Fundamentals of Engineering Reference Handbook

UNITS

This handbook uses the metric system of units. Ultimately, the FE examination will be entirely metric. However, currently some __the problems use both metric and U.S. Customary System (USCS). In the USCS system of units, both force and mass are called pounds. Therefore, one must distinguish the pound-force (lbf) from the pound-mass (lbm).

The pound-force is that force which accelerates one pound-mass at 32.174 ft/s^2. Thus, 1 lbf = 32.174 lbm-ft/s^2. The expression 32.174 lbm-ft/(lbf-s^2) is designated as g_c and is used to resolve expressions involving both mass and force expressed as pounds. For instance, in writing Newton's second law, the equation would be written as $F = ma/g_c$, where F is in lbf, m in lbm, and a is in ft/s^2.

Similar expressions exist for other quantities. Kinetic Energy: $KE = mv^2/2g_c$, with KE in (ft-lbf); Potential Energy: $PE = mgh/g_c$, with PE in (ft-lbf); Fluid Pressure: $p = \rho gh/g_c$, with p in (lbf/ft^2); Specific Weight: $SW = \rho g/g_c$, in (lbf/ft^3); Shear Stress: $\tau = (\mu/g_c)(dv/dy)$, with shear stress in (lbf/ft^2). In all these examples, g_c should be regarded as a unit conversion factor. It is frequently not written explicitly in engineering equations. However, its use is required to produce a consistent set of units.

Note that the conversion factor g_c [lbm-ft/(lbf-s^2)] should not be confused with the local acceleration of gravity g, which has different units (m/s^2) and may be either its standard value (9.807 m/s^2) or some other local value.

If the problem is presented in USCS units, it may be necessary to use the constant g_c in the equation to have a consistent set of units.

METRIC PREFIXES		
Multiple	Prefix	Symbol
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{1}	deka	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E

COMMONLY USED EQUIVALENTS	
1 gallon of water weighs	8.34 lbf
1 cubic foot of water weighs	62.4 lbf
1 cubic inch of mercury weighs	0.491 lbf
The mass of one cubic meter of water is 1,000 kilograms	

TEMPERATURE CONVERSIONS

°F = 1.8 (°C) + 32

°C = (°F – 32)/1.8

°R = °F + 459.69

K = °C + 273.15

FUNDAMENTAL CONSTANTS

ntity		Symbol	Value	Units
ron charge		e	1.6022×10^{-19}	C (coulombs)
day constant		$\mathcal{F}$	96,485	coulombs/(mol)
constant	metric	$\bar{R}$	8,314	J/(kmol·K)
constant	metric	$\bar{R}$	8.314	kPa·m^3/(kmol·K)
constant	USCS	$\bar{R}$	1,545	ft-lbf/(lb mole-°R)
		$\bar{R}$	0.08206	L-atm/mole-K
itation - newtonian constant		G	6.673×10^{-11}	m^3/(kg·s^2)
itation - newtonian constant		G	6.673×10^{-11}	N·m^2/kg^2
ty acceleration (standard)	metric	g	9.807	m/s^2
ty acceleration (standard)	USCS	g	32.174	ft/s^2
r volume (ideal gas), T = 273.15K, p = 101.3 kPa		V_m	22,414	L/kmol
d of light in vacuum		c	299,792,000	m/s

CONVERSION FACTORS

Multiply	By	To Obtain
acre	43,560	square feet (ft^2)
ampere-hr (A-hr)	3,600	coulomb (C)
ångström (Å)	1×10^{-10}	meter (m)
atmosphere (atm)	76.0	cm, mercury (Hg)
atm, std	29.92	in, mercury (Hg)
atm, std	14.70	lbf/in^2 abs (psia)
atm, std	33.90	ft, water
atm, std	1.013×10^{5}	pascal (Pa)
bar	1×10^{5}	Pa
barrels–oil	42	gallons–oil
Btu	1,055	joule (J)
Btu	2.928×10^{-4}	kilowatt-hr (kWh)
Btu	778	ft-lbf
Btu/hr	3.930×10^{-4}	horsepower (hp)
Btu/hr	0.293	watt (W)
Btu/hr	0.216	ft-lbf/sec
calorie (g-cal)	3.968×10^{-3}	Btu
cal	1.560×10^{-6}	hp-hr
cal	4.186	joule (J)
cal/sec	4.186	watt (W)
centimeter (cm)	3.281×10^{-2}	foot (ft)
cm	0.394	inch (in)
centipoise (cP)	0.001	pascal·sec (Pa·s)
centistokes (cSt)	1×10^{-6}	m^2/sec (m^2/s)
cubic feet/second (cfs)	0.646317	million gallons/day (mgd)
cubic foot (ft^3)	7.481	gallon
cubic meters (m^3)	1,000	Liters
electronvolt (eV)	1.602×10^{-19}	joule (J)
foot (ft)	30.48	cm
ft	0.3048	meter (m)
ft-pound (ft-lbf)	1.285×10^{-3}	Btu
ft-lbf	3.766×10^{-7}	kilowatt-hr (kWh)
ft-lbf	0.324	calorie (g-cal)
ft-lbf	1.356	joule (J)
ft-lbf/sec	1.818×10^{-3}	horsepower (hp)
gallon (US Liq)	3.785	liter (L)
gallon (US Liq)	0.134	ft^3
gallons of water	8.3453	pounds of water
gamma (γ, Γ)	1×10^{-9}	tesla (T)
gauss	1×10^{-4}	T
gram (g)	2.205×10^{-3}	pound (lbm)
hectare	1×10^{4}	square meters (m^2)
hectare	2.47104	acres
horsepower (hp)	42.4	Btu/min
hp	745.7	watt (W)
hp	33,000	(ft-lbf)/min
hp	550	(ft-lbf)/sec
hp-hr	2,544	Btu
hp-hr	1.98×10^{6}	ft-lbf
hp-hr	2.68×10^{6}	joule (J)
hp-hr	0.746	kWh
inch (in)	2.540	centimeter (cm)
in of Hg	0.0334	atm
in of Hg	13.60	in of H_2O

Multiply	By	To Obtain
joule (J)	9.478×10^{-4}	Btu
J	0.7376	ft-lbf
J	1	newton·m (N·m)
J/s	1	watt (W)
kilogram (kg)	2.205	pound (lbm)
kgf	9.8066	newton (N)
kilometer (km)	3,281	feet (ft)
km/hr	0.621	mph
kilopascal (kPa)	0.145	lbf/in^2 (psi)
kilowatt (kW)	1.341	horsepower (hp)
kW	3,413	Btu/hr
kW	737.6	(ft-lbf)/sec
kW-hour (kWh)	3,413	Btu
kWh	1.341	hp-hr
kWh	3.6×10^{6}	joule (J)
kip (K)	1,000	lbf
K	4,448	newton (N)
liter (L)	61.02	in^3
L	0.264	gal (US Liq)
L	10^{-3}	m^3
L/second (L/s)	2.119	ft^3/min (cfm)
L/s	15.85	gal (US)/min (gpm)
meter (m)	3.281	feet (ft)
m	1.094	yard
m/second (m/s)	196.8	feet/min (ft/min)
mile (statute)	5,280	feet (ft)
mile (statute)	1.609	kilometer (km)
mile/hour (mph)	88.0	ft/min (fpm)
mph	1.609	km/h
mm of Hg	1.316×10^{-3}	atm
mm of H_2O	9.678×10^{-5}	atm
newton (N)	0.225	lbf
N·m	0.7376	ft-lbf
N·m	1	joule (J)
pascal (Pa)	9.869×10^{-6}	atmosphere (atm)
Pa	1	$newton/m^2$ (N/m^2)
Pa·sec (Pa·s)	10	poise (P)
pound (lbm,avdp)	0.454	kilogram (kg)
lbf	4.448	N
lbf-ft	1.356	N·m
lbf/in^2 (psi)	0.068	atm
psi	2.307	ft of H_2O
psi	2.036	in of Hg
psi	6,895	Pa
radian	$180/\pi$	degree
stokes	1×10^{-4}	m^2/s
therm	1×10^{5}	Btu
watt (W)	3.413	Btu/hr
W	1.341×10^{-3}	horsepower (hp)
W	1	joule/sec (J/s)
weber/m^2 (Wb/m^2)	10,000	gauss

MATHEMATICS

STRAIGHT LINE

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is $\quad y - y_1 = m(x - x_1)$

Given two points: slope, $\quad m = (y_2 - y_1)/(x_2 - x_1)$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\,[(m_2 - m_1)/(1 + m_2 \cdot m_1)]$$

Two lines are perpendicular if $\quad m_1 = -1/m_2$

The distance between two points is

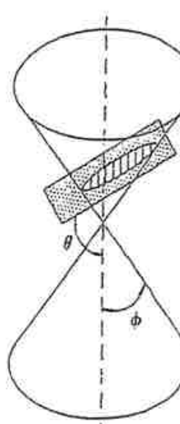
$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

QUADRATIC EQUATION

$$ax^2 + bx + c = 0$$

$$\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

CONIC SECTIONS



$$e = \text{eccentricity} = \cos\theta/(\cos\phi)$$

[Note: X' and Y', in the following cases, are translated axes.]

Case 1. Parabola $\quad e = 1$:

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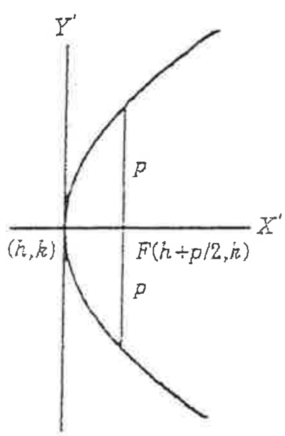

$$(y - k)^2 = 2p(x - h); \text{ Center at } (h,k)$$

is the standard form of the equation. Then, when $h = k = 0$,

Focus: $(p/2, 0)$; Directrix: $x = -p/2$

Case 2. Ellipse $\quad e < 1$:

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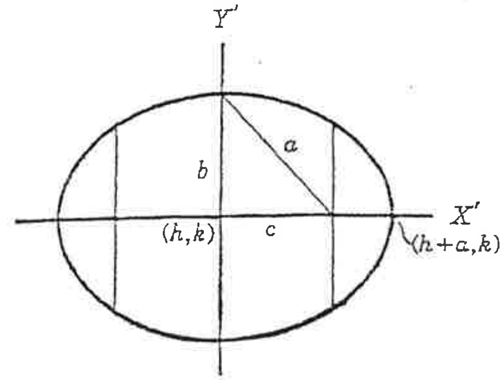

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1; \text{ Center at } (h,k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 - (b^2/a^2)} = c/a$

$b = a\sqrt{1 - e^2}$;

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Case 3. Hyperbola $\quad e > 1$:

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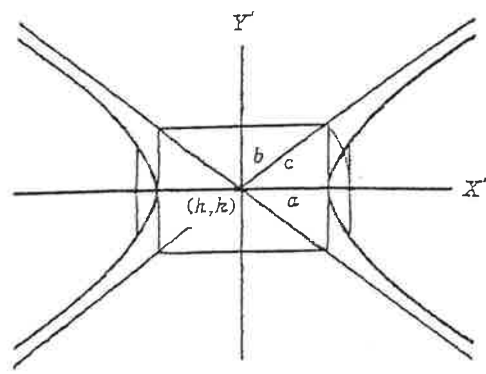


$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1; \text{ Center at } (h,k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 + (b^2/a^2)} = c/a$

$b = a\sqrt{e^2 - 1}$;

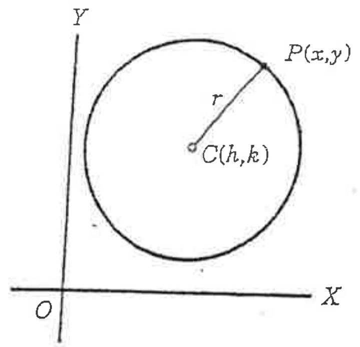
Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Case 4. Circle $\quad e = 0$:

$$(x - h)^2 + (y - k)^2 = r^2; \text{ Center at } (h,k)$$

is the general form of the equation with radius

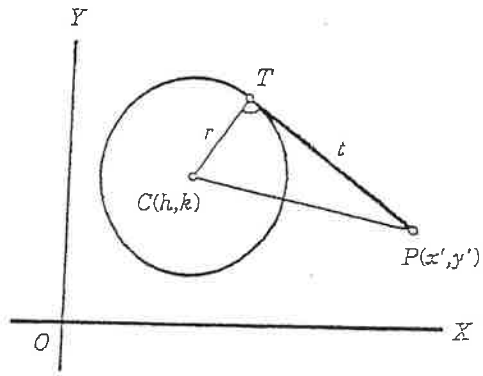
$$r = \sqrt{(x - h)^2 + (y - k)^2}$$



Length of the tangent from a point. Using the general form of the equation of a circle, the length of the tangent is found from

$$t^2 = (x'-h)^2 + (y'-k)^2 - r^2$$

by substituting the coordinates of a point $P(x',y')$ and the coordinates of the center of the circle into the equation and computing.



Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

where not both A and C are zero.
If $B^2 - AC < 0$, an *ellipse* is defined.
If $B^2 - AC > 0$, a *hyperbola* is defined.
If $B^2 - AC = 0$, the conic is a *parabola*.
If $A = C$ and $B = 0$, a *circle* is defined.
If $A = B = C = 0$, a *straight line* is defined.

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$$h = -a; \quad k = -b$$

$$r = \sqrt{a^2 + b^2 - c}$$

If $a^2 + b^2 - c$ is positive, a *circle*, center $(-a,-b)$.
If $a^2 + b^2 - c$ equals zero, a *point* at $(-a,-b)$.
If $a^2 + b^2 - c$ is negative, locus is *imaginary*.

QUADRIC SURFACE (SPHERE)

The general form of the equation is

$$(x-h)^2 + (y-k)^2 + (z-m)^2 = r^2$$

with center at (h,k,m).

In a three-dimensional space, the distance between two points is

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

LOGARITHMS

The logarithm of x to the Base b is defined by

$$\log_b(x) = c\text{, where} \quad b^c = x$$

Special definitions for $b = e$ or $b = 10$ are:

$$\ln x, \text{ Base} = e$$

$$\log x, \text{ Base} = 10$$

To change from one Base to another:

$$\log_b x = (\log_a x)/(\log_a b)$$

e.g., $\ln x = (\log_{10} x)/(\log_{10} e) = 2.302585\ (\log_{10} x)$

Identities $\log_b b^n = n$

$$\log x^c = c \log x; \quad x^c = \text{antilog}\,(c \log x)$$

$$\log xy = \log x + \log y$$

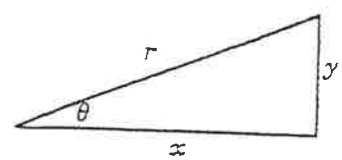
$$\log_b b = 1; \quad \log 1 = 0$$

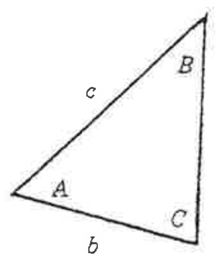
$$\log x/y = \log x - \log y$$

TRIGONOMETRY

Trigonometric functions are defined using a right triangle.

$\sin\ \theta = y/r$, $\cos\ \theta = x/r$
$\tan\ \theta = y/x$, $\cot\ \theta = x/y$
$\csc\ \theta = r/y$, $\sec\ \theta = r/x$





Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\ \cos A$$

$$b^2 = a^2 + c^2 - 2ac\ \cos B$$

$$c^2 = a^2 + b^2 - 2ab\ \cos C$$

Identities

$\csc\ \theta = 1/\sin\ \theta$; $\quad \tan\ \theta = \sin\ \theta/\cos\ \theta$
$\sec\ \theta = 1/\cos\ \theta$; $\quad \cot\ \theta = 1/\tan\ \theta$
$\sin^2\theta + \cos^2\theta = 1$; $\quad \tan^2\theta + 1 = \sec^2\theta$
$\cot^2\theta + 1 = \csc^2\theta$

$$\sin(\alpha + \beta) = \sin\ \alpha\ \cos\ \beta + \cos\ \alpha\ \sin\ \beta$$

$$\cos(\alpha + \beta) = \cos\ \alpha\ \cos\ \beta - \sin\ \alpha\ \sin\ \beta$$

$$\sin 2\alpha = 2 \sin\ \alpha\ \cos\ \alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\tan 2\alpha = (2 \tan\ \alpha)/(1 - \tan^2\alpha)$$

$$\cot 2\alpha = (\cot^2\alpha - 1)/(2 \cot\ \alpha)$$

$$\tan(\alpha + \beta) = (\tan\ \alpha + \tan\ \beta)/(1 - \tan\ \alpha\ \tan\ \beta)$$

$$\cot(\alpha + \beta) = (\cot\ \alpha\ \cot\ \beta - 1)/(\cot\ \alpha + \cot\ \beta)$$

$$\sin(\alpha - \beta) = \sin\ \alpha\ \cos\ \beta - \cos\ \alpha\ \sin\ \beta$$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\tan(\alpha-\beta) = (\tan\alpha - \tan\beta)/(1+\tan\alpha\tan\beta)$

$\cot(\alpha-\beta) = (\cot\alpha\cot\beta + 1)/(\cot\beta - \cot\alpha)$

$\sin(\alpha/2) = \pm\sqrt{(1-\cos\alpha)/2}$

$\cos(\alpha/2) = \pm\sqrt{(1+\cos\alpha)/2}$

$\tan(\alpha/2) = \pm\sqrt{(1-\cos\alpha)/(1+\cos\alpha)}$

$\cot(\alpha/2) = \pm\sqrt{(1+\cos\alpha)/(1-\cos\alpha)}$

$\sin\alpha\sin\beta = (1/2)[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\cos\alpha\cos\beta = (1/2)[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

$\sin\alpha\cos\beta = (1/2)[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\sin\alpha + \sin\beta = 2\sin(1/2)(\alpha+\beta)\cos(1/2)(\alpha-\beta)$

$\sin\alpha - \sin\beta = 2\cos(1/2)(\alpha+\beta)\sin(1/2)(\alpha-\beta)$

$\cos\alpha + \cos\beta = 2\cos(1/2)(\alpha+\beta)\cos(1/2)(\alpha-\beta)$

$\cos\alpha - \cos\beta = -2\sin(1/2)(\alpha+\beta)\sin(1/2)(\alpha-\beta)$

COMPLEX NUMBERS

Definition $i = \sqrt{-1}$

$(a+ib) + (c+id) = (a+c) + i(b+d)$

$(a+ib) - (c+id) = (a-c) + i(b-d)$

$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$(a+ib) + (a-ib) = 2a$

$(a+ib) - (a-ib) = 2ib$

$(a+ib)(a-ib) = a^2+b^2$

Polar Coordinates

$x = r\cos\theta;\quad y = r\sin\theta;\quad \theta = \arctan(y/x)$

$r = |x+iy| = \sqrt{x^2+y^2}$

$x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$(x+iy)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

Euler's Identity $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2},\quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Roots

If k is any positive integer, any complex number (other than zero) has k distinct roots. The k roots of $r(\cos\theta + i\sin\theta)$ can be found by substituting successively $n = 0, 1, 2, \ldots, (k-1)$ in the formula

$$w = \sqrt[k]{r}\left[\cos\left(\frac{\theta}{k} + n\frac{360°}{k}\right) + i\sin\left(\frac{\theta}{k} + n\frac{360°}{k}\right)\right]$$

MATRICES

A matrix is an ordered rectangular array of numbers with m rows and n columns. The element a_{ij} refers to row i and column j.

Multiplication

If $A = (a_{ik})$ is an $m \times n$ matrix and $B = (b_{kj})$ is an $n \times s$ matrix, the matrix product AB is an $m \times s$ matrix

$$C = (c_{ij}) = \left(\sum_{l=1}^{n} a_{il}b_{lj}\right)$$

where n is the common integer representing the number of columns of A and the number of rows of B (l and $k = 1, 2, \ldots, n$).

Addition

If $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices of the same size $m \times n$, the sum $A + B$ is the $m \times n$ matrix $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$.

Identity

The matrix $I = (a_{ij})$ is a square $n \times n$ identity matrix where $a_{ii} = 1$ for $i = 1, 2, \ldots, n$ and $a_{ij} = 0$ for $i \neq j$.

Transpose

The matrix B is the transpose of the matrix A if each entry b_{ji} in B is the same as the entry a_{ij} in A and conversely. In equation form, the transpose is

$$B = A^T$$

Inverse

The inverse B of a square $n \times n$ matrix A is

$$B = A^{-1} = \frac{\text{adj}(A)}{|A|},\quad \text{where}$$

adj(A) = adjoint of A (obtained by replacing A^T elements with their cofactors, see **DETERMINANTS**) and

$|A|$ = determinant of A.

DETERMINANTS

A *determinant of order* n consists of n^2 numbers, called the *elements* of the determinant, arranged in n rows and n columns and enclosed by two vertical lines. In any determinant, the *minor* of a given element is the determinant that remains after all of the elements are struck out that lie in the same row and in the same column as the given element. Consider an element which lies in the hth column and the kth row. The *cofactor* of this element is the value of the minor of the element (if $h + k$ is *even*), and it is the negative of the value of the minor of the element (if $h + k$ is *odd*).

If n is greater than 1, the *value* of a determinant of order n is the sum of the n products formed by multiplying each element of some specified row (or column) by its cofactor. This sum is called the *expansion of the determinant* [according to the elements of the specified row (or column)].

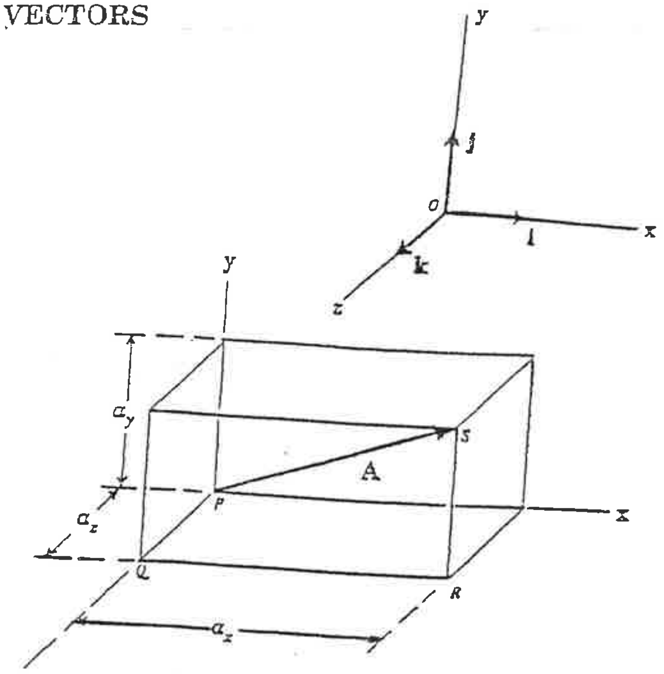
For a second-order determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

For a third-order determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

VECTORS



$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Addition and *subtraction*:

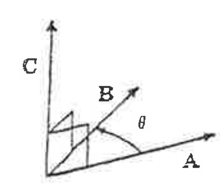
$$\mathbf{A} + \mathbf{B} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$$
$$\mathbf{A} - \mathbf{B} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k}$$

The *dot product* is a *scalar product* and represents the projection of $\mathbf{B}$ onto $\mathbf{A}$ times $|\mathbf{A}|$. It is given by

$$\mathbf{A}\cdot\mathbf{B} = a_xb_x + a_yb_y + a_zb_z$$
$$= |\mathbf{A}||\mathbf{B}| \cos\theta = \mathbf{B}\cdot\mathbf{A}$$

The *cross product* is a *vector product* of magnitude $|\mathbf{B}||\mathbf{A}| \sin\theta$ which is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$. The product is

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{B}\times\mathbf{A}$$



The sense of $\mathbf{A}\times\mathbf{B}$ is determined by the right-hand rule.

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\,\mathbf{n}\sin\theta,\quad \text{where}$$

$\mathbf{n}$ = unit vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$.

Gradient, Divergence, and Curl

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi$$

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

The Laplacian of a scalar function ϕ is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Identities

$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A};\quad \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$

$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2$

$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$

$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$

If $\mathbf{A}\cdot\mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$

$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$

$(\mathbf{B}+\mathbf{C})\times\mathbf{A} = \mathbf{B}\times\mathbf{A} + \mathbf{C}\times\mathbf{A}$

$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$

$\mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i};\quad \mathbf{j}\times\mathbf{k} = \mathbf{i} = -\mathbf{k}\times\mathbf{j}$

$\mathbf{k}\times\mathbf{i} = \mathbf{j} = -\mathbf{i}\times\mathbf{k}$

If $\mathbf{A}\times\mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or $\mathbf{A}$ is parallel to $\mathbf{B}$.

$\nabla^2\phi = \nabla\cdot(\nabla\phi) = (\nabla\cdot\nabla)\phi$

$\nabla\times\nabla\phi = 0$

$\nabla\cdot(\nabla\times A) = 0$

$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$

PROGRESSIONS AND SERIES

Arithmetic Progression

To determine whether a given finite sequence of numbers is an arithmetic progression, subtract each number from the following number. If the differences are equal, the series is arithmetic.

1. The first term is a.
2. The common difference is d.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = a + (n-1)d$$
$$S = n(a+l)/2 = n[2a + (n-1)d]/2$$

Geometric Progression

To determine whether a given finite sequence is a geometric progression, divide each number after the first by the preceding number. If the quotients are equal, the series is geometric.

1. The first term is a.
2. The common ratio is r.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = ar^{n-1}$$
$$S = a(1-r^n)/(1-r); \quad r \neq 1$$
$$S = (a-rl)/(1-r); \quad r \neq 1$$
$$\lim_{n\to\infty} S_n = a/(1-r); \quad r < 1$$

A G.P. converges if $|r| < 1$ and it diverges if $|r| \geq 1$.

Properties of Series

$$\sum_{i=1}^{n} c = nc; \quad c = \text{constant}$$
$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$
$$\sum_{i=1}^{n} (x_i + y_i - z_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i$$
$$\sum_{x=1}^{n} x = (n + n^2)/2$$

1. A power series in x, or in $x-a$, which is convergent in the interval $-1 < x < 1$ (or $-1 < x - a < 1$), defines a function of x which is continuous for all values of x within the interval and is said to represent the function in that interval.
2. A power series may be differentiated term by term, and the resulting series has the same interval of convergence as the original series (except possibly at the end points of the interval).
3. A power series may be integrated term by term provided the limits of integration are within the interval of convergence of the series.
4. Two power series may be added, subtracted, or multiplied, and the resulting series in each case is convergent, at least, in the interval common to the two series.
5. Using the process of long division (as for polynomials), two power series may be divided one by the other.

Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

is called *Taylor's series*. The function $f(x)$ is said to be expanded about the point a in a Taylor's series.

If $a = 0$, the Taylor's series becomes a *Maclaurin's series*.

PROBABILITY AND STATISTICS

Permutations and Combinations

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of n distinct objects *taken r at a time* is
$$P(n,r) = \frac{n!}{(n-r)!}$$
2. The number of different *combinations* of n distinct objects *taken r at a time* is
$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$
3. The number of different *permutations* of n objects *taken n at a time*, given that n_i are of type i,

where $i = 1, 2, \ldots, k$ and $\Sigma n_i = n$, is

$$P(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!n_2!\ldots n_k!}$$

Laws of Probability

***PROPERTY* 1** (General Character of Probability) The probability $P(E)$ of an event E is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

***PROPERTY* 2** (Law of Total Probability)

$$P(A+B) = P(A) + P(B) - P(A,B), \quad \text{where}$$

$P(A+B)$ = the probability that either A or B occur alone or that both occur together,
$P(A)$ = the probability that A occurs,
$P(B)$ = the probability that B occurs, and
$P(A,B)$ = the probability that both A and B occur simultaneously.

***PROPERTY* 3** (Law of Compound or Joint Probability) If neither $P(A)$ nor $P(B)$ is zero,

$$P(A,B) = P(A)P(B \mid A) = P(B)P(A \mid B),$$

where

$P(B \mid A)$ = the probability that B occurs given the fact that A has occurred, and
$P(A \mid B)$ = the probability that A occurs given the fact that B has occurred.

If either $P(A)$ or $P(B)$ is zero, then

$$P(A,B) = 0$$

Probability Functions

A random variable x has a probability associated with each of its values. The probability is termed a discrete probability if x can only assume the discrete values

$$x = X_1, X_2, \ldots, X_i, \ldots, X_N$$

The *discrete probability* of the event $X = x_i$ occurring is defined as $P(X_i)$.

Probability Density Functions

If x is continuous, then the *probability density function* $f(x)$ is defined so that

$\int_{x_1}^{x_2} f(x)\,dx$ = the probability that x lies between x_1 and x_2. The probability is determined by defining the equation for $f(x)$ and integrating between the values of x required.

Probability Distribution Functions

The *probability distribution function* $F(X_n)$ of the discrete probability function $P(X_i)$ is defined by

$$F(X_n) = \sum_{k=1}^{n} P(X_k) = P(X_i \le X_n)$$

When x is continuous, the *probability distribution function* $F(x)$ is defined by

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

which implies that $F(a)$ is the probability that $x \le a$.

The *expected value* $g(x)$ of any function is defined as

$$E\{g(x)\} = \int_{-\infty}^{x} g(t)\, f(t)\, dt$$

BINOMIAL DISTRIBUTION

$F(x)$ is the probability that x will occur in n trials. If p = probability of success and q = probability of failure $= 1 - p$, then

$$F(x) = C(n,x)\, p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

where

$x = 0, 1, 2, \ldots, n$,
$C(n,x)$ = the number of combinations, and
n, p = parameters.

NORMAL DISTRIBUTION (Gaussian Distribution)

This is a unimodal distribution, the mode being $x = \mu$, with two points of inflection (each located at a distance σ to either side of the mode). The averages of n observations tend to become normally distributed as n increases. The variate x is said to be normally distributed if its density function $f(x)$ is given by an expression of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad \text{where}$$

μ = the population mean,
σ = the standard deviation of the population, and
$-\infty \le x \le \infty$.

When $\mu = 0$ and $\sigma^2 = 1 = \sigma$, the distribution is called a *unit normal* distribution. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{where}$$

$-\infty \le x \le \infty$.

A unit normal distribution table is included in this section. In the table, the following notations are utilized:

$F(x)$ = the area under the curve from $-\infty$ to x,
$R(x)$ = the area under the curve from x to ∞, and
$W(x)$ = the area under the curve between $-x$ and x.

DISPERSION, MEAN, MEDIAN, AND MODE VALUES

If $X_1, X_2, \ldots, X_n$ represent the values of n items or observations from a population, the *arithmetic mean* of these items or observations, denoted $\bar{X}$, is defined as

$$\bar{X} = (1/n)(X_1 + X_2 + \ldots + X_n) = (1/n)\sum_{i=1}^{n} X_i$$

$\bar{X} \to \mu$ for sufficiently large values of n.

The *weighted arithmetic mean* is

$$\bar{X}_w = \frac{\Sigma\, w_i X_i}{\Sigma\, w_i}, \quad \text{where}$$

$\bar{X}_w$ = the weighted arithmetic mean,
X_i = the values of the observations to be averaged, and
w_i = the weight applied to the X_i value.

The *variance* of the observations is the *arithmetic mean of the squared deviations from the population mean*. In symbols, $X_1, X_2, \ldots, X_n$ represent the values of the n *sample* observations of a *population of size* N. If μ is the arithmetic mean of the population, the *population variance* is defined by

$$\sigma^2 = (1/N)[(X_1-\mu)^2 + (X_2-\mu)^2 + \ldots + (X_N-\mu)^2] = (1/N)\sum_{i=1}^{N}(X_i - \mu)^2$$

The *standard deviation* of a population is

$$\sigma = \sqrt{(1/N)\,\Sigma\,(X_i - \mu)^2}$$

The *sample variance* is

$$s^2 = [1/(n-1)]\sum_{i=1}^{n}(X_i - \bar{X})^2$$

The *sample standard deviation* is

$$s = \sqrt{[1/(n-1)]\,\Sigma\,(X_i - \bar{X})^2}$$

The *coefficient of variation* $= CV = s/\bar{X}$

The *geometric mean* $= \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$

The *root-mean-squared value* $= \sqrt{(1/n)\,\Sigma\, x_i^2}$

The *median* is defined as the *value of the middle item* when the data are *rank-ordered* and the number of items is *odd*. The *median* is the *average of the middle two items* when the rank-ordered data consists of an *even* number of items.

The *mode* of a set of data is the *value that occurs with greatest frequency*.

t-DISTRIBUTION

The variate t is defined as the quotient of two independent variates x and r where x *is unit normal* and r *is the root mean square* of n other independent *unit normal variates*; that is, $t = x/r$. The following is the t-distribution with n degrees of freedom:

$$F(t) = \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\sqrt{n\pi}} \frac{1}{(1+t^2/n)^{(n+1)/2}}$$

where $-\infty \le t \le \infty$.

A table is available at the end of this section which gives the values of $t_{\alpha,n}$ for values of α and n. Note that in view of the symmetry of the t-distribution, $t_{1-\alpha,n} = -t_{\alpha,n}$. The function for α follows:

$$\alpha = \int_{t_{\alpha,n}}^{\infty} f(t)\, dt$$

A table showing "Pertinent Equations From Probability and Statistics" is included in the **INDUSTRIAL ENGINEERING SECTION** of this handbook.

CONFIDENCE INTERVALS

Confidence Interval for the Mean μ of a Normal Distribution

(a) Standard deviation σ is known

$$\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

(b) Standard deviation σ is not known

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ corresponds to n-1 degrees of freedom

Confidence Interval for the Difference Between two Means μ_1 and μ_2

(a) Standard deviations σ_1 and σ_2 known

$$\bar{x}_1 - \bar{x}_2 - Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

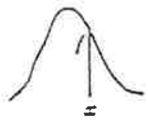
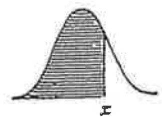
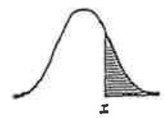
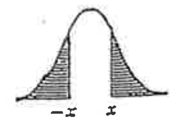
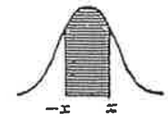
(b) Standard deviations $\sigma_1 = \sigma_2 = \sigma$ are not known

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2}\sqrt{\frac{(\frac{1}{n_1} + \frac{1}{n_2})(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2}}$$

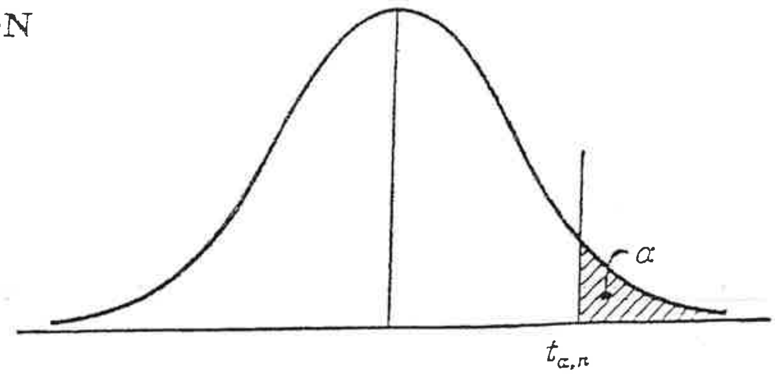
$$\le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + t_{\alpha/2}\sqrt{\frac{(\frac{1}{n_1} + \frac{1}{n_2})(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2}}$$

where $t_{\alpha/2}$ corresponds to $n_1 + n_2 - 2$ degrees of freedom.

UNIT NORMAL DISTRIBUTION

					
x	$f(x)$	$F(x)$	$R(x)$	$2R(x)$	$W(x)$
0.0	.3989	.5000	.5000	1.0000	0.0000
0.1	.3970	.5398	.4602	.9203	.0797
0.2	.3910	.5793	.4207	.8415	.1585
0.3	.3814	.6179	.3821	.7642	.2358
0.4	.3683	.6554	.3446	.6892	.3108
0.5	.3521	.6915	.3085	.6171	.3829
0.6	.3332	.7257	.2743	.5485	.4515
0.7	.3123	.7580	.2420	.4839	.5161
0.8	.2897	.7881	.2119	.4237	.5763
0.9	.2661	.8159	.1841	.3681	.6319
1.0	.2420	.8413	.1587	.3173	.6827
1.1	.2179	.8643	.1357	.2713	.7287
1.2	.1942	.8849	.1151	.2301	.7699
1.3	.1714	.9032	.0968	.1936	.8064
1.4	.1497	.9192	.0808	.1615	.8385
1.5	.1295	.9332	.0668	.1336	.8664
1.6	.1109	.9452	.0548	.1096	.8904
1.7	.0940	.9554	.0446	.0891	.9109
1.8	.0790	.9641	.0359	.0719	.9281
1.9	.0656	.9713	.0287	.0574	.9426
2.0	.0540	.9772	.0228	.0455	.9545
2.1	.0440	.9821	.0179	.0357	.9643
2.2	.0355	.9861	.0139	.0278	.9722
2.3	.0283	.9893	.0107	.0214	.9786
2.4	.0224	.9918	.0082	.0164	.9836
2.5	.0175	.9938	.0062	.0124	.9876
2.6	.0136	.9953	.0047	.0093	.9907
2.7	.0104	.9965	.0035	.0069	.9931
2.8	.0079	.9974	.0026	.0051	.9949
2.9	.0060	.9981	.0019	.0037	.9963
3.0	.0044	.9987	.0013	.0027	.9973
Fractiles					
1.2816	.1755	.9000	.1000	.2000	.8000
1.6449	.1031	.9500	.0500	.1000	.9000
1.9600	.0584	.9750	.0250	.0500	.9500
2.0537	.0484	.9800	.0200	.0400	.9600
2.3263	.0267	.9900	.0100	.0200	.9800
2.5758	.0145	.9950	.0050	.0100	.9900

t-DISTRIBUTION

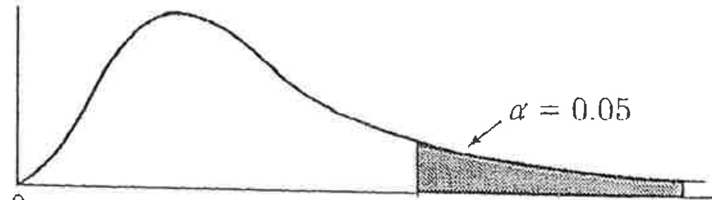


VALUES OF $t_{\alpha,n}$

n	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$	n
1	3.078	6.314	12.706	31.821	63.657	1
2	1.886	2.920	4.303	6.965	9.925	2
3	1.638	2.353	3.182	4.541	5.841	3
4	1.533	2.132	2.776	3.747	4.604	4
5	1.476	2.015	2.571	3.365	4.032	5
6	1.440	1.943	2.447	3.143	3.707	6
7	1.415	1.895	2.365	2.998	3.499	7
8	1.397	1.860	2.306	2.896	3.355	8
9	1.383	1.833	2.262	2.821	3.250	9
10	1.372	1.812	2.228	2.764	3.169	10
11	1.363	1.796	2.201	2.718	3.106	11
12	1.356	1.782	2.179	2.681	3.055	12
13	1.350	1.771	2.160	2.650	3.012	13
14	1.345	1.761	2.145	2.624	2.977	14
15	1.341	1.753	2.131	2.602	2.947	15
16	1.337	1.746	2.120	2.583	2.921	16
17	1.333	1.740	2.110	2.567	2.898	17
18	1.330	1.734	2.101	2.552	2.878	18
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20
21	1.323	1.721	2.080	2.518	2.831	21
22	1.321	1.717	2.074	2.508	2.819	22
23	1.319	1.714	2.069	2.500	2.807	23
24	1.318	1.711	2.064	2.492	2.797	24
25	1.316	1.708	2.060	2.485	2.787	25
26	1.315	1.706	2.056	2.479	2.779	26
27	1.314	1.703	2.052	2.473	2.771	27
28	1.313	1.701	2.048	2.467	2.763	28
29	1.311	1.699	2.045	2.462	2.756	29
inf.	1.282	1.645	1.960	2.326	2.576	inf.

Critical Values of F

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of F corresponding to a specified upper tail area (α).



Denominator df_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	**6.61**	**5.79**	**5.41**	**5.19**	**5.05**	**4.95**	**4.88**	**4.82**	**4.77**	**4.74**	**4.68**	**4.62**	**4.56**	**4.53**	**4.50**	**4.46**	**4.43**	**4.40**	**4.36**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	**4.96**	**4.10**	**3.71**	**3.48**	**3.33**	**3.22**	**3.14**	**3.07**	**3.02**	**2.98**	**2.91**	**2.85**	**2.77**	**2.74**	**2.70**	**2.66**	**2.62**	**2.58**	**2.54**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	**4.54**	**3.68**	**3.29**	**3.06**	**2.90**	**2.79**	**2.71**	**2.64**	**2.59**	**2.54**	**2.48**	**2.40**	**2.33**	**2.29**	**2.25**	**2.20**	**2.16**	**2.11**	**2.07**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	**4.35**	**3.49**	**3.10**	**2.87**	**2.71**	**2.60**	**2.51**	**2.45**	**2.39**	**2.35**	**2.28**	**2.20**	**2.12**	**2.08**	**2.04**	**1.99**	**1.95**	**1.90**	**1.84**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	**4.24**	**3.39**	**2.99**	**2.76**	**2.60**	**2.49**	**2.40**	**2.34**	**2.28**	**2.24**	**2.16**	**2.09**	**2.01**	**1.96**	**19.2**	**1.87**	**1.82**	**1.77**	**1.71**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	**4.17**	**3.32**	**2.92**	**2.69**	**2.53**	**2.42**	**2.33**	**2.27**	**2.21**	**2.16**	**2.09**	**2.01**	**1.93**	**1.89**	**1.84**	**1.79**	**1.74**	**1.68**	**1.62**
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

DIFFERENTIAL CALCULUS

The Derivative

For any function $y = f(x)$,

the derivative $= D_x y = dy/dx = y'$

$$y' = \lim_{\Delta x \to 0} [(\Delta y)/(\Delta x)]$$

$$= \lim_{\Delta x \to 0} \{[f(x + \Delta x) - f(x)]/(\Delta x)\}$$

$y' =$ the slope of the curve $f(x)$.

TEST FOR A MAXIMUM

$y = f(x)$ is a maximum for

$x = a$, if $f'(a) = 0$ and $f''(a) < 0$.

TEST FOR A MINIMUM

$y = f(x)$ is a minimum for

$x = a$, if $f'(a) = 0$ and $f''(a) > 0$.

TEST FOR A POINT OF INFLECTION

$y = f(x)$ has a point of inflection at $x = a$,

if $f''(a) = 0$, and

if $f''(x)$ changes sign as x increases through $x = a$.

The Partial Derivative

In a function of two independent variables x and y, a derivative with respect to one of the variables may be found if the other variable is *assumed* to remain constant. If *y is kept fixed*, the function

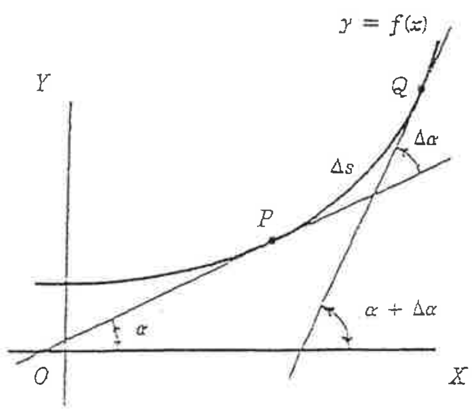
$$z = f(x, y)$$

becomes a function of the *single variable* x, and its derivative (if it exists) can be found. This derivative is called the *partial derivative of z with respect to x*. The partial derivative with respect to x is denoted as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

The Curvature of Any Curve

♦



The curvature K of a curve at P is the limit of its average curvature for the arc PQ as Q approaches P. This is also expressed as: the curvature of a curve at a given point is the rate-of-change of its inclination with respect to its arc length.

$$K = \lim_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}$$

CURVATURE IN RECTANGULAR COORDINATES

$$K = \frac{y''}{[1 + (y')^2]^{3/2}}$$

When it may be easier to differentiate the function with respect to y rather than x, the notation x' will be used for the derivative.

$$x' = dx/dy$$

$$K = \frac{-x''}{[1 + (x')^2]^{3/2}}$$

THE RADIUS OF CURVATURE

The *radius of curvature* R at any point on a curve is defined as the absolute value of the reciprocal of the curvature K at that point.

$$R = \frac{1}{|K|} \qquad (K \neq 0)$$

$$R = \frac{[1 + (y')^2]^{3/2}}{|y''|} \qquad (y'' \neq 0)$$

L'Hospital's Rule (L'Hôpital's Rule)

If the fractional function $f(x)/g(x)$ assumes one of the indeterminate forms $0/0$ or ∞/∞ (where a is finite or infinite), then

$$\lim_{x \to a} f(x)/g(x)$$

is equal to the first of the expressions

$$\lim_{x \to a} \frac{f'(x)}{g'(x)},\ \lim_{x \to a} \frac{f''(x)}{g''(x)},\ \lim_{x \to a} \frac{f'''(x)}{g'''(x)}$$

which is not indeterminate, provided such first indicated limit exists.

INTEGRAL CALCULUS

Fundamental Theorem

The *fundamental theorem* of the integral calculus is:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx$$

Also, $\Delta x_i \to 0$ for all i.

A table of derivatives and integrals is available on the next page. The integral equations can be used along with the following methods of integration:

A. Integration by Parts (integral equation #6);

B. Integration by Substitution, and

C. Separation of Rational Fractions into Partial Fractions.

♦ Wade, Thomas L., *Calculus*, Copyright © 1953 by Ginn & Company. Diagram reprinted by permission of Simon & Schuster Publishers.

DERIVATIVES and INDEFINITE INTEGRALS

In these formulas, u, v, and w represent functions of x. Also, a, c, and n represent constants. All arguments of the trigonometric functions are in radians. A constant of integration should be added to the integrals. To avoid terminology difficulty, the following definitions are followed: $\arcsin u = \sin^{-1} u$, $(\sin u)^{-1} = 1/\sin u$.

1. $dc/dx = 0$
2. $dx/dx = 1$
3. $d(cu)/dx = c\, du/dx$
4. $d(u + v - w)/dx = du/dx + dv/dx - dw/dx$
5. $d(uv)/dx = u\, dv/dx + v\, du/dx$
6. $d(uvw)/dx = uv\, dw/dx + uw\, dv/dx + vw\, du/dx$
7. $\dfrac{d(u/v)}{dx} = \dfrac{v\, du/dx - u\, dv/dx}{v^2}$
8. $d(u^n)/dx = nu^{n-1}\, du/dx$
9. $d[f(u)]/dx = \{d[f(u)]/du\}\, du/dx$
10. $du/dx = 1/(dx/du)$
11. $\dfrac{d(\log_a u)}{dx} = (\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$
12. $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$
13. $\dfrac{d(a^u)}{dx} = (\ln a)\, a^u \dfrac{du}{dx}$
14. $d(e^u)/dx = e^u\, du/dx$
15. $d(u^v)/dx = vu^{v-1}\, du/dx + (\ln u)\, u^v\, dv/dx$
16. $d(\sin u)/dx = \cos u\, du/dx$
17. $d(\cos u)/dx = -\sin u\, du/dx$
18. $d(\tan u)/dx = \sec^2 u\, du/dx$
19. $d(\cot u)/dx = -\csc^2 u\, du/dx$
20. $d(\sec u)/dx = \sec u \tan u\, du/dx$
21. $d(\csc u)/dx = -\csc u \cot u\, du/dx$
22. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad (-\pi/2 \le \sin^{-1} u \le \pi/2)$
23. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad (0 \le \cos^{-1} u \le \pi)$
24. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad (-\pi/2 < \tan^{-1} u < \pi/2)$
25. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad (0 < \cot^{-1} u < \pi)$
26. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$ $\quad (0 \le \sec^{-1} u < \pi/2)(-\pi \le \sec^{-1} u < -\pi/2)$
27. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$ $\quad (0 < \csc^{-1} u \le \pi/2)(-\pi < \csc^{-1} u \le -\pi/2)$

1. $\int df(x) = f(x)$
2. $\int dx = x$
3. $\int a f(x)\, dx = a \int f(x)\, dx$
4. $\int [u(x) \pm v(x)]\, dx = \int u(x)\, dx \pm \int v(x)\, dx$
5. $\int x^m\, dx = \dfrac{x^{m+1}}{m+1}$ $\quad (m \ne -1)$
6. $\int u(x)\, dv(x) = u(x)v(x) - \int v(x)\, du(x)$
7. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\ln|ax+b|$
8. $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$
9. $\int a^x\, dx = \dfrac{a^x}{\ln a}$
10. $\int \sin x\, dx = -\cos x$
11. $\int \cos x\, dx = \sin x$
12. $\int \sin^2 x\, dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$
13. $\int \cos^2 x\, dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4}$
14. $\int x \sin x\, dx = \sin x - x\cos x$
15. $\int x \cos x\, dx = \cos x + x \sin x$
16. $\int \sin x \cos x\, dx = (\sin^2 x)/2$
17. $\int \sin ax \cos bx\, dx = -\dfrac{\cos(a-b)x}{2(a-b)} - \dfrac{\cos(a+b)x}{2(a+b)}$ $\quad (a^2 \ne b^2)$
18. $\int \tan x\, dx = -\ln|\cos x| = \ln|\sec x|$
19. $\int \cot x\, dx = -\ln|\csc x| = \ln|\sin x|$
20. $\int \tan^2 x\, dx = \tan x - x$
21. $\int \cot^2 x\, dx = -\cot x - x$
22. $\int e^{ax}\, dx = (1/a)\, e^{ax}$
23. $\int x e^{ax}\, dx = (e^{ax}/a^2)(ax - 1)$
24. $\int \ln x\, dx = x[\ln(x) - 1]$ $\quad (x > 0)$
25. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$ $\quad (a \ne 0)$
26. $\int \dfrac{dx}{ax^2+c} = \dfrac{1}{\sqrt{ac}}\tan^{-1}\left(x\sqrt{\dfrac{a}{c}}\right),$ $\quad (a > 0, c > 0)$

27a. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\dfrac{2ax+b}{\sqrt{4ac-b^2}}$ $\quad (4ac - b^2 > 0)$

27b. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|$ $\quad (b^2 - 4ac > 0)$

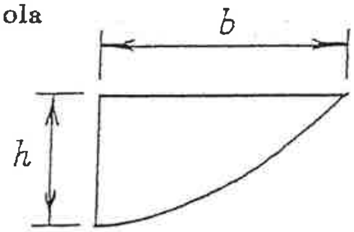
27c. $\int \dfrac{dx}{ax^2+bx+c} = -\dfrac{2}{2ax+b},$ $\quad (b^2 - 4ac = 0)$

MENSURATION OF AREAS AND VOLUMES

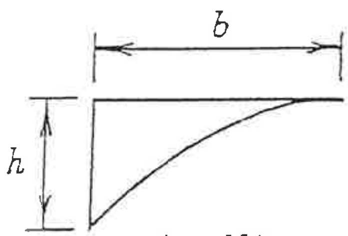
Nomenclature

A = total surface area
p = perimeter
V = volume

Parabola



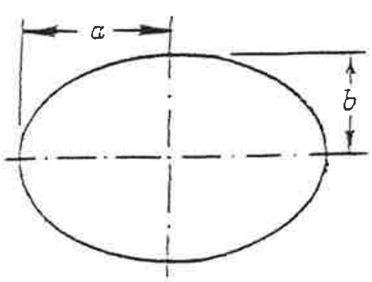
$$A = 2bh/3$$



$$A = bh/3$$

Ellipse

♦



$$A = \pi a b$$

$$p_{\text{approx}} = 2\pi\sqrt{(a^2 + b^2)/2}$$

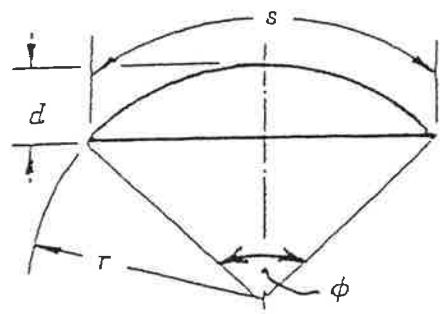
$$p = \pi(a+b)[1 + (^1/_2)^2\lambda^2 + (^1/_2 \times {}^1/_4)^2\lambda^4 + (^1/_2 \times {}^1/_4 \times {}^3/_6)^2\lambda^6 + (^1/_2 \times {}^1/_4 \times {}^3/_6 \times {}^5/_8)^2\lambda^8 + (^1/_2 \times {}^1/_4 \times {}^3/_6 \times {}^5/_8 \times {}^7/_{10})^2\lambda^{10} + \ldots],$$

where

$$\lambda = (a-b)/(a+b)$$

Circular Segment

♦

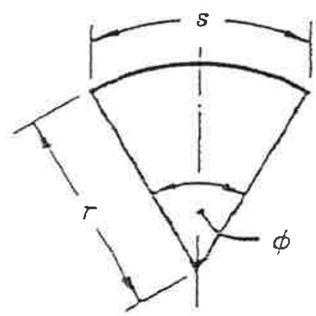


$$A = [r^2(\phi - \sin\phi)]/2$$

$$\phi = s/r = 2\{\arccos[(r-d)/r]\}$$

Circular Sector

♦

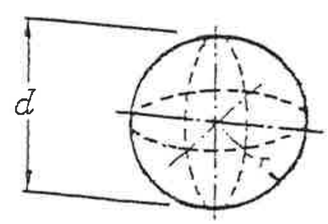


$$A = \phi r^2/2 = sr/2$$

$$\phi = s/r$$

Sphere

♦

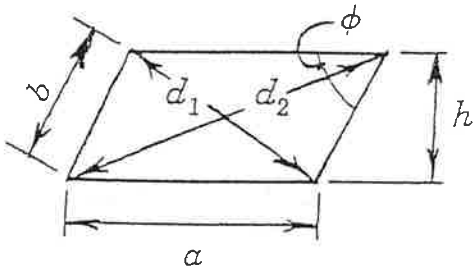


$$V = 4\pi r^3/3 = \pi d^3/6$$

$$A = 4\pi r^2 = \pi d^2$$

MENSURATION OF AREAS AND VOLUMES

Parallelogram



$$p = 2(a + b)$$

$$d_1 = \sqrt{a^2 + b^2 - 2ab(\cos\phi)}$$

$$d_2 = \sqrt{a^2 + b^2 + 2ab(\cos\phi)}$$

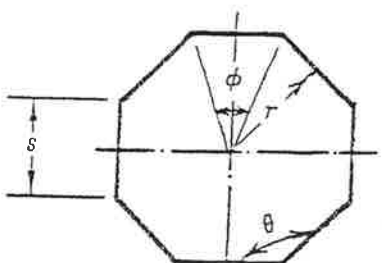
$$d_1^2 + d_2^2 = 2(a^2 + b^2)$$

$$A = ah = ab(\sin\phi)$$

If $a = b$, the parallelogram is a rhombus.

Regular Polygon (n equal sides)

♦



$$\phi = 2\pi/n$$

$$\theta = [\pi(n-2)]/n = \pi - \phi$$

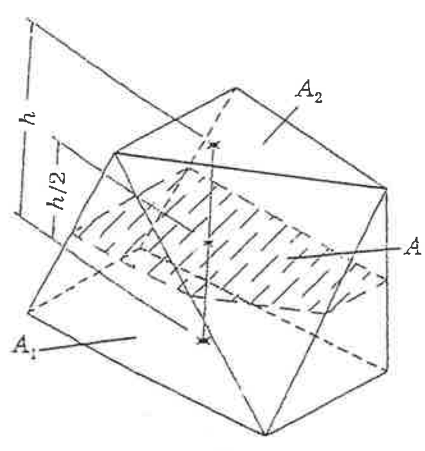
$$p = ns$$

$$s = 2r[\tan(\phi/2)]$$

$$A = (nsr)/2$$

Prismoid

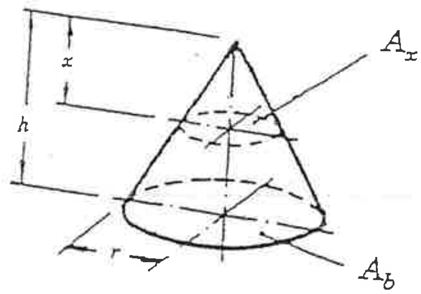
♦



$$V = (h/6)(A_1 + A_2 + 4A)$$

Right Circular Cone

♦



$$V = (\pi r^2 h)/3$$

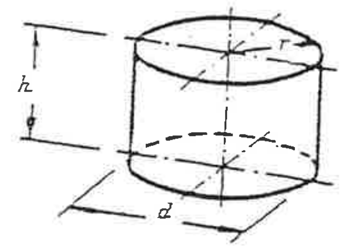
$$A = \text{side area} + \text{base area}$$

$$= \pi r\left(r + \sqrt{r^2 + h^2}\right)$$

$$A_x : A_b = x^2 : h^2$$

Right Circular Cylinder

♦

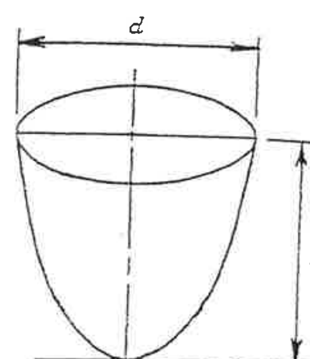


$$V = \pi r^2 h = \pi d^2 h/4$$

$$A = \text{side area} + \text{end areas}$$

$$= 2\pi r(h + r)$$

Paraboloid of Revolution



$$V = \pi d^2 h/8$$

♦ Gieck, K. & R. Gieck, *Engineering Formulas*, 6th Ed., Copyright © 1967 by Gieck Publishing. Diagrams reprinted by permission of Kurt Gieck.

CENTROIDS AND MOMENTS OF INERTIA

The *location of the centroid of an area*, bounded by the axes and the function $y = f(x)$, can be found by integration.

$$x_c = \frac{\int x\,dA}{A}$$

$$y_c = \frac{\int y\,dA}{A}$$

$$A = \int f(x)\,dx$$

$$dA = f(x)\,dx = g(y)\,dy$$

The *first moment of area* with respect to the y-axis and the x-axis, respectively, are:

$$M_y = \int x\,dA = x_c A$$

$$M_x = \int y\,dA = y_c A$$

when either $\bar{x}$ or $\bar{y}$ is of finite dimension then $\int x\,dA$ or $\int y\,dA$ refer to the centroid x or y of dA in these integrals.

The *moment of inertia (second moment of area)* with respect to the y-axis and the x-axis, respectively, are:

$$I_y = \int x^2\,dA$$

$$I_x = \int y^2\,dA$$

The moment of inertia taken with respect to an axis passing through the area's centroid is the *centroidal moment of inertia*. The *parallel axis theorem* for the moment of inertia with respect to another axis parallel with and located d units from the centroidal axis is expressed by

$$I_{\text{parallel axis}} = I_c + Ad^2$$

Values for standard shapes are presented in a table in the DYNAMICS section.

DIFFERENTIAL EQUATIONS

A common class of ordinary linear differential equations is

$$b_N \frac{d^N y(x)}{dx^N} + \ldots + b_1 \frac{dy(x)}{dx} + b_0 y(x) = f(x)$$

where $b_N, \ldots, b_n, \ldots, b_1, b_0$ are constants.

When the equation is a homogeneous differential equation, $f(x) = 0$, the solution is

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \ldots + C_n e^{r_n x} + \ldots + C_N e^{r_N x}$$

where r_n is the nth distinct root of the characteristic polynomial $P(x)$ with

$$P(r) = b_N r^N + b_{N-1} r^{N-1} + \ldots + b_1 r + b_0$$

If the root $r_1 = r_2$, then $C_2 e^{r_2 x}$ is replaced with $C_2 x e^{r_1 x}$. Higher orders of multiplicity imply higher powers of x. The complete solution for the differential equation is

$$y(x) = y_h(x) + y_p(x),$$

where $y_p(x)$ is any solution with $f(x)$ present. If $f(x)$ has $e^{r_n x}$ terms, then resonance is manifested. Furthermore, specific $f(x)$ forms result in specific $y_p(x)$ forms, some of which are:

$f(x)$	$y_p(x)$
A	B
$Ae^{\alpha x}$	$Be^{\alpha x}$, $\alpha \neq r_n$
$A_1 \sin \omega x + A_2 \cos \omega x$	$B_1 \sin \omega x + B_2 \cos \omega x$

If the independent variable is time t, then transient dynamic solutions are implied.

First Order Linear Homogeneous Differential Equations With Constant Coefficients

$y' + ay = 0$, where a is a real constant:

Solution, $y = Ce^{-at}$, where

C = a constant that satisfies the initial conditions.

Second Order Linear Homogeneous Differential Equations With Constant Coefficients

An equation of the form

$$y'' + 2ay' + by = 0$$

can be solved by the method of undetermined coefficients where a solution of the form $y = Ce^{rx}$ is sought. Substitution of this solution gives

$$(r^2 + 2ar + b)Ce^{rx} = 0$$

and since Ce^{rx} cannot be zero, the characteristic equation must vanish or

$$r^2 + 2ar + b = 0$$

The roots of the characteristic equation are

$$r_{1,2} = -a \pm \sqrt{a^2 - b}$$

and can be real and distinct for $a^2 > b$, real and equal for $a^2 = b$, and complex for $a^2 < b$.

If $a^2 > b$, the solution is of the form (overdamped)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $a^2 = b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2 x)e^{r_1 x}$$

If $a^2 < b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

where

$$\alpha = -a$$

$$\beta = \sqrt{b - a^2}$$

FOURIER SERIES

Every function $F(t)$, which has the period $\tau = 2\pi/\omega$ and satisfies certain continuity conditions, can be represented by a series plus a constant.

$$F(t) = a_o/2 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

The above equation holds if $F(t)$ has a continuous derivative $F'(t)$ for all t. Multiply both sides of the equation by $\cos m\omega t$ and integrate from 0 to τ.

$$\int_0^{\tau} F(t) \cos m\omega t\,dt = \int_0^{\tau} (a_o/2) \cos m\omega t\,dt$$

$$\int_0^\tau F(t)\cos m\omega t\,dt = \int_0^\tau (a_o/2)\cos m\omega t\,dt$$
$$+\sum_{n=1}^{\infty}\Big[a_n\int_0^\tau \cos n\omega t\cos m\omega t\,dt$$
$$+\,b_n\int_0^\tau \sin n\omega t\cos m\omega t\,dt\Big]$$

Term-by-term integration of the series can be justified if $F(t)$ is continuous. The *coefficients* are

$$a_n = (2/\tau)\int_0^\tau F(t)\cos n\omega t\,dt \quad \text{and}$$
$$b_n = (2/\tau)\int_0^\tau F(t)\sin n\omega t\,dt, \quad \text{where}$$

$\tau = 2\pi/\omega$. The constants a_n, b_n are the *Fourier coefficients* of $F(t)$ for the interval 0 to τ, and the corresponding series is called the *Fourier series* of $F(t)$ over the same interval. The integrals have the same value over any interval of length τ.

If a Fourier series representing a periodic function is truncated after term n=N the mean square value F_N^2 of the truncated series is given by the Parseval relation. This relation says that the mean square value is the sum of the mean square values of the Fourier components, or

$$F_N^2 = (a_0/2)^2 + (1/2)\sum_{n=1}^{N}(a_n^2 + b_n^2)$$

and the RMS value is then defined to be the square root of this quantity or F_N

FOURIER TRANSFORM

The Fourier transform pair, one form of which is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt$$
$$f(t) = [1/(2\pi)]\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$$

can be used to characterize a broad class of signal models in terms of their frequency or spectral content. Some useful transform pairs are:

$f(t)$	$F(\omega)$
$\delta(t)$	1
$u(t)$	$(1/2)\delta(\omega) + 1/j\omega$
$u\left(t+\frac{\tau}{2}\right)-u\left(t-\frac{\tau}{2}\right) = r_{rect}\left(\frac{t}{\tau}\right)$	$\tau\,\dfrac{\sin(\omega\tau/2)}{\omega\tau/2}$
$e^{j\omega_o t}$	$2\pi\,\delta(\omega - \omega_o)$

Some mathematical liberties are required to obtain the second and fourth form. Other Fourier transforms are derivable from the Laplace transform by replacing s with $j\omega$ provided

$$f(t) = 0,\ t < 0$$
$$\int_0^\infty |f(t)|\,dt < \infty$$

LAPLACE TRANSFORMS

The unilateral Laplace transform pair

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}\,ds$$

represents a powerful tool for the transient and fre quency response of linear time invariant system: Some useful Laplace transform pairs are [Note: Th last two transforms represent the Final Value Theo rem (F.V.T.) and Initial Value Theorem (I.V.T.) re spectively. It is assumed that the limits exist.]:

$f(t)$	$F(s)$
$\delta(t)$, Impulse at $t = 0$	1
$u(t)$, Step at $t = 0$	$1/s$
$t[u(t)]$, Ramp at $t = 0$	$1/s^2$
$e^{-\alpha t}$	$1/(s+\alpha)$
$te^{-\alpha t}$	$1/(s+\alpha)^2$
$e^{-\alpha t}\sin\beta t$	$\beta/[(s+\alpha)^2+\beta^2]$
$e^{-\alpha t}\cos\beta t$	$(s+\alpha)/[(s+\alpha)^2+\beta^2]$
$\mathcal{L}\left\{\dfrac{d^n f(t)}{dt^n}\right\}$	$s^n F(s) - \sum_{m=0}^{n-1} s^{n-m-1}\dfrac{d^m f(($ dt
$\int_0^t f(\tau)\,d\tau$	$(1/s)F(s)$
$\int_0^t x(t-\tau)\,h(t)\,d\tau$	$H(s)X(s)$
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$

DIFFERENCE EQUATIONS

Difference equations are used to model discrete sys tems. Systems which can be described by differenc equations include: computer program variable iteratively evaluated in a loop, sequential circuits, cas flows, recursive processes, systems with time-dela components, etc. Any system whose input $v(t)$ an output $y(t)$ are defined only at the equally-space intervals $t = kT$ can be described by a differenc equation.

First Order Linear Difference Equation

The difference equation

$$P_k = P_{k-1}(1+i) - A$$

represents the balance P of a loan after the kth pay ment A. If P_k is defined as $y(k)$, the model become

$$y(k) - (1+i)y(k-1) = -A$$

Second Order Linear Difference Equation

The Fibonacci number sequence can be generated by

$$y(k) = y(k-1) + y(k-2)$$

where $y(-1) = 1$ and $y(-2) = 1$. An alternate forr for this model is

$$f(x+2) = f(k+1) + f(k)$$

with $f(0) = 1$ and $f(1) = 1$.

z - Transforms

The z-transform pair

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$f(k) = \frac{1}{2\pi i} \oint_{\Gamma} F(z) z^{n-1}\, dz$$

represents a powerful tool for solving linear shift invariant difference equations. A limited unilateral list of z-transform pairs follows [Note: The last two transform pairs represent the Initial Value Theorem (I.V.T.) and the Final Value Theorem (F.V.T.) respectively.]:

$f(k)$	$F(z)$
$\delta(k)$, Impulse at $k = 0$	1
$u(k)$, Step at $k = 0$	$1/(1 - z^{-1})$
β^k	$1/(1 - \beta z^{-1})$
$y(k-1)$	$z^{-1} Y(z) + y(-1)$
$y(k-2)$	$z^{-2} Y(z) + y(-2) + y(-1) z^{-1}$
$y(k+1)$	$z Y(z) - z y(0)$
$y(k+2)$	$z^2 Y(z) - z^2 y(0) - z y(1)$
$\sum_{m=0}^{\infty} X(k-m) h(m)$	$H(z) X(z)$
$\lim_{k \to 0} f(k)$	$\lim_{z \to \infty} F(z)$
$\lim_{k \to \infty} f(k)$	$\lim_{z \to 1} (1 - z^{-1}) F(z)$

EULER'S APPROXIMATION

$$x_{i+1} = x_i + \Delta t (dx_i/dt)$$

NUMERICAL METHODS

Newton's Method of Root Extraction

Given a polynomial $P(x)$ with n simple roots, $a_1, a_2, \cdots, a_n$ where

$$P(x) = \prod_{m=1}^{n} (x - a_m)$$
$$= x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_{n-1}$$

and $P(a_i) = 0$. A root a_i can be computed by the iterative algorithm

$$a_i^{j+1} = a_i^j - \left. \frac{P(x)}{\partial P(x)/\partial x} \right|_{x = a_i^j}$$

with $|P(a_i^{j+1})| \le |P(a_i^j)|$. Convergence is quadratic.

Newton's Method of Minimization

Given a scalar value function

$$h(x) = h(x_1, x_2, \cdots, x_n)$$

find a vector $x^* \in R_n$ such that

$$h(x^*) \le h(x) \text{ for all } x$$

Newton's algorithm is

$$x_{K+1} = x_K - \left(\left. \frac{\partial^2 h}{\partial x^2} \right|_{x = x_K} \right)^{-1} \left. \frac{\partial h}{\partial x} \right|_{x = x_K}$$

where

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \\ \cdots \\ \cdots \\ \frac{\partial h}{\partial x_n} \end{bmatrix}$$

and

$$\frac{\partial^2 h}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \cdots & \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 h}{\partial x_2^2} & \cdots & \cdots & \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & & \cdots \\ \frac{\partial^2 h}{\partial x_1 \partial x_n} & \frac{\partial^2 h}{\partial x_2 \partial x_n} & \cdots & \cdots & \frac{\partial^2 h}{\partial x_n^2} \end{bmatrix}$$

Numerical Integration

Three of the more common numerical integration algorithms used to evaluate the integral

$$\int_a^b f(x)\, dx$$

with $\Delta x = (b - a)/n$ are:

Euler's or Forward Rectangular Rule

$$\int_a^b f(x)\, dx \approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x)$$

Trapezoidal Rule

for $n = 1$

$$\int_a^b f(x)\, dx \approx \Delta x \left[\frac{f(a) + f(b)}{2} \right]$$

for $n > 1$

$$\int_a^b f(x)\, dx \approx \frac{\Delta x}{2} \left[f(a) + 2 \sum_{k=1}^{n-1} f(a + k\Delta x) + f(b) \right]$$

Simpson's Rule/Parabolic Rule (n must be an even integer)

for $n = 2$

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

for $n \geq 4$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(a) + 2\sum_{k=2,4,6,\cdots}^{n-2} f(a + k\Delta x) + 4\sum_{k=1,3,5,\cdots}^{n-1} f(a + k\Delta x) + f(b)\right]$$

Numerical Solution of Ordinary Differential Equations - Euler's Method

Given a differential equation

$$dy/dt = f(y,t) \quad \text{with} \quad y(0) = y_o$$

At some general time $k\Delta t$

$$y[(k+1)\Delta t] \cong y(k\Delta t) + \Delta t\, f[y(k\Delta t), k\Delta t]$$

which can be used with starting condition y_o to solve recursively for $y(\Delta t), y(2\Delta t), \cdots, y(n\Delta t)$.

The method can be extended to nth order differential equations by recasting them as n first order equations.

ELECTRIC CIRCUITS

UNITS

The basic electrical units are: coulombs for charge, volts for voltage, amperes for current, and ohms for resistance and impedance.

ELECTROSTATICS

$$F_2 = \frac{Q_1 Q_2}{4\pi \epsilon r^2} \boldsymbol{a}_{r12}, \text{ where}$$

F_2 = the force on charge 2 due to charge 1,
Q_i = the ith point charge,
r = the distance between charges 1 and 2,
$\boldsymbol{a}_{r12}$ = a unit vector directed from 1 to 2, and
ϵ = the permittivity (or dielectric constant) of the medium.

For free space or air:

$$\epsilon = \epsilon_0 = 8.85 \times 10^{-12} \text{ Farads/meter}$$

Electrostatic Fields

Electric field intensity $\boldsymbol{E}$ (volts/meter) at point 2 due to a point charge Q_1 at point 1 is

$$\boldsymbol{E} = \frac{Q_1}{4\pi \epsilon r^2} \boldsymbol{a}_{r12}$$

For a line charge with density ρ_L C/m on the z-axis, the radial electric field is

$$\boldsymbol{E}_L = \frac{\rho_L}{2\pi \epsilon r} \boldsymbol{a}_r$$

For a sheet charge of density ρ_s C/m^2 in the x-y plane:

$$\boldsymbol{E}_s = \frac{\rho_s}{2\epsilon} \boldsymbol{a}_z, \ z > 0$$

Gauss' law states that the integral of the electric flux density $\boldsymbol{D} = \epsilon \boldsymbol{E}$ over a closed surface is equal to the charge enclosed or

$$Q_{encl} = \oint_A \epsilon \boldsymbol{E} \cdot d\boldsymbol{A}$$

The force on a point charge Q in an electric field with intensity $\boldsymbol{E}$ is $\boldsymbol{F} = Q\boldsymbol{E}$.

The work done by an external agent in moving a charge Q in an electric field from point r_1 to point r_2 is

$$W = -Q\int_{r_1}^{r_2} \boldsymbol{E} \cdot d\boldsymbol{L}$$

The energy stored $\mathcal{E}_E$ in an electric field $\boldsymbol{E}$ is

$$\mathcal{E}_E = (1/2) \iiint_V \epsilon |\boldsymbol{E}|^2 \, dv$$

Voltage

The potential difference V between two points r_1 and r_2 is the work W required per unit charge to move a charge Q from r_1 to r_2; i.e., $V = W/Q$.

For two parallel plates with potential difference V, separated by distance d, the strength of the E field between the plates is

$$E = \frac{V}{d}$$

directed from the + plate to the − plate.

Current

Electric current $i(t)$ through a surface is defined as the rate of charge transport across that surface or

$i(t) = dq(t)/dt$, which is a function of time t

since $q(t)$ denotes instantaneous charge.

A constant $i(t)$ is written as I, and the vector current density in amps/m^2 is defined as $\boldsymbol{J}$.

Magnetic Fields

For a current carrying wire on the z-axis

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu} = \frac{I \boldsymbol{a}_\theta}{2\pi r}, \text{ where}$$

$\boldsymbol{H}$ = the magnetic field strength (amps/meter),
$\boldsymbol{B}$ = the magnetic flux density (tesla),
$\boldsymbol{a}_\theta$ = the unit vector in positive θ direction in cylindrical coordinates,
I = the current, and
μ = the permeability of the medium.

For air: $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m

Force on a current carrying conductor in a uniform magnetic field is

$$\boldsymbol{F} = I\boldsymbol{L} \times \boldsymbol{B}, \text{ where}$$

$\boldsymbol{L}$ = the length vector of a conductor.

The energy stored $\mathcal{E}_H$ in a magnetic field $\boldsymbol{H}$ is

$$\mathcal{E}_H = (1/2) \iiint_V \mu |\boldsymbol{H}|^2 \, dv$$

Induced Voltage

Faraday's Law states that for a coil of N turns enclosing flux ϕ, the induced voltage e is given by

$$e = -N \, d\phi/dt, \text{ where}$$

ϕ = the flux (webers) enclosed by the N conductor turns and

$$\phi = \int_A \boldsymbol{B} \cdot d\boldsymbol{A}$$

Resistivity

For a conductor of length L, electrical resistivity ρ, and area A, the resistance is

$$R = \frac{\rho L}{A}$$

For metallic conductors, the resistivity and resistance vary linearly with changes in temperature according to the following relationships

$$\rho = \rho_0[1 + \alpha(T - T_0)], \quad \text{and}$$
$$R = R_0[1 + \alpha(T - T_0)], \quad \text{where}$$

ρ_0 is resistivity at T_0 and
α is the temperature coefficient.

Ohm's Law is $V = IR$; $v(t) = i(t)R$

Resistors in Series and Parallel

For series connections, current in resistors is the same and the equivalent resistance for n resistors in series is

$$R_T = R_1 + R_2 + \ldots + R_n$$

For parallel connections of resistors, the voltage drop across the resistors is the same and the resistance for n resistors in parallel is

$$R_T = 1/(1/R_1 + 1/R_2 + \ldots + 1/R_n)$$

For two resistors R_1 and R_2 in parallel

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

Power in a Resistive Element

$$P = VI = \frac{V^2}{R} = I^2 R$$

Kirchhoff's Laws

Kirchhoff's voltage law for a closed loop is

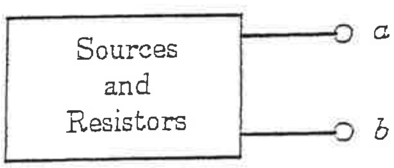
$$\Sigma V_{\text{rises}} = \Sigma V_{\text{drops}}$$

Kirchhoff's current law for a closed surface is

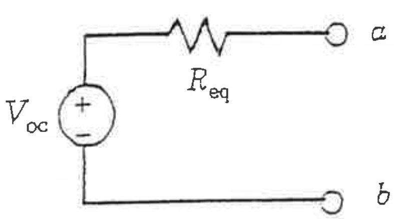
$$\Sigma I_{\text{in}} = \Sigma I_{\text{out}}$$

SOURCE EQUIVALENTS

For an arbitrary circuit



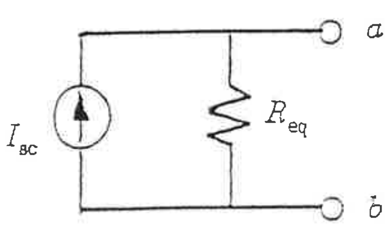
The Thevenin equivalent is



$$R_{eq} = \frac{V_{oc}}{I_{sc}}$$

The open circuit voltage V_{oc} is $V_a - V_b$, and the short circuit current is I_{sc} from a to b.

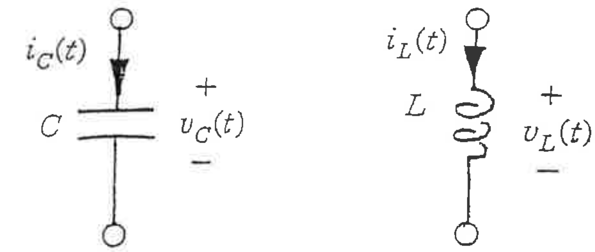
The Norton equivalent circuit is



where I_{sc} and R_{eq} are defined above.

A load resistor R_L connected across terminals a and b will draw maximum power when $R_L = R_{eq}$.

CAPACITORS AND INDUCTORS



The charge $q_C(t)$ and voltage $v_C(t)$ relationship for a capacitor C in farads is

$$C = q_C(t)/v_C(t) \quad \text{or} \quad q_C(t) = Cv_C(t)$$

A parallel plate capacitor of area A separated a distance d by an insulator with a permittivity ϵ has a capacitance

$$C = \frac{\epsilon A}{d}$$

The current-voltage relationships for a capacitor are

$$v_C(t) = v_C(0) + \frac{1}{C}\int_0^t i_C(\tau)d\tau$$

and $\quad i_C(t) = C(dv_C/dt)$

The energy stored in a capacitor is expressed in joules and

$$\text{Energy} = Cv_C^2/2 = q_C^2/2C = q_C v_C/2$$

The inductance L of a coil is

$$L = N\phi/i_L$$

and using Faraday's law, the voltage-current relations for an inductor are

$$v_L(t) = L(di_L/dt)$$

$$i_L(t) = i_L(0) + \frac{1}{L}\int_0^t v_L(\tau)d\tau, \text{ where}$$

v_L = inductor voltage,
L = inductance (henrys), and
i = current (amps).

The energy stored in an inductor is expressed in joules and

$$\text{Energy} = Li_L^2/2$$

Capacitors & Inductors in Parallel and Series

Capacitors in Parallel

$$C_{eq} = C_1 + C_2 + \ldots + C_n$$

Capacitors in Series

$$C_{eq} = \frac{1}{1/C_1 + 1/C_2 + \ldots + 1/C_n}$$

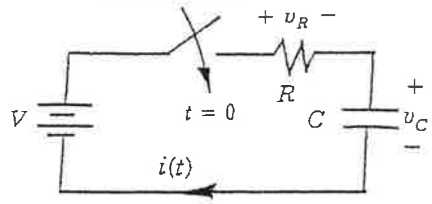
Inductors In Parallel

$$L_{eq} = \frac{1}{1/L_1 + 1/L_2 + \ldots + 1/L_n}$$

Inductors In Series

$$L_{eq} = L_1 + L_2 + \ldots + L_n$$

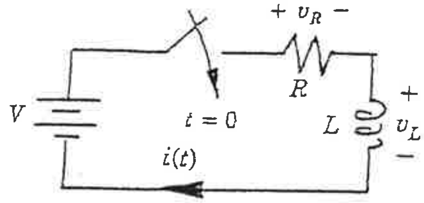
RC AND RL TRANSIENTS



$$t \geq 0;\ v_C(t) = v_C(0)e^{-t/RC} + V(1 - e^{-t/RC})$$

$$i(t) = \{[V - v_C(0)]/R\}e^{-t/RC}$$

$$v_R(t) = i(t)R = [V - v_C(0)]e^{-t/RC}$$



$$t \geq 0;\ i(t) = i(0)e^{-Rt/L} + \frac{V}{R}(1 - e^{-Rt/L})$$

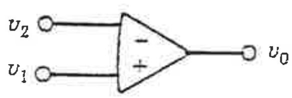
$$v_R(t) = i(t)R = i(0)R\,e^{-Rt/L} + V(1 - e^{-Rt/L})$$

$$v_L(t) = L(di/dt) = -i(0)R\,e^{-Rt/L} + V e^{-Rt/L}$$

$v(0)$ and $i(0)$ denote the initial conditions and the parameters RC and L/R are termed the respective circuit time constants.

OPERATIONAL AMPLIFIERS

$v_o = A(v_1 - v_2)$, where

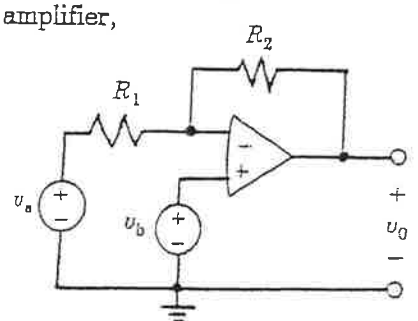


A is large ($>10^4$) and

$v_1 - v_2$ is small enough so as not to saturate the amplifier.

For the ideal operational amplifier, assume that the input currents are zero and that the gain A is infinite so when operating linearly, $v_2 - v_1 = 0$.

For the two-source configuration with an ideal operational amplifier,



$$v_o = -\frac{R_2}{R_1}v_a + \left(1 + \frac{R_2}{R_1}\right)v_b$$

If $v_a = 0$, the non-inverting amplifier output is

$$v_o = \left(1 + \frac{R_2}{R_1}\right)v_b$$

If $v_b = 0$, the inverting amplifier output is

$$v_o = -\frac{R_2}{R_1}v_a$$

AC CIRCUITS

For a sinusoidal voltage or current of frequency f (Hz) and period T (seconds),

$$f = 1/T = \omega/(2\pi), \text{ where}$$

ω = the angular frequency in radians/s.

Average Value

For a periodic waveform (either voltage or current) with period T,

$$X_{ave} = (1/T)\int_0^T x(t)\,dt$$

The average value of a full-wave rectified sine wave is

$$X_{ave} = (2X_{max})/\pi$$

and half this for a half-wave rectification, where

X_{max} = the amplitude of the waveform.

Effective or RMS Values

For a periodic waveform with period T, the rms or effective value is

$$X_{rms} = \left[(1/T)\int_0^T x^2(t)\,dt\right]^{1/2}$$

For a sinusoidal waveform and full-wave rectified sine wave,

$$X_{rms} = X_{max}/\sqrt{2}$$

For a half-wave rectified sine wave,

$$X_{rms} = X_{max}/2$$

Sine-Cosine Relations

$$\cos(\omega t) = \sin(\omega t + \pi/2) = -\sin(\omega t - \pi/2)$$

$$\sin(\omega t) = \cos(\omega t - \pi/2) = -\cos(\omega t + \pi/2)$$

Phasor Transforms of Sinusoids

$$\mathcal{P}[V_{max}\cos(\omega t + \phi)] = V_{rms}\angle\phi = \mathbf{V}$$

$$\mathcal{P}[I_{max}\cos(\omega t + \theta)] = I_{rms}\angle\theta = \mathbf{I}$$

For a circuit element, the impedance is defined as the ratio of phasor voltage to phasor current.

$$Z = \frac{\mathbf{V}}{\mathbf{I}}$$

For a Resistor,

$$Z_R = R$$

For a Capacitor,

$$Z_C = \frac{1}{j\omega C} = jX_C$$

For an Inductor,

$$Z_L = j\omega L = jX_L, \text{ where}$$

X_C and X_L are the capacitive and inductive reactances respectively defined as

$$X_C = -\frac{1}{\omega C} \quad \text{and} \quad X_L = \omega L$$

Impedances in series combine additively while those in parallel combine according to the reciprocal rule just as in the case of resistors.

Complex Power

Real power P (watts) is defined by

$$P = (½)V_{max} I_{max} \cos \theta$$
$$= V_{rms} I_{rms} \cos \theta$$

where θ is the angle measured from V to I. If I leads (lags) V, then the power factor ($p.f.$),

$$p.f. = \cos \theta$$

is said to be a leading (lagging) $p.f.$

Reactive power Q (vars) is defined by

$$Q = (½)V_{max} I_{max} \sin \theta$$
$$= V_{rms} I_{rms} \sin \theta$$

Complex power S (volt-amperes) is defined by

$$S = P + jQ$$

For resistors, $\theta = 0$, so the real power is

$$P = V_{rms} I_{rms} = V_{rms}^2/R = I_{rms}^2 R$$

RESONANCE

The radian resonant frequency for both parallel and series LC combinations is

$$\omega_o = \frac{1}{\sqrt{LC}} = 2\pi f_o \text{ (rad/s)}$$

Series Resonance

$$\omega_o L = \frac{1}{\omega_o C} \quad \text{and}$$

$Z = R$ at resonance.

Quality factor

$$Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o CR}$$

Bandwidth

$$BW = \omega_o/Q \text{ (rad/s)}$$

Parallel Resonance

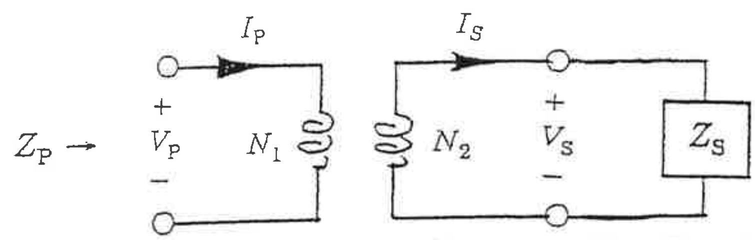
$$\omega_o L = \frac{1}{\omega_o C} \quad \text{and}$$

$Z = R$ at resonance.

$$Q = \omega_o RC = \frac{R}{\omega_o L}$$

$$BW = \omega_o/Q \text{ (rad/s)}$$

TRANSFORMERS



Turns Ratio

$$a = N_1/N_2$$

$$a = \frac{|V_P|}{|V_S|} = \frac{|I_S|}{|I_P|}$$

The impedance seen at input is

$$Z_P = a^2 Z_S$$

ALGEBRA OF COMPLEX NUMBERS

Complex numbers may be designated in rectangular form or polar form. In rectangular form, a complex number is written in terms of its real and imaginary components.

$$z = a + jb, \quad \text{where}$$

a = the real component,
b = the imaginary component, and
$j = \sqrt{-1}$.

In polar form

$$z = c\angle\theta, \quad \text{where}$$

$c = \sqrt{a^2 + b^2}$,
$\theta = \tan^{-1}(b/a)$,
$a = c \cos \theta$, and
$b = c \sin \theta$.

Complex numbers are added and subtracted in rectangular form. If

$$z_1 = a_1 + jb_1 = c_1(\cos \theta_1 + j\sin \theta_1)$$
$$= c_1 \angle \theta_1 \quad \text{and}$$
$$z_2 = a_2 + jb_2 = c_2(\cos \theta_2 + j\sin \theta_2)$$
$$= c_2 \angle \theta_2, \quad \text{then}$$

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \quad \text{and}$$
$$z_1 - z_2 = (a_1 - a_2) + j(b_1 - b_2)$$

While complex numbers can be multiplied or divided in rectangular form, it is more convenient to perform these operations in polar form.

$$z_1 \times z_2 = (c_1 \times c_2) \angle \theta_1 + \theta_2$$

$$z_1/z_2 = (c_1/c_2) \angle \theta_1 - \theta_2$$

ELECTRICAL AND COMPUTER ENGINEERING

ELECTROMAGNETIC DYNAMIC FIELDS

The integral and point form of Maxwell's equations are

$$\oint E \cdot dl = -\iint_S (\partial B/\partial t) \cdot dS$$

$$\oint H \cdot dl = I_{enc} + \iint_S (\partial D/\partial t) \cdot dS$$

$$\iint_{S_V} D \cdot dS = \iiint_{V_S} \rho \, dv$$

$$\iint_{S_V} B \cdot dS = 0$$

$$\nabla \times E = -\partial B/\partial t$$

$$\nabla \times H = J + \partial D/\partial t$$

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

The sinusoidal wave equation in E for an isotropic homogeneous medium is given by

$$\nabla^2 E = -\omega^2 \mu \epsilon E$$

The EM energy flow of a volume V_S enclosed by the surface S_V can be expressed in terms of the Poynting's Theorem

$$-\iint_{S_V} (E \times H) \cdot dS = \iiint_{V_S} J \cdot E \, dv + \partial/\partial t \iiint_{V_S} (\epsilon E^2/2 + \mu H^2/2) \, dv$$

where the left-side term represents the energy flow per unit time or power flow into the volume V_S, whereas the $J \cdot E$ represents the loss in V_S and the last term represents the rate of change of the energy stored in the E and H fields.

LOSS LESS TRANSMISSION LINES

The wavelength, λ, of a sinusoidal signal is defined as the distance the signal will travel in one period.

$$\lambda = \frac{V}{f}$$

where V is the velocity of propagation and f is the frequency of the sinusoid.

The charactersitic impedance, Z_o, of a transmission line is the input impedance of an infinite length of the line and is given by

$$Z_o = \sqrt{L/C}$$

where L and C are the per unit length inductance and capacitance of the line.

Standing wave ratio, SWR, is given by

$$SWR = \frac{|Z_L|}{|Z_o|}$$

where Z_L is the load impedance.

The reflection coefficient, ρ, is a measure of the percentage of the voltage arriving at the load which is reflected towards the source. The reflection coefficent is related to SWR, Z_L and Z_O by the following equations:

$$SWR = \frac{\rho + 1}{1 - \rho}$$

$$\rho = \left|\frac{Z_L - Z_o}{Z_L + Z_o}\right|$$

AC MACHINES

The synchronous speed n_s for ac motors is given by

$n_s = 120f/p$ (in rpm), where

f = the line voltage frequency in Hz and

p = the number of poles.

The slip for an induction motor is

slip $= (n_s - n)/n_s$, where

n = the rotational speed (rpm).

BALANCED THREE PHASE SYSTEMS

The three phase line-phase relations are

$I_L = \sqrt{3}\, I_P$ (for delta)

$V_L = \sqrt{3}\, V_P$ (for wye)

where subscripts L/P denote line/phase respectively.

Three phase complex power is defined by

$$VA = P + jQ$$

$$VA = \sqrt{3}\, V_L I_L \, (\cos\theta_P + j\sin\theta_P)$$

where

VA = total complex volt-amperes,

P = real power,

Q = reactive volt-amperes, and

θ_P = power factor angle of each phase.

SIGNAL PROCESSING

Signal processing concepts include circuits, transform, communication, and other concepts which are covered in other sections of this reference. Two concepts of importance not covered elsewhere include convolution and correlation. The convolution $v(t)$ of two functions $x(t)$ and $y(t)$ can be written as

$$v(t) = x(t) \otimes y(t)$$

where

$$x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) \, d\tau$$

One form for the correlation $r(\tau)$ of nonperiodic functions $x(t)$ and $y(t)$ is

$$r(\tau) = \int_{-\infty}^{\infty} x(t) y(\tau + t) \, dt$$

Fourier transforms of correlation functions generally lead to power density functions.

COMMUNICATION THEORY CONCEPTS

Spectral characterization of communication signals can be represented by mathematical transform theory. An amplitude modulated AM signal form is

$$v(t) = A_c[1 + m(t)]\cos \omega_c t, \text{ where}$$

A_c = carrier signal amplitude.

If the modulation baseband signal $m(t)$ is the sinusoidal form with frequency ω_m or

$$m(t) = m\cos \omega_m t$$

then m is the index of modulation with $m > 1$ implying overmodulation. One form of a frequency modulated FM signal form is

$$v(t) = A\cos[\omega_c t + \phi(t)]$$

where the $\phi(t)$ angle modulation is a function of the baseband signal. The angle modulation form

$$\phi(t) = k_p m_i(t)$$

is termed phase modulation since angle variations are proportional to the baseband signal $m_i(t)$. Alternately

$$\phi(t) = k_f \int_{-\infty}^{t} m(\tau)\, d\tau$$

is termed frequency modulation since $\omega t = \phi(t)$ implies $d\phi(t)/dt = \omega$. Therefore, the instantaneous frequency ω_i associated with $v(t)$ is defined by

$$\omega_i t = \omega_c t + k_f \int_{-\infty}^{t} m(\tau)\, d\tau$$

from which

$$\omega_i = d(\omega_i t)/dt = \omega_c + k_f m(t) = \omega_c + \Delta\omega(t)$$

where the frequency deviation is proportional to the baseband signal or

$$\Delta\omega(t) = k_f m(t)$$

These fundamental concepts form the basis of analog communication theory. Alternately, sampling theory, conversion, and PCM (Pulse Code Modulation) are fundamental concepts of digital communication.

FOURIER SERIES

If $f(t)$ satisfies certain continuity conditions and the relationship for periodicity given by

$$f(t) = f(t + T)$$

then $f(t)$ can be represented in the trigonometric and complex Fourier series given by

$$f(t) = A_o + \sum_{n=1}^{\infty} A_n \cos n\omega_o t + \sum_{n=1}^{\infty} B_n \sin n\omega_o t$$

and

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_o t}$$

where

$$\omega_o = 2\pi/T$$

$$A_o = (1/T)\int_t^{t+T} f(\tau)\, d\tau$$

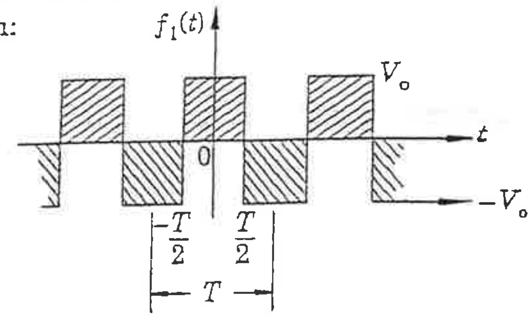
$$A_n = (2/T)\int_t^{t+T} f(\tau)\cos n\omega_o\tau\, d\tau$$

$$B_n = (2/T)\int_t^{t+T} f(\tau)\sin n\omega_o\tau\, d\tau$$

$$C_n = (1/T)\int_t^{t+T} f(\tau)\, e^{-jn\omega_o\tau}\, d\tau$$

Three useful and common Fourier series forms are defined in terms of the following graphs (with $\omega_o = 2\pi/T$).

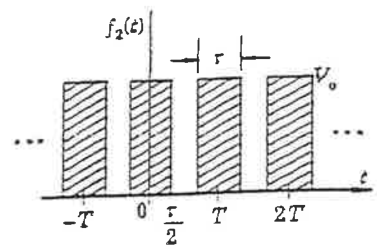
Given:



then

$$f_1(t) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} (-1)^{(n-1)/2}(4V_o/n\pi)\cos n\omega_o t$$

Given:

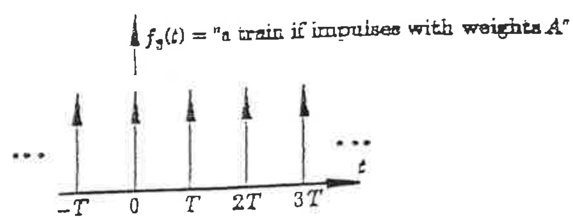


then

$$f_2(t) = \frac{V_o\tau}{T} + \frac{2V_o\tau}{T}\sum_{n=1}^{\infty}\frac{\sin(n\pi\tau/T)}{(n\pi\tau/T)}\cos n\omega_o t$$

$$f_2(t) = \frac{V_o\tau}{T}\sum_{n=-\infty}^{\infty}\frac{\sin(n\pi\tau/T)}{(n\pi\tau/T)} e^{jn\omega_o t}$$

Given:



then

$$f_3(t) = \sum_{n=-\infty}^{\infty} A\delta(t - nT)$$

$$f_3(t) = (A/T) + (2A/T)\sum_{n=1}^{\infty}\cos n\omega_o t$$

$$f_3(t) = (A/T)\sum_{n=-\infty}^{\infty} e^{jn\omega_o t}$$

SOLID STATE ELECTRONICS & DEVICES

Conductivity of a semiconductor material:

$$\sigma = q(n\mu_n + p\mu_p), \text{ where}$$

μ_n = electron mobility,
μ_p = hole mobility,
n = electron concentration,
p = hole concentration, and
q = charge on an electron.

Doped material:

p-type material; $p_p \approx N_a$

n-type material; $n_n \approx N_d$

Carrier concentrations at equilibrium

$(p)(n) = n_i^2$ where

$n_i \equiv$ intrinsic concentration.

Built-in potential (contact potential) of a p-n junction:

$$V_o = \frac{kT}{q} \ln \frac{N_a N_d}{n_i^2}$$, where

Thermal voltage

$$V_T = \frac{kT}{q}$$

N_a = acceptor concentration,

N_d = donor concentration,

T = temperature (K), and

k = Boltzmann's Constant = 1.38×10^{-23} ζ/K.

Resistance R of a diffused layer is

$$R = R_\square \frac{L}{W}$$, where

$R_\square$ = sheet resistance = ρ/T

ρ = resistivity,

T = thickness,

L = length of diffusion, and

W = width of diffusion.

TABULATED CHARACTERISTICS FOR:

Diodes

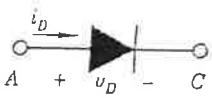
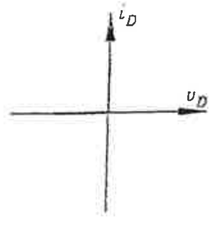
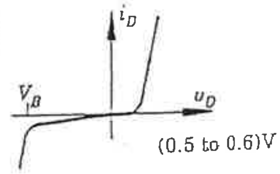
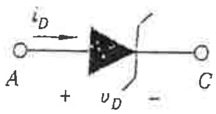
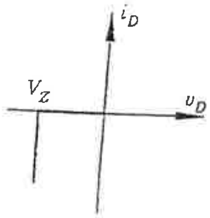
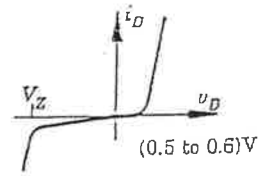
Bipolar Junction Transistors

N-Channel JFET and MOSFET

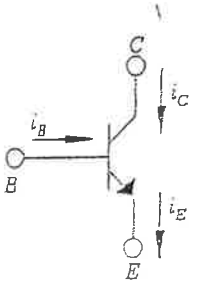
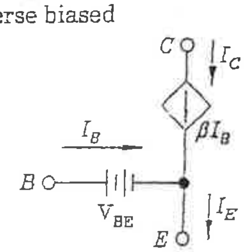
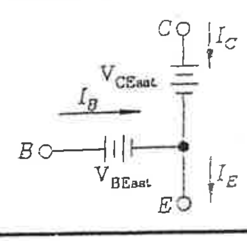
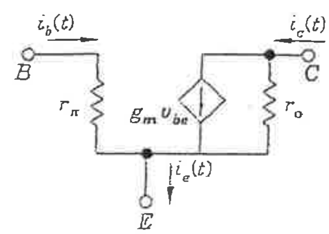
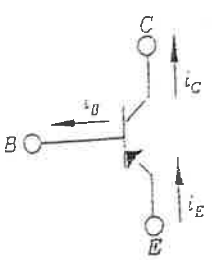
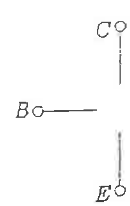
Enhancement MOSFETs

follow on pages 102-103.

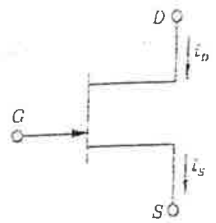
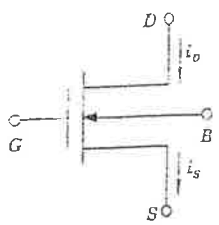
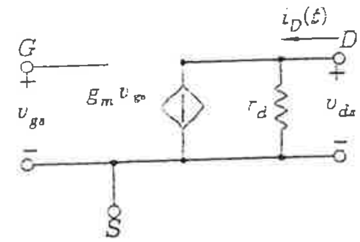
DIODES

Device and Schematic Symbol	Ideal $I-V$ Characteristics	Real $I-V$ Characteristics	Mathematical $I-V$ Relationship
(Junction Diode) 		 V_B = breakdown voltage	Shockley Equation $i_D \approx I_s[e^{(v_D/\eta V_T)} - 1]$, where I_s = saturation current η = emission coefficient, typically 1 for Si V_T = thermal voltage (The Shockley equation is good for $v_D > 0$)
(Zener Diode) 		 V_Z = Zener voltage	Same as above.

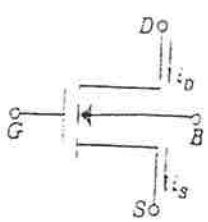
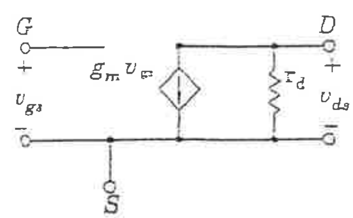
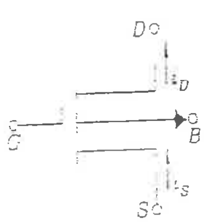
Bipolar Junction Transistor (BJT)

Schematic Symbol	Mathematical Relationships	Large-Signal (DC) Simplified Equivalent Circuits	Low-Frequency Small-Signal (AC) Equivalent Circuits
 NPN - Transister	$i_E = i_B + i_C$ $i_C \approx \beta i_B$ $i_C = \alpha i_E$ $\alpha = \beta/(\beta + 1)$ $i_C \approx I_S e^{(v_{BE}/V_T)}$ I_S = emitter saturation current V_T = thermal voltage Note: These relationships are valid in the active mode of operation.	Active Region: base emitter junction forward biased; base collector junction reverse biased  Saturation Region: both junctions forward biased 	Low Frequency: $g_m \approx I_C/V_T$ $r_\pi \approx \beta/g_m$, where I_C = DC collector current at the Q_{point} $r_o = \left[\frac{\partial v_{CE}}{\partial i_c}\right]_{Q_{point}}$ 
 PNP - Transister	Same as for NPN with currents and voltage polarities reversed.	Cutoff Region: both junctions reverse biased  Same as for NPN with currents and voltage polarities reversed.	Same as for NPN.

N-Channel Junction Field Effect Transistors (JFET) and Depletion MOSFET (Low and Medium Frequency)

Schematic Symbol	Mathematical Relationships	Small-Signal (AC) Equivalent Circuit
JFET  Depletion MOSFET 	Cutoff Region: $v_{GS} < V_p$ $i_D = 0$ Triode Region: $v_{GS} > V_p$ and $v_{GD} > V_p$ $i_D = (I_{DSS}/V_p^2)[2v_{DS}(v_{GS} - V_p) - v_{DS}^2]$ Saturation Region: $v_{GS} > V_p$ and $v_{GD} < V_p$ $i_D = I_{DSS}(1 - v_{GS}/V_p)^2$, where I_{DSS} = drain current with $v_{GS} = 0$ (in the saturation region) $= KV_p^2$ K = conductivity factor V_p = pinch-off voltage	$g_m = \frac{2\sqrt{I_{DSS}I_D}}{\lvert V_p \rvert}$ in saturation region  where $r_d = \left.\frac{\partial v_{ds}}{\partial i_d}\right\rvert_{Q_{point}}$

Enhancement MOSFET (Low and Medium Frequency)

Schematic Symbol	Mathematical Relationships	Small-Signal (AC) Equivalent Circuit
 N - channel	Cutoff Region: $v_{GS} < V_t$ $i_D = 0$ Triode Region: $v_{GS} > V_t$ and $v_{GD} > V_t$ $i_D = K[2v_{DS}(v_{GS} - V_t) - v_{DS}^2]$ Saturation Region: $v_{GS} > V_t$ and $v_{GD} < V_t$ $i_D = K(v_{GS} - V_t)^2$, where K = conductivity factor V_t = threshold voltage	$g_m = 2K(v_{GS} - V_t)$ in saturation region  where $r_d = \left.\frac{\partial v_{ds}}{\partial i_d}\right\rvert_{Q_{point}}$
 P - channel	Same as for N - channel with current and voltage polarities reversed.	Same as for N - channel.

NUMBER SYSTEMS AND CODES

An unsigned number of base-r has a decimal equivalent D defined by

$$D = \sum_{K=0}^{n} a_K r^K + \sum_{i=1}^{m} a_i r^{-i} \text{ , where}$$

a_K = the $(K+1)$ digit to the left of the radix point and
a_i = the ith digit to the right of the radix point.

Signed numbers of base-r are often represented by the radix complement operation. If M is an N-digit value of base-r, the radix complement $R(M)$ is defined by

$$R(M) = r^N - M$$

The 2's complement of an N-bit binary integer can be written

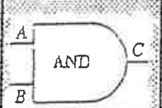
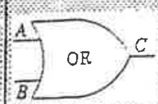
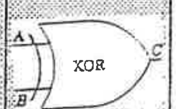
$$\text{2's Complement } (M) = \sum_{K=0}^{N-1} b_K 2^K - b_{N-1} 2^{N-1}$$

The following table contains equivalent codes for a four-bit binary value.

Binary Base-2	Decimal Base-10	Hexa-decimal Base-16	Octal Base-8	BCD Code	Gray Code
0000	0	0	0	0	0000
0001	1	1	1	1	0001
0010	2	2	2	2	0011
0011	3	3	3	3	0010
0100	**4**	**4**	**4**	**4**	**0110**
0101	5	5	5	5	0111
0110	6	6	6	6	0101
0111	7	7	7	7	0100
1000	8	8	10	8	1100
1001	**9**	**9**	**11**	**9**	**1101**
1010	10	A	12	---	1111
1011	11	B	13	---	1110
1100	12	C	14	---	1010
1101	13	D	15	---	1011
1110	**14**	**E**	**16**	**---**	**1001**
1111	15	F	17	---	1000

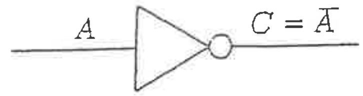
LOGIC OPERATIONS AND BOOLEAN ALGEBRA

Three basic logic operations are the "AND ($\cdot$)," "OR (+)," and "Exclusive-OR ($\oplus$)" functions. The definition of each function, its logic symbol, and its Boolean expression are given in the following table.

Function / Inputs				
A	B	$C = A \cdot B$	$C = A + B$	$C = A \oplus B$
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

As commonly used, A AND B is often written AB or $A \cdot B$.

The not operator inverts the sense of a binary value $(0 \to 1, 1 \to 0)$



Logic Symbol

NOT OPERATOR

Input A	Output $C = \overline{A}$
0	1
1	0

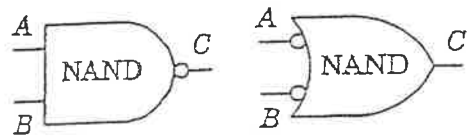
DeMorgan's Theorem

first theorem: $\overline{A + B} = \overline{A} \cdot \overline{B}$

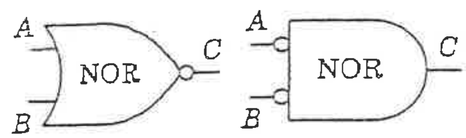
second theorem: $\overline{A \cdot B} = \overline{A} + \overline{B}$

These theorems define the NAND gate and the NOR gate. Logic symbols for these gates are shown below.

NAND Gates: $\overline{A \cdot B} = \overline{A} + \overline{B}$



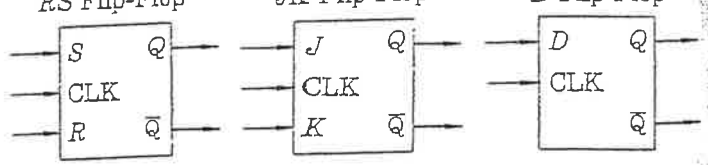
NOR Gates: $\overline{A + B} = \overline{A} \cdot \overline{B}$



FLIP-FLOPS

A flip-flop is a device whose output can be placed in one of two states, 0 or 1. The flip-flop output is synchronized with a clock (CLK) signal. Q_n represents the value of the flip-flop output before CLK is applied and Q_{n+1} represents the output after CLK has been applied. Three basic flip-flops are described below.

RS Flip-Flop　　*JK* Flip-Flop　　*D* Flip-Flop



SR	Q_{n+1}
00	Q_n no change
01	0
10	1
11	x invalid

JK	Q_{n+1}
00	Q_n no change
01	0
10	1
11	$\overline{Q}_n$ toggle

D	Q_{n+1}
0	0
1	1

Composite Flip-Flop State Transition

Q_n	Q_{n+1}	S	R	J	K	D
0	0	0	x	0	x	0
0	1	1	0	1	x	1
1	0	0	1	x	1	0
1	1	x	0	x	0	1

Switching Function Terminology

Minterm - A product term which contains an occurrence of every variable in the function.

Maxterm - A sum term which contains an occurrence of every variable in the function.

Implicant - A Boolean algebra term, either in sum or product form, which contains one or more minterms or maxterms of a function.

Prime Implicant - An implicant which is not entirely contained in any other implicant.

Essential Prime Implicant - A prime implicant which contains a minterm or maxterm which is not contained in any other prime implicant.

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Tables

Table T.1

Fourier Transforms

Definitions

Transform
$$V(f) = \mathscr{F}[v(t)] = \int_{-\infty}^{\infty} v(t)e^{-j2\pi ft}\,dt$$

Inverse transform
$$v(t) = \mathscr{F}^{-1}[V(f)] = \int_{-\infty}^{\infty} V(f)e^{j2\pi ft}\,df$$

Integral theorem

$$\int_{-\infty}^{\infty} v(t)w^*(t)\,dt = \int_{-\infty}^{\infty} V(f)W^*(f)\,df$$

Theorems

Operation	Function	Transform
Superposition	$a_1v_1(t) + a_2v_2(t)$	$a_1V_1(f) + a_2V_2(f)$
Time delay	$v(t - t_d)$	$V(f)e^{-j\omega t_d}$
Scale change	$v(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}V\left(\frac{f}{\alpha}\right)$
Conjugation	$v^*(t)$	$V^*(-f)$
Duality	$V(t)$	$v(-f)$
Frequent translation	$v(t)e^{j\omega_c t}$	$V(f - f_c)$
Modulation	$v(t)\cos(\omega_c t + \phi)$	$\frac{1}{2}[V(f - f_c)e^{j\phi} + V(f + f_c)e^{-j\phi}]$
Differentiation	$\frac{d^nv(t)}{dt^n}$	$(j2\pi f)^nV(f)$
Integration	$\int_{-\infty}^{t} v(\lambda)\,d\lambda$	$\frac{1}{j2\pi f}V(f) + \frac{1}{2}V(0)\,\delta(f)$
Convolution	$v * w(t)$	$V(f)W(f)$
Multiplication	$v(t)w(t)$	$V * W(f)$
Multiplication by t^n	$t^nv(t)$	$(-j2\pi)^{-n}\frac{d^nV(f)}{df^n}$

Transforms

Function	$v(t)$	$V(f)$
Rectangular	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} f\tau$
Triangular	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 f\tau$
Gaussian	$e^{-\pi(bt)^2}$	$(1/b)\, e^{-\pi(f/b)^2}$
Causal exponential	$e^{-bt}u(t)$	$\dfrac{1}{b + j2\pi f}$
Symmetric exponential	$e^{-b\|t\|}$	$\dfrac{2b}{b^2 + (2\pi f)^2}$
Sinc	$\operatorname{sinc} 2Wt$	$\dfrac{1}{2W}\Pi\left(\dfrac{f}{2W}\right)$
Sinc squared	$\operatorname{sinc}^2 2Wt$	$\dfrac{1}{2W}\Lambda\left(\dfrac{f}{2W}\right)$
Constant	1	$\delta(f)$
Phasor	$e^{j(\omega_c t + \phi)}$	$e^{j\phi}\,\delta(f - f_c)$
Sinusoid	$\cos(\omega_c t + \phi)$	$\frac{1}{2}[e^{j\phi}\,\delta(f - f_c) + e^{-j\phi}\delta(f + f_c)]$
Impulse	$\delta(t - t_d)$	$e^{-j\omega t_d}$
Sampling	$\sum_{k=-\infty}^{\infty} \delta(t - kT_s)$	$f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$
Signum	$\operatorname{sgn} t$	$1/j\pi f$
Step	$u(t)$	$\dfrac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)$