Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 05, Issue 2 (December 2010), pp. 612 – 620
(Previously, Vol. 05, Issue 10, pp. 1709 – 1717)

On the Eigenvalue and Inertia Problems for Descriptor Systems

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Received: February 14, 2010; Accepted: October 12, 2010

Abstract

The present study is intended to demonstrate that for a descriptor system with matrix pencil (C,G), there exists a matrix F such that matrix FG and matrix pencil (C,G) have the same positive and negative eigenvalues. It is also shown that matrix F can be calculated as a contour integral. On the other hand, different representations for matrix F are introduced.

Keywords: Descriptor system; matrix pencil; Weierstrass canonical normal form; generalized eigenvalues; inertia

MSC 2000 No.: 45CXX; 34HXX

1. Introduction

Matrix pencils appear in the process of analyzing descriptor systems

$$\begin{cases} C\dot{x}(t) = Gx(t) + Bu(t) \\ y = Lx(t) + Du(t) \end{cases}. \tag{1}$$

where C, $G \in R^{n\times n}$, $x \in R^{n}$, $u \in R^{m}$, $B \in R^{n\times m}$, $L \in R^{p\times n}$, $D \in R^{p\times m}$. It is generally known that the stability properties of system (1) can be characterized in terms of the eigenvalues of pencil $\lambda C - G$ and that the controllability and observability properties of system (1) depend on the rank of matrix pencils $\lambda C - G$ and $\lambda C^T - G^T$, respectively. See Cobb (1984), Mehrmann and Stykel (2005) and Yip and Sincovec (1981). Note that the eigenvalues of $\lambda C - G$ may be ill-conditioned in the sense that they may change largely even for small perturbations in C and G. On the other hand, there are different methods for various types of eigenvalues problems, see Mehrmann and Voss (2005) and Van den Eshof (2002). For a historical background of eigenproblems, one can refer to Ciarlet (2001).

First, some basic concepts that are later needed for the purposes of the present paper are to be introduced. Matrix $\lambda C - G$ is called a matrix pencil where $C, G, \in R^{n\times n}$ and $\lambda \in \mathbb{C}$. Matrix pencil is also shown by matrix pair (C, G). Pencil (C, G) is called *regular* if there exists a $\lambda \in \mathbb{C}$ such that $\det(\lambda C - G) \neq 0$; otherwise, it is called *singular*. Descriptor system (1) is called regular if pencil (C, G) is regular. The analyses in this paper are based on regular descriptor systems. Eigenvalues of matrix pencil (C, G) are defined to be the roots of characteristic polynomial $\det(\lambda C - G) = 0$. $sp(C, G)$ is the set of eigenvalues of pencil (C, G). Value $\lambda_0 \in \mathbb{C}$ is called a finite eigenvalue of $\lambda C - G$, if $\det(\lambda_0 C - G) = 0$. If matrix C is singular, then $\lambda C - G$ is said to have an eigenvalue at infinity. Two matrix pencils of $(C_1 G_1)$ and $(C_2 G_2)$ are called *equivalent* if there exist nonsingular matrices T and S such that $\lambda C_1 - G_1 = \lambda T C_2 S - T G_2 S$. Let $\lambda C - G$ be a regular pencil with $C, G \in R^{n\times n}$. There exist two nonsingular matrices T and S$\in C^{n\times n}$ such that

$$TCS = diag(I_{n_f}, N), \quad TGS = diag(J, I_{n_\infty}), \tag{2}$$

where matrices J and N are in Jordan canonical form and matrix N is a nilpotent matrix; that is, there exists an integer v such that $N^v = 0$ but $N^{v-1} \neq 0$.

The representation (TCS, TGS) of pencil (C, G) is called Weierstrass canonical normal form. See Dai (1989).The inertia of a regular pencil $\lambda C - G$ is defined by a quadruple of integers

$$In(C, G) = \{\pi_-(C, G), \pi_+(C, G), \pi_0(C, G), \pi_\infty(C, G)\},$$

where $\pi_-(C, G), \pi_+(C, G)$ and $\pi_0(C, G)$ are the numbers of eigenvalues with negative, positive and zero real parts, respectively and where $\pi_\infty(C, G)$ is the number of infinite eigenvalues of $\lambda C - G$.

2. Reduction Generalized Eigenvalue and Inertia Problem to Standard Form

Theorem1. There exists a matrix F such that

$$\pi_+(C, G) = \pi_+(FG) \qquad \pi_-(C, G) = \pi_-(FG).$$

Proof:

Assume that matrix pencil (C, G) is in canonical form (TCS, TGS),

$$TCS = diag(I_{n_f}, N)\ , \qquad TGS = diag(J, I_{n_\infty}),$$

where matrices J and N are in Jordan canonical form and matrix N is a nilpotent matrix; then we define the matrix as

$$F = S diag(I_{n_f}, 0) T.$$

Therefore,

$$FG = S diag\left(I_{n_f}, 0\right) TG = S diag\left(I_{n_f}, 0\right) TGSS^{-1}$$
$$= S diag\left(I_{n_f}, 0\right) diag(J, I_{n_\infty}) S^{-1} = S diag(J, 0) S^{-1}$$

Thus,

$$sp(FG) = sp(J) \cup \{0\}.$$

Also, it is generally known that the finite eigenvalues of $\lambda C - G$ are the eigenvalues of matrix J, see Malyshev (1990) and Lewis(1985), so

$$\pi_+(C,G) = \pi_+(J) = \pi_+(FG) \qquad\qquad \pi_-(C,G) = \pi_-(J) = \pi_-(FG).$$

In fact, it is hence proved that matrix pencil (C, G) and matrix FG have the same positive and negative eigenvalues.

In the next step, we want to show the relation of F with coefficients of Laurent series of $(\lambda C - G)^{-1}$ at infinity. It is generally known that Laurent series of $(\lambda C - G)^{-1}$ at infinity are in the following form

$$(\lambda C - G)^{-1} = \sum_{n=-\infty}^{+\infty} h_n \lambda^{-n-1},$$

where

$$h_n = S diag(J^n, 0) T \qquad\qquad n = 0, 1, 2, \ldots$$

and

$$h_n = S diag(0, -N^{-n-1}) T \qquad n = -1, -2, \ldots.$$

See Lewis (1985).

Theorem 2. Let c be a closed simple curve such that the finite eigenvalues of $\lambda C - G$ lie inside c, then

$$h_n = \frac{1}{2\pi i}\oint_c \lambda^n(\lambda C - G)^{-1}\, d\lambda \qquad n \geq 0.$$

Proof:

We have

$$(\lambda C - G)^{-1} = \left[\lambda T^{-1}\, diag\left(I_{n_f}, N\right) S^{-1} - T^{-1}\, diag(J, I_{n_\infty}) S^{-1}\right]^{-1} =$$

$$S diag(\lambda I_{n_f} - J, \lambda N - I_{n_\infty})^{-1} T = S diag[\left(\lambda I_{n_f} - J\right)^{-1}, (\lambda N - I_{n_\infty})^{-1}] T.$$

So,

$$\frac{1}{2\pi i}\oint_c \lambda^n(\lambda C - G)^{-1}\, d\lambda =$$

$$\frac{1}{2\pi i} S diag\left[\oint_c \lambda^n(\lambda I_{n_f} - J)^{-1}\, d\lambda, \oint_c \lambda^n(\lambda N - I_{n_\infty})^{-1}\, d\lambda\right] T$$

But, c includes all finite eigenvalues of $\lambda C - G$. Hence,

$$\frac{1}{2\pi i}\oint_c \lambda^n(\lambda I_{n_f} - J)^{-1}\, d\lambda = J^n$$

and since N is nilpotent,

$$\frac{1}{2\pi i}\oint_c \lambda^n(\lambda N - I_{n_\infty})^{-1}\, d\lambda = 0.$$

Thus,

$$\frac{1}{2\pi i}\oint_c \lambda^n(\lambda C - G)^{-1}\, d\lambda = S diag(J^n, 0) T = h_n \qquad n \geq 0.$$

Theorem 3. Let c be a closed simple curve such that the finite eigenvalues of $\lambda C - G$ lie inside c. Furthermore, c includes origin. Then,

$$h_n = \frac{-1}{2\pi i}\oint_c \lambda^n(\lambda C - G)^{-1}\, d\lambda \qquad n < 0.$$

Proof:

Since c includes origin,

$$\frac{-1}{2\pi i}\oint_c \lambda^n(\lambda I_{n_f} - J)^{-1}\,d\lambda = 0$$

and

$$\frac{-1}{2\pi i}\oint_c \lambda^n(\lambda N - I_{n_\infty})^{-1}\,d\lambda = -N^{-n-1}$$

Thus,

$$\frac{-1}{2\pi i}\oint_c \lambda^n(\lambda C - G)^{-1}\,d\lambda = S\,diag(0, -N^{-n-1})T \qquad n < 0$$

So far, a generalized eigenvalue and inertia problem have been reduced to a standard eigenvalue and inertia problem by using Weierstrass canonical normal form. Also the coefficients of Laurent series of $(\lambda C - G)^{-1}$ at infinity have been described as contour integrals. In sequel it will be shown that matrix F can be calculated as a contour integral. Then, different representations for matrix F will be obtained.

Corollary 1. F is the coefficient of λ^{-1}in the Laurent series of $(\lambda C - G)^{-1}$ at infinity.

Proof:

The coefficient of λ^{-1}in the Laurent series of $(\lambda C - G)^{-1}$ at infinity is equal to

$$h_0 = \frac{1}{2\pi i}\oint_c(\lambda C - G)^{-1}\,d\lambda = S\,diag\left(I_{n_f}, 0\right)T = F\ .$$

3. Some Representations for Matrix F

In the previous section, matrix F was defined in terms of two matrices T, S. This section attempts to find different representations for F. One of them is a representation that just depends on matrix S; the other describes F in terms of matrix T; and the third representation describes the relation between F and the constant term of Laurent series of $(\lambda C - G)^{-1}$ at infinity.

Lemma1. There exists a representation for F that just depends on matrix S.

Proof:

We define matrix T_r as follows:

$$T_r = S\,diag(I_{n_f}, 0)S^{-1},$$

where S is the matrix defined in canonical normal form. We would then have

$$CT_r = T^{-1}TCSS^{-1}T_r = T^{-1}\,diag\left(I_{n_f}, N\right)S^{-1}S\,diag\left(I_{n_f}, 0\right)S^{-1} =$$

$$T^{-1}diag\left(I_{n_f}, 0\right)S^{-1} \quad . \tag{3}$$

Also,

$$I - T_r = SS^{-1} - S\,diag\left(I_{n_f}, 0\right)S^{-1} = S\,diag(0, I_{n_\infty})S^{-1}$$

and

$$\begin{aligned} G\,(I - T_r) &= T^{-1}TGSS^{-1}(I - T_r) \\ &= T^{-1}\,diag(J, I_{n_\infty})S^{-1}S\,diag(0, I_{n_\infty})S^{-1} \\ &= T^{-1}diag(0, I_{n_\infty})S^{-1} \quad . \end{aligned} \tag{4}$$

By adding relations (3) and (4)

$$CT_r + G\,(I - T_r) = T^{-1}diag\left(I_{n_f}, 0\right)S^{-1} + T^{-1}diag(0, I_{n_\infty})S^{-1} = T^{-1}diag\left(I_{n_f}, I_{n_\infty}\right)S^{-1},$$

So,

$$[CT_r + G\,(I - T_r)]^{-1} = S\,diag\left(I_{n_f}, I_{n_\infty}\right)T.$$

Thus,

$$T_r[CT_r + G\,(I - T_r)]^{-1} = S\,diag\left(I_{n_f}, 0\right)S^{-1}S\,diag\left(I_{n_f}, I_{n_\infty}\right)T = S\,diag\left(I_{n_f}, 0\right)T.$$

But, the last term is equal to F and the proof is then complete.

Lemma2. There exists a representation for F that just depends on matrix T.

Proof:

The proof is similar to Lemma1. It is sufficient to define

$$T_l = T^{-1} diag(I_{n_f}, 0) T.$$

It, then, follows that

$$F = [T_l C + (I - T_l) G]^{-1} T_l.$$

Theorem 4. There exists a matrix E such that

$$I - T_r = -EG, \qquad I - T_l = -GE.$$

Proof:

Let E be the constant term of Laurent series of $(\lambda C - G)^{-1}$ at infinity.

$$E = h_{-1} = \frac{-1}{2\pi i} \oint_c \lambda^{-1} (\lambda C - G)^{-1} \, d\lambda.$$

According to computations in Theorem 2,

$$\lambda^{-1} (\lambda C - G)^{-1} = S diag[\lambda^{-1} (\lambda I_{n_f} - J)^{-1}, \lambda^{-1} (\lambda N - I_{n_\infty})^{-1}] T.$$

So,

$$E = \frac{-1}{2\pi i} S diag[\oint_c \lambda^{-1} (\lambda I_{n_f} - J)^{-1} \, d\lambda, \oint_c \lambda^{-1} (\lambda N - I_{n_\infty})^{-1} \, d\lambda] T.$$

Since c includes the origin

$$\frac{-1}{2\pi i} \oint_c \lambda^{-1} (\lambda I_{n_f} - J)^{-1} \, d\lambda = 0,$$

and

$$\frac{-1}{2\pi i} \oint_c \lambda^{-1} (\lambda N - I_{n_\infty})^{-1} \, d\lambda = -I_{n_\infty}.$$

Therefore,

$$E = -S diag(0, I_{n_\infty}) T$$

and

$$-EG = S\,diag(0, I_{n_\infty})TT^{-1}\,diag(J, I_{n_\infty})S^{-1} = S\,diag(0, I_{n_\infty})S^{-1}.$$

But,

$$I - T_r = I - S\,diag\left(I_{n_f}, 0\right)S^{-1} = S\,diag(0, I_{n_\infty})S^{-1}.$$

Hence,

$$I - T_r = -EG.$$

Similar computations show that

$$I - T_l = -GE.$$

Corollary 2. The following equalities hold

$$F = (I + EG)[C(I + EG) - GEG]^{-1} = [(I + GE)C - GEG]^{-1}(I + GE).$$

Proof:

It follows immediately from Lemma 1 and Lemma 2 and Theorem 4.

4. Conclusion

In this paper, a method for the reduction of a generalized eigenvalue and inertia problem to a standard eigenvalue and inertia problem is presented. The method is based on the use of Weierstrass canonical normal form and matrix F. In fact, it is demonstrated that matrix FG and matrix pencil (C, G) have the same positive and negative eigenvalues. Also according to the analyses presented, it is stated that matrix F can be described in terms of coefficients of Laurent series of $(\lambda C - G)^{-1}$ at infinity. As part of this study, other representations for F have also been offered. It is to be noted that the regularity of matrix pencil (C, G) was the only assumption used in this paper and the results are valid even if C and G are singular matrices.

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