of the system is given by

$$\rho = \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2) + \theta E(V)], \tag{89}$$

where $\rho < 1$, is the stability condition under which the steady state exists. Equation (88) gives the probability that the server is idle.

By knowing Q from (88), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

6. The average queue size and average waiting time

Let L_q denote the mean number of customers in the queue. Then,

$$L_q = \frac{d}{dz} W_q(z) \;\text{ at } z = 1,$$

since this formula gives $0/0$ form. Then we write $W_q(z)$ given in (86) as

$$W_q(z) = \frac{N(z)}{D(z)} Q,$$

where

$$\begin{aligned} N(z) =& p_1(\bar{B}_1(b) - 1)(1 + r_1\bar{B}_1(b)) + p_2(\bar{B}_2(b) - 1)(1 + r_2\bar{B}_2(b)) \\ & + \theta[p_1\bar{B}_1(b)(1 - r_1 + r_1\bar{B}_1(b)) + p_2\bar{B}_2(b)(1 - r_2 + r_2\bar{B}_2(b))] \\ & \times (\bar{V}(b) - 1), \end{aligned}$$

and $D(z)$ given in equation (85). Then, we use

$$
\begin{aligned}
L_q &= \lim_{z\to 1}\frac{d}{dz}W_q(z) \\
&= \lim_{z\to 1}\left[\frac{D'(z)N''(z)-N'(z)D''(z)}{2(D'(z))^2}\right]Q \\
&= \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q,
\end{aligned}
\tag{90}
$$

where primes and double primes in equation (90) denote first and second derivative at $z = 1$ respectively. Carrying out the derivative at $z = 1$, we have

$$
\begin{aligned}
N'(1) &= \lambda E(I)[p_1(1+r_1)E(B_1)+p_2(1+r_2)E(B_2)+\theta E(V)], \\
N''(1) &= \lambda^2(E(I))^2[p_1(1+r_1)E(B_1^2)+p_2(1+r_2)E(B_2^2)+\theta E(V^2)] \\
&\quad +\lambda E(I(I-1))[p_1(1+r_1)E(B_1)+p_2(1+r_2)E(B_2)+\theta E(V)] \\
&\quad +2\lambda^2(E(I))^2[p_1r_1(E(B_1))^2+p_2r_2(E(B_2))^2] \\
&\quad +2\theta\lambda^2(E(I))^2E(V)[p_1(1+r_1)E(B_1)+p_2(1+r_2)E(B_2)], \\
D'(1) &= 1-\lambda E(I)[p_1(1+r_1)E(B_1)+p_2(1+r_2)E(B_2)+\theta E(V)], \\
D''(1) &= -2\lambda^2(E(I))^2\theta E(V)[p_1(1+r_1)E(B_1)+p_2(1+r_2)E(B_2)] \\
&\quad -\lambda^2(E(I))^2[\theta E(V^2)+p_1(1+r_1)E(B_1^2)+p_2(1+r_2)E(B_2^2)] \\
&\quad -\lambda E(I(I-1))[\theta E(V)+p_1(1+r_1)E(B_1)+p_2(1+r_2)E(B_2)] \\
&\quad -2\lambda^2(E(I))^2[p_1r_1(E(B_1))^2+p_2r_2(E(B_2))^2],
\end{aligned}
$$

where $E(B_1^2)$, $E(B_2^2)$ and $E(V^2)$ are the second moment of type 1 service, type 2 service and vacation time respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Further, we find the average system size L by using Little's formula. Thus, we have

$$
L = L_q + \rho, \tag{91}
$$

where L_q is obtained from equation (90) and ρ is obtained from equation (89).

Let W_q and W denote the average waiting time in the queue and in the system respectively. Then, by using Little's formula, we obtain,

$$
W_q = \frac{L_q}{\lambda},
$$

$$
W = \frac{L}{\lambda},
$$

where L_q and L have been found in equations (90) and (91).

7. Particular cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

7.1. Case 1

If server has no vacation i.e., θ=0. Then, our model reduces to the $M^{[X]}/G/1$ queue with two types of service and optional re-service.

Using this in the main result of $(88), (89)$ and (90), we can find the the idle probability Q, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions

$$\begin{aligned}
Q &= 1 - \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)], \\
\rho &= \lambda E(I)[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)],
\end{aligned}$$

$$\begin{aligned}
L_q = &\frac{1}{2[1-\lambda E(I)(p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2))]} \\
&\Big\{\lambda^2(E(I))^2[p_1(1+r_1)E(B_1^2) + p_2(1+r_2)E(B_2^2)] \\
&+ \lambda E(I(I-1))[p_1(1+r_1)E(B_1) + p_2(1+r_2)E(B_2)] \\
&+ 2\lambda^2(E(I))^2[p_1r_1(E(B_1))^2 + p_2r_2(E(B_2))^2]\Big\}.
\end{aligned}$$

The above result coincides with results given by Madan et al. (2004).

7.2. Case 2

If there is no second type of service i.e., p_2= 0. Then, our model reduces to $M^{[X]}/G/1$ queue with re-service and Bernoulli vacation.

Using this in the main result of $(88), (89)$ and (90) we can find the idle probability Q, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions

$$\begin{aligned}
Q &= 1 - \lambda E(I)[(1+r_1)E(B_1) + \theta E(V)], \\
\rho &= \lambda E(I)[(1+r_1)E(B_1) + \theta E(V)], \\
L_q &= \frac{1}{2[1-\lambda E(I)((1+r_1)E(B_1) + \theta E(V))]} \\
&\quad \Big\{\lambda^2(E(I))^2[(1+r_1)E(B_1^2) + \theta E(V^2)] + \lambda E(I(I-1)) \\
&\quad \times [(1+r_1)E(B_1) + \theta E(V)] + 2\lambda^2(E(I))^2 r_1(E(B_1))^2\Big\}.
\end{aligned}$$

7.3. Case 3

If there is no second type of service, re-service, no first type reservice, no vacation and $C(z) = z$. i.e., $p_2 = 0$, $r_1 = 0$ and $\theta = 0$, $E(I) = 1$ and $E(I(I-1)) = 0$. Then, our model reduces $M/G/1$ queueing system.

Using this in the main result of $(88), (89)$ and (90), we can find the idle probability Q, utilization factor ρ and the mean queue size L_q can be simplified to the following expressions

$$Q = 1 - \lambda E(B_1),$$
$$\rho = \lambda E(B_1),$$
$$L_q = \frac{\lambda^2 E(B_1^2)}{2(1 - \lambda E(B_1))}.$$

We note that the above results coincide with the results given by Kashyap and Chaudhry (1988).

8. Numerical results

The most common distributions which are frequently used in the real time systems are the exponential and Poisson distributions. In-fact, the exponential distribution is widely accepted because the queueing models having exponential distribution are of practical utility and very easy to be handled. The main target of selecting a proper distribution and estimating their parameters is to provide a tractable analytical model giving a close approximation to the real life system under consideration.

To numerically illustrate the results obtained in this work, we consider that the service times and vacation times are exponentially distributed with rates μ_1, μ_2 and γ.

In order to see the effect of various parameters on server's idle time Q, utilization factor ρ and various other queue characteristics such as L_q, L, W_q, W.

We base our numerical example on the result found in case 1. For this purpose in Table I, we choose the following arbitrary values: $E(I) = 0.3$, $E(I(I-1)) = 0.04$, $r_1 = 0.4$, $r_2 = 0.5$, $\mu_1 = 6$, $\mu_2 = 4$ and $p_1 = 0.4, p_2 = 0.6$ while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the average queue size, system size of our queueing model are all increases.

We base our numerical example on the result found in case 2. For this purpose in Table II we choose the following arbitrary values: $E(I) = 0.3$, $E(I(I-1)) = 0.04$, $r_1 = 0.3$, $\mu_1 = 4$, $\theta = 0.6$, $\lambda = 2$ while γ varies from 1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, average queue size, system size of our queueing model are all decreases.

For the effect of the parameters λ, γ, W_q, W on the system performance measures, two dimen-

Table I: Computed values of various queue characteristics

λ	Q	ρ	L_q	L
0.1	0.990450	0.009550	0.000729	0.010279
0.2	0.980900	0.019100	0.001647	0.020747
0.3	0.971350	0.028650	0.002759	0.031409
0.4	0.961800	0.038200	0.004070	0.042270
0.5	0.952250	0.047750	0.005588	0.053338
0.6	0.942700	0.057300	0.007317	0.064617
0.7	0.933150	0.066850	0.009266	0.076116
0.8	0.923600	0.076400	0.011439	0.087839
0.9	0.914050	0.085950	0.013846	0.099796
1.0	0.904500	0.095500	0.016492	0.111992

Table II: Computed values of various queue characteristics

γ	Q	ρ	L_q	L
1	0.445000	0.555000	0.537798	1.092798
2	0.625000	0.375000	0.207760	0.582760
3	0.685000	0.315000	0.141372	0.450372
4	0.715000	0.285000	0.114969	0.399969
5	0.733000	0.267000	0.101191	0.368191
6	0.745000	0.255000	0.092846	0.347846
7	0.753570	0.246430	0.087290	0.333718
8	0.760000	0.240000	0.083342	0.323342
9	0.765000	0.235000	0.080401	0.315401
10	0.769000	0.231000	0.078129	0.309129

sional graphs are drawn.

Figure 1 shows that the increasing arrival rate λ waiting time in the queue and system also increases. Figure 2 shows that when vacation rate γ increases then waiting time in the queue and system decreases.

9. Conclusion

In this paper, we investigated a single server, batch arrival queueing systems with phase service and Bernoulli vacation. The probability generating functions of the number of customers in the queue and system, waiting time in the queue and system on Bernoulli vacation is found by using the supplementary variable technique. The phase service queueing models have found wide applications in the modeling and analysis of day-to-day as well as several industrial systems. The main aim of the present survey is to suggest a unified framework for analyzing the phase service models via queue theoretic approach. The phase service queueing models with the combination of different concepts have been reviewed. Queueing models with phase service are helpful for resolving the problem of congestion and can be treated as an effective tool, for reducing the blocking and delay of the concerned system and are preferred by the system analysts, engineers, and managers for depicting the more realistic scenario of congestion problems. Simulation of the given phenomenon has been done with the help of a real life situation. The numerical illustrations

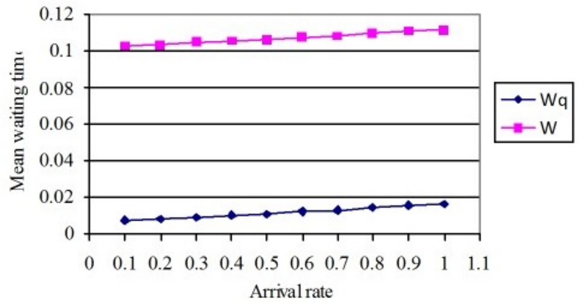

Figure 1: Arrival rate versus mean waiting time

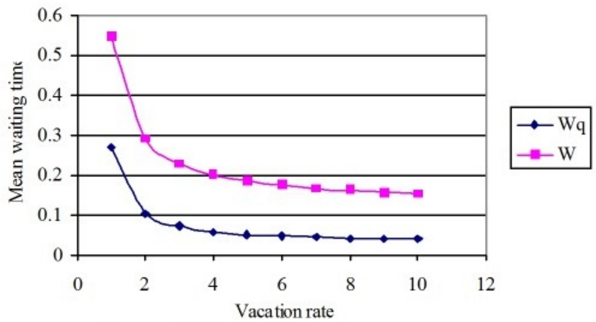

Figure 2: Vacation rate versus mean waiting time

given provide an insight regarding computational tractability of the analytical results established for the concerned model.

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