

## COMMON FOURIER TRANSFORM PAIRS

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$$1, \quad -\infty < t < \infty \leftrightarrow 2\pi\delta(\omega)$$

$$-0.5 + u(t) \leftrightarrow \frac{1}{j\omega}$$

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\delta(t) \leftrightarrow 1$$

$$\delta(t - c) \leftrightarrow e^{-j\omega c}, \quad c \text{ any real number}$$

$$e^{-bt}u(t) \leftrightarrow \frac{1}{j\omega + b}, \quad b > 0$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0), \quad \omega_0 \text{ any real number}$$

$$p_r(t) \leftrightarrow \tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$$

$$\tau \operatorname{sinc} \frac{\tau t}{2\pi} \leftrightarrow 2\pi p_r(\omega)$$

$$\left(1 - \frac{2|t|}{\tau}\right)p_r(t) \leftrightarrow \frac{\tau}{2} \operatorname{sinc}^2 \left( \frac{\tau\omega}{4\pi} \right)$$

$$\frac{\tau}{2} \operatorname{sinc}^2 \left( \frac{\tau t}{4\pi} \right) \leftrightarrow 2\pi \left(1 - \frac{2|\omega|}{\tau}\right)p_r(\omega)$$

$$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

$$\cos(\omega_0 t + \theta) \leftrightarrow \pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$$

$$\sin \omega_0 t \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

$$\sin(\omega_0 t + \theta) \leftrightarrow j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$$


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### COMMON LAPLACE TRANSFORM PAIRS

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$$u(t) \leftrightarrow \frac{1}{s}$$

$$u(t) - u(t - c) \leftrightarrow \frac{1 - e^{-cs}}{s}, \quad c > 0$$

$$t^N u(t) \leftrightarrow \frac{N!}{s^{N+1}}, \quad N = 1, 2, 3, \dots$$

$$\delta(t) \leftrightarrow 1$$

$$\delta(t - c) \leftrightarrow e^{-cs}, \quad c > 0$$

$$e^{-bt} u(t) \leftrightarrow \frac{1}{s + b}, \quad b \text{ real or complex}$$

$$t^N e^{-bt} u(t) \leftrightarrow \frac{N!}{(s + b)^{N+1}}, \quad N = 1, 2, 3, \dots$$

$$(\cos \omega t) u(t) \leftrightarrow \frac{s}{s^2 + \omega^2}$$

$$(\sin \omega t) u(t) \leftrightarrow \frac{\omega}{s^2 + \omega^2}$$

$$(\cos^2 \omega t) u(t) \leftrightarrow \frac{s^2 + 2\omega^2}{s(s^2 + 4\omega^2)}$$

$$(\sin^2 \omega t) u(t) \leftrightarrow \frac{2\omega^2}{s(s^2 + 4\omega^2)}$$

$$(e^{-bt} \cos \omega t) u(t) \leftrightarrow \frac{s + b}{(s + b)^2 + \omega^2}$$

$$(e^{-bt} \sin \omega t) u(t) \leftrightarrow \frac{\omega}{(s + b)^2 + \omega^2}$$

$$(t \cos \omega t) u(t) \leftrightarrow \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

$$(t \sin \omega t) u(t) \leftrightarrow \frac{2\omega s}{(s^2 + \omega^2)^2}$$

$$(t e^{-bt} \cos \omega t) u(t) \leftrightarrow \frac{(s + b)^2 - \omega^2}{[(s + b)^2 + \omega^2]^2}$$

$$(t e^{-bt} \sin \omega t) u(t) \leftrightarrow \frac{2\omega(s + b)}{[(s + b)^2 + \omega^2]^2}$$


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TABLE 7.3 Common z-Transform Pairs

$$\delta[n] \leftrightarrow 1$$

$$\delta[n - q] \leftrightarrow \frac{1}{z^q}, \quad q = 1, 2, \dots$$

$$u[n] \leftrightarrow \frac{z}{z - 1}$$

$$u[n] - u[n - q] \leftrightarrow \frac{z^q - 1}{z^{q-1}(z - 1)}, \quad q = 1, 2, \dots$$

$$a^n u[n] \leftrightarrow \frac{z}{z - a}, \quad a \text{ real or complex}$$

$$a^n u[n - q] = \frac{z^q}{z - a}$$

$$n u[n] \leftrightarrow \frac{z}{(z - 1)^2}$$

$$(n + 1) u[n] \leftrightarrow \frac{z^2}{(z - 1)^2}$$

$$n^2 u[n] \leftrightarrow \frac{z(z + 1)}{(z - 1)^3}$$

$$n a^n u[n] \leftrightarrow \frac{az}{(z - a)^2}$$

$$n^2 a^n u[n] \leftrightarrow \frac{az(z + a)}{(z - a)^3}$$

$$n(n + 1)a^n u[n] \leftrightarrow \frac{2az^2}{(z - a)^3}$$

$$(\cos \Omega n) u[n] \leftrightarrow \frac{z^2 - (\cos \Omega)z}{z^2 - (2 \cos \Omega)z + 1}$$

$$(\sin \Omega n) u[n] \leftrightarrow \frac{(\sin \Omega)z}{z^2 - (2 \cos \Omega)z + 1}$$

$$a^n (\cos \Omega n) u[n] \leftrightarrow \frac{z^2 - (a \cos \Omega)z}{z^2 - (2a \cos \Omega)z + a^2}$$

$$a^n (\sin \Omega n) u[n] \leftrightarrow \frac{(a \sin \Omega)z}{z^2 - (2a \cos \Omega)z + a^2}$$

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67

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TABLE 4.2 Properties of the DTFT

Property	Transform Pair/Property
Linearity	$ax[n] + bv[n] \leftrightarrow aX(\Omega) + bV(\Omega)$
Right or left shift in time	$x[n - q] \leftrightarrow X(\Omega)e^{-iq\Omega}$ , $q$ any integer
Time reversal	$x[-n] \leftrightarrow X(-\Omega) = \overline{X(\Omega)}$
Multiplication by $n$	$nx[n] \leftrightarrow j \frac{d}{d\Omega} X(\Omega)$
Multiplication by a complex exponential	$x[n]e^{j\pi f_0 n} \leftrightarrow X(\Omega - \Omega_0)$ , $\Omega_0$ real
Multiplication by $\sin \Omega_0 n$	$x[n] \sin \Omega_0 n \leftrightarrow \frac{j}{2}[X(\Omega + \Omega_0) - X(\Omega - \Omega_0)]$
Multiplication by $\cos \Omega_0 n$	$x[n] \cos \Omega_0 n \leftrightarrow \frac{1}{2}[X(\Omega + \Omega_0) + X(\Omega - \Omega_0)]$
Convolution in the time domain	$x[n] * v[n] \leftrightarrow \underbrace{X(\Omega)V(\Omega)}$
Summation	$\sum_{i=0}^n x[i] \leftrightarrow \frac{1}{1 - e^{-j\Omega}} X(\Omega) + \sum_{n=-\infty}^{\infty} \pi X(2\pi n) \delta(\Omega - 2\pi n)$
Multiplication in the time domain	$x[n]v[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega - \lambda)V(\lambda) d\lambda$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x[n]v[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{X(\Omega)}V(\Omega) d\Omega$
Special case of Parseval's theorem	$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi}  X(\Omega) ^2 d\Omega$
Relationship to inverse CTFT	If $x[n] \leftrightarrow X(\Omega)$ and $\gamma(t) \leftrightarrow X(\omega)p_{2\pi}(\omega)$ , then $x[n] = \gamma(t) _{t=n} = \gamma(n)$

## Table T.3

### Mathematical Relations

Certain of the mathematical relationship encountered in this text are listed below for convenient reference. However, this table is not intended as a substitute for more comprehensive handbooks.

#### Trigonometric identities

$$\begin{aligned} e^{\pm j\theta} &= \cos \theta \pm j \sin \theta \\ e^{j2\alpha} + e^{j2\beta} &= 2 \cos (\alpha - \beta) e^{j(\alpha+\beta)} \\ e^{j2\alpha} - e^{j2\beta} &= j2 \sin (\alpha - \beta) e^{j(\alpha+\beta)} \\ \cos \theta &= \frac{1}{2} (e^{j\theta} + e^{-j\theta}) = \sin (\theta + 90^\circ) \\ \sin \theta &= \frac{1}{2j} (e^{j\theta} - e^{-j\theta}) = \cos (\theta - 90^\circ) \\ \sin^2 \theta + \cos^2 \theta &= 1 \\ \cos^2 \theta - \sin^2 \theta &= \cos 2\theta \\ \cos^2 \theta &= \frac{1}{2} (1 + \cos 2\theta) \\ \cos^3 \theta &= \frac{1}{4} (3 \cos \theta + \cos 3\theta) \end{aligned}$$

$$\begin{aligned} \sin^2 \theta &= \frac{1}{2} (1 - \cos 2\theta) \\ \sin^3 \theta &= \frac{1}{4} (3 \sin \theta - \sin 3\theta) \end{aligned}$$

$$\begin{aligned} \sin (\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos (\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \end{aligned}$$

$$\tan (\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

$$\begin{aligned} \sin \alpha \sin \beta &= \frac{1}{2} \cos (\alpha - \beta) - \frac{1}{2} \cos (\alpha + \beta) \\ \cos \alpha \cos \beta &= \frac{1}{2} \cos (\alpha - \beta) + \frac{1}{2} \cos (\alpha + \beta) \\ \sin \alpha \cos \beta &= \frac{1}{2} \sin (\alpha - \beta) + \frac{1}{2} \sin (\alpha + \beta) \end{aligned}$$

$$A \cos (\theta + \alpha) + B \cos (\theta + \beta) = C \cos \theta - S \sin \theta = R \cos (\theta + \phi)$$

where

$$C = A \cos \alpha + B \cos \beta$$

$$S = A \sin \alpha + B \sin \beta$$

$$R = \sqrt{C^2 + S^2} = \sqrt{A^2 + B^2 + 2AB \cos (\alpha - \beta)}$$

$$\phi = \arctan \frac{S}{C} = \arctan \frac{A \sin \alpha + B \sin \beta}{A \cos \alpha + B \cos \beta}$$

Series expansions and approximations

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots \quad |nx| < 1$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \dots$$

$$a^x = 1 + x \ln a + \frac{1}{2!}(x \ln a)^2 + \dots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

$$\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots$$

$$\arctan x = \begin{cases} x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots & |x| < 1 \\ \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \dots & x > 1 \end{cases}$$

$$\text{sinc } x = 1 - \frac{1}{3!}(\pi x)^2 + \frac{1}{5!}(\pi x)^4 - \dots$$

$$J_n(x) = \frac{1}{n!} \left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!} \left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!} \left(\frac{x}{2}\right)^{n+4} - \dots$$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \quad x \gg 1$$

$$I_0(x) \approx \begin{cases} e^{x^2/4} & x^2 \ll 1 \\ e^x / \sqrt{2\pi x} & x \gg 1 \end{cases}$$

Summations

$$\sum_{m=1}^M m = \frac{M(M+1)}{2}$$

$$\sum_{m=1}^M m^2 = \frac{M(M+1)(2M+1)}{6}$$

$$\sum_{m=1}^M m^3 = \frac{M^2(M+1)^2}{4}$$

$$\sum_{m=0}^M x^m = \frac{x^{M+1} - 1}{x - 1}$$

The inverse Laplace transform of  $X(s)$  is

$$\text{Left: } \frac{1}{s} - \frac{1}{s+T} \quad x(t) = 1 - e^{-t}, \quad 0 \leq t \quad \checkmark$$

Hence,

$$\begin{aligned} X(z) &= \mathcal{Z}[1 - e^{-t}] = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}} \\ &= \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})} \\ &= \frac{(1 - e^{-T})z}{(z - 1)(z - e^{-T})} \end{aligned}$$

*Comments.* Just as in working with the Laplace transformation, a table of  $z$  transforms of commonly encountered functions is very useful for solving problems in the field of discrete-time systems. Table 2-1 is such a table.

TABLE 2-1 TABLE OF  $z$  TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1	—	—	Kronecker delta $\delta_0(k)$ 1, $k = 0$ 0, $k \neq 0$	1
2	—	—	$\delta_0(n - k)$ 1, $n = k$ 0, $n \neq k$	$z^{-k}$
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1 - z^{-1}}$
4	$\frac{1}{s + a}$	$e^{-at}$	$e^{-akT}$	$\frac{1}{1 - e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	$t$	$kT$	$\frac{Tz^{-1}}{(1 - z^{-1})^2}$
6	$\frac{2}{s^3}$	$t^2$	$(kT)^2$	$\frac{T^2 z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} = \frac{T^2 z^2 (2 + 1)}{(z - 1)^3}$
7.	$\frac{6}{s^4}$	$t^3$	$(kT)^3$	$\frac{T^3 z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$
8.	$\frac{a}{s(s + a)}$	$1 - e^{-at}$	$1 - e^{-akT}$	$\frac{(1 - e^{-aT})z^{-1}}{(1 - z^{-1})(1 - e^{-aT}z^{-1})} \quad \checkmark$
9	$\frac{b - a}{(s + a)(s + b)}$	$e^{-at} - e^{-bt}$	$e^{-akT} - e^{-bkT}$	$\frac{(e^{-aT} - e^{-bT})z^{-1}}{(1 - e^{-aT}z^{-1})(1 - e^{-bT}z^{-1})}$
10	$\frac{1}{(s + a)^2}$	$te^{-at}$	$kTe^{-akT}$	$\frac{Tz^{-1}}{(1 - e^{-aT}z^{-1})^2}$
11	$\frac{s}{(s + a)^3}$	$(1 - at)e^{-at}$	$(1 - akT)e^{-akT}$	$\frac{1 - (1 + aT)e^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$

$$\frac{a}{s(s+a)} \rightarrow 1 - e^{-at} \rightarrow \frac{z^2 (1 - e^{-aT})}{(z-1)(z - e^{-aT})} = \frac{(1 - e^{-aT})z^{-1}}{(1 - z^{-1})(1 - e^{-aT}z^{-1})} \quad \checkmark$$

TABLE 2-2 IMPORTANT PROPERTIES AND THEOREMS OF THE  $z$  TRANSFORM

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z) - zx(0)$
4.	$x(t+2T)$	$z^2 X(z) - z^2 x(0) - zx(T)$
5.	$x(k+2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
6.	$x(t+kT)$	$z^k X(z) - z^k x(0) - z^{k-1} x(1) - \dots - zx(kT - T)$
7.	$x(t-kT)$	$z^{-k} X(z)$
8.	$x(n+k)$	$z^k X(z) - z^k x(0) - z^{k-1} x(1) - \dots - zx(k-1)$
9.	$x(n-k)$	$z^{-k} X(z)$
10.	$t x(t)$	$-Tz \frac{d}{dz} X(z)$
11.	$k x(k)$	$-z \frac{d}{dz} X(z)$
12.	$e^{-at} x(t)$	$X(z e^{aT})$
13.	$e^{-ak} x(k)$	$X(z e^a)$
14.	$a^k x(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^k x(k)$	$-z \frac{d}{dz} X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z \rightarrow \infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z \rightarrow 1} [(1 - z^{-1}) X(z)]$ if $(1 - z^{-1}) X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k-1)$	$(1 - z^{-1}) X(z)$
19.	$\Delta x(k) = x(k+1) - x(k)$	$(z - 1) X(z) - zx(0)$
20.	$\sum_{k=0}^n x(k)$	$\frac{1}{1 - z^{-1}} X(z)$
21.	$\frac{\partial}{\partial a} x(t, a)$	$\frac{\partial}{\partial a} X(z, a)$
22.	$k^m x(k)$	$\left(-z \frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^n x(kT) y(nT - kT)$	$X(z) Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

$$\begin{aligned} \mathcal{Z}\{x(k-n)\} &= z^{-n} X(z) \\ \mathcal{Z}\{x(k+n)\} &= z^n \left\{ X(z) - \sum_{k=0}^{n-1} x(k) z^{-k} \right\} \end{aligned}$$

### Sec. 2-6 z Transform Method for Solving Difference Equations

53

TABLE 2-3 z TRANSFORMS OF  $x(k+m)$  AND  $x(k-m)$

Discrete function	$z$ Transform
$x(k+4)$	$z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - zx(3)$
$x(k+3)$	$z^3 X(z) - z^3 x(0) - z^2 x(1) - zx(2)$
$x(k+2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1} X(z)$
$x(k-2)$	$z^{-2} X(z)$
$x(k-3)$	$z^{-3} X(z)$
$x(k-4)$	$z^{-4} X(z)$

#### Example 2-18

Solve the following difference equation by use of the  $z$  transform method:

$$x(k+2) + 3x(k+1) + 2x(k) = 0, \quad x(0) = 0, \quad x(1) = 1$$

First note that the  $z$  transforms of  $x(k+2)$ ,  $x(k+1)$ , and  $x(k)$  are given, respectively, by

$$\mathcal{Z}[x(k+2)] = z^2 X(z) - z^2 x(0) - zx(1)$$

$$\mathcal{Z}[x(k+1)] = zX(z) - zx(0)$$

$$\mathcal{Z}[x(k)] = X(z)$$

Taking the  $z$  transforms of both sides of the given difference equation, we obtain

$$z^2 X(z) - z^2 x(0) - zx(1) + 3zX(z) - 3zx(0) + 2X(z) = 0$$

Substituting the initial data and simplifying gives

$$\begin{aligned} X(z) &= \frac{z}{z^2 + 3z + 2} = \frac{z}{(z+1)(z+2)} = \frac{z}{z+1} - \frac{z}{z+2} \\ &= \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}} \end{aligned}$$

Noting that

$$\mathcal{Z}^{-1}\left[\frac{1}{1+z^{-1}}\right] = (-1)^k, \quad \mathcal{Z}^{-1}\left[\frac{1}{1+2z^{-1}}\right] = (-2)^k$$

(see Fig. 2-1)  
from (1)  
Fig. 2-1

we have

$$x(k) = (-1)^k - (-2)^k, \quad k = 0, 1, 2, \dots$$

#### Example 2-19

Obtain the solution of the following difference equation in terms of  $x(0)$  and  $x(1)$ :

$$x(k+2) + (a+b)x(k+1) + abx(k) = 0$$

where  $a$  and  $b$  are constants and  $k = 0, 1, 2, \dots$

## Boolean Algebra

- i)  $0 \cdot 0 = 0 \quad (x \cdot 0 = 0)$
- ii)  $1+1=1 \quad (x+1=1)$
- iii)  $1 \cdot 1 = 1 \quad (x \cdot 1 = x)$
- v)  $0+0=0 \quad (x+0=x)$
- vi)  $0 \cdot 1 = 1 \cdot 0 = 0$
- vii)  $1+0=0+1=1$
- viii) if  $x=0$  then  $\bar{x}=1$
- ix) if  $x=1$  then  $\bar{x}=0$



$$(8) x \cdot x = x$$

$$(9) x+x = x$$

$$(10) x \cdot \bar{x} = 0$$

$$\therefore x + \bar{x} = 1 \quad (12) \bar{\bar{x}} = x$$

# Excitation Table

of J-K, R-S, D & T FF

S.	R	$Q(t+1)$
0	0	$Q(t)$
0	1	0
1	0	1
1	1	X

$\Leftarrow$  characteristic  
table

$Q(t)$	$Q(t+1)$	S	R
0	0	0	X
0	1	1	0
1	0	0	1
1	1	X	0

$\Rightarrow$  excitation  
table

D	$Q(t+1)$
0	0
1	1

$Q(t)$	$Q(t+1)$	D
0	0	0
0	1	1
1	0	0
1	1	1

