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Finite Element Analysis in Porous Media for Incompressible Flow of Contamination from Nuclear Waste

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Abstract

A non-linear parabolic system is used to describe incompressible nuclear waste disposal contamination in porous media, in which both molecular diffusion and dispersion are considered. The Galerkin method is applied for the pressure equation. For the brine, radionuclide and heat, a kind of partial upwind finite element scheme is constructed. Examples are included to demonstrate certain aspects of the theory and illustrate the capabilities of the kind of partial upwind finite element approach.

Keywords: Finite element method; Galerkin method; incompressible flow; nuclear waste

MSC (2000) No.: 65L60, 74S05, 82D75

1. Introduction

The proposed disposal of high-level nuclear waste in underground repositories is an important environmental topic for many countries. Decisions on the feasibility and safety of the various sites and disposal methods is based, in part, on numerical models for describing the flow of contaminated brines and groundwater through porous or fractured media under severe thermal regimes caused by the radioactive contaminants. A fully discrete formulation is given in some detail to present key ideas that are essential in code development. The non-linear couplings between the unknowns are important in modeling the correct physics of flow.

In this model one obtains a convection-diffusion equations which represent a mathematical model for a case of diffusion phenomena in which underlying flow is present;

Δw *and* $b\nabla w$ correspond to the transport of w through the diffusion process and the convection effects, respectively, where ∇ *and* Δ denoted respectively the gradient operator and the Laplacian operator in the spatial coordinates.

The prediction of an accurate numerical solution for the convection dominated flow problem is one of the more difficult tasks in computational fluid dynamics. The central difference method and the conventional Galerkin method have consistently produced unphysical oscillatory solutions. One successful technique for solving such a problem is known as the upwind algorithm, which was originally devised for the finite difference method. In the finite element method, a popular technique known as the streamline upwind Petrov-Galerkin (SUPG) method is used. This method modifies the weighting functions by using the local velocity to dictate an upstream direction of these functions. Such modification eliminates the oscillatory behavior for some convection problems. For more details, see Wansophark and Pramote (2008).

In this paper, we have considered the fluid flow in porous media using a Galerkin method for the pressure equation and a kind of partial upwind finite element scheme is constructed for the convection dominated saturation (or concentration) equation, the trace concentration of i^{th} radionuclide and the heat equation. For more details of this subject see Douglas (2001, 2002), Huang (2000) and Chen et al. (2009).

2. Model Equations

The model for incompressible flow and transport of contaminated brine in porous media can be described by a differential system that can be put into the following form, Gaohong and Cheng (1999).

Fluid:

$$\begin{cases} (a)\,\nabla.u = -q + R_s', \\ (b)\ u = -\dfrac{k(x)}{\mu(c)}\nabla P = -a(c)\nabla P. \end{cases} \tag{1}$$

Brine:

$$\varphi\frac{\partial c}{\partial t} + u.\nabla c - \nabla.(E_c\nabla c) = g(c). \tag{2}$$

Heat:

$$d_2\frac{\partial T}{\partial t} + c_p u.\nabla T - \nabla.(E_H\nabla T) = Q(u,T,c,p). \tag{3}$$

Radionuclide:

$$\varphi K_i\frac{\partial c_i}{\partial t} + u.\nabla c_i - \nabla.(E_c\nabla c_i) = f_i(c,c_1,...,c_N), \tag{4}$$

with the boundary conditions

$$\begin{cases} (a)\ \ u.n = 0, & on \quad \Gamma \\ (b)\ \ (E_c \nabla c - cu).n = 0, & on \quad \Gamma \\ (c)\ \ (E_c \nabla c_i - c_i u).n = 0, & on \quad \Gamma \\ (d)\ \ (E_H \nabla T - c_p T u).n = 0, & on \quad \Gamma \\ (e)\ \ \dfrac{\partial p}{\partial n} = 0, & (x,t) \in \Gamma \times (0,T] \end{cases} \tag{5}$$

and the initial conditions

$$\begin{cases} (a)\ \ p(x,0) = p_0(x); & x \in \Omega, \\ (b)\ \ c(x,0) = c_0(x); & x \in \Omega, \\ (c)\ \ c_i(x,0) = c_{0i}(x); & x \in \Omega, \\ (d)\ \ T(x,0) = T_0; & x \in \Omega, \end{cases} \tag{6}$$

where *n* is the unit outer normal to Γ, $x \in \Omega \subset R^2, t \in (0,T]$; *u* is the Darcy velocity; *P* is the pressure; $\phi_1 = \phi c_w, q = q(x,t)$ is the production term; $R_s^{'} = R_s^{'}(c) = [c_s \phi K_s f_s / (1+c_s)](1-c)$ is the salt dissolution term; *k*(*x*) is the permeability of the rock; $\mu(c)$ is the viscosity of the fluid, c dependent upon c, the concentration of the brine in the fluid and *T* is the temperature of the fluid,

$$\begin{aligned} d_2 &= \varphi\ c_p + (1-\varphi)\rho_R c_{pR}, \\ E_H\ &= D c_{pw} + K_m^{'} I, K_m^{'} = k_m / \rho_0, D = (D_{ij}) \\ &= (\alpha_T |u| \delta_{ij} + (\alpha_L - \alpha_T) u_i u_j / |u|), \end{aligned}$$

and

$$Q(u,T,c,p) = -\{[\nabla U_0 - c_p \nabla T_0].u + [U_0 + c_p (T - T_0)) + (p/\rho)][-q + R_s^{'}]\} - q_L - qH - q_H .$$

$$E_c = D + D_m I, \text{and}\ \ \mathrm{g(c) = -c}\{[\mathrm{c_s}\phi K_s f_s / (1+c_s)](1-c)\} - q_c + R_s^{'},$$

c_i is the trace concentration of the *i-th* radionuclide, and

$$\begin{aligned} f_i(c, c_1, c_2, ..., c_N) = c_i \{q - [c_s \phi K_s f_s / (1+c_s)](1-c)\} - q c_i - q_{ci} + q_{oi} \\ + \sum_{j=1}^{N} k_{ij} \lambda_j K_j \phi c_j - \lambda_i K_i \phi c_i . \end{aligned}$$

The reservoir Ω will be taken to be of unit thickness and will be identified with a bounded domain in R^2. We shall omit gravitational terms for simplicity of exposition, no significant mathematical questions arises the lower order terms are included.

We assume that:

(A1) $a(c), R_s^{'}(c), g(c), f_i(c, c_1, ..., c_N), Q(u, T, c, p) \in C_0^1(R)$,
$\phi(x), K_i, d_2 \in H'(\Omega), q \in L^\infty(0, T; H^1(\Omega))$
$0 < c_0 \le a(c), R_s^{'}(c), g(c), \phi(x), f_i(c, c_1, ..., c_n), K_i, d_2, Q(u, T, c, p) \le c_1$, $\forall c \in R, x \in \Omega$
$D_m > 0$, $\alpha_l \ge \alpha_t > 0$

(A2) The solution of the problem (1-6) are regular:
$C(x,t) \in L^2(0,T; H^2(\Omega)) \bigcap L^\infty(0,T; w_\infty^{'}(\Omega))$
$C_i(x,t) \in L^2(0,T; H^2(\Omega)) \bigcap L^\infty(0,T; w_\infty^{'}(\Omega))$
$P(x,t) \in L^\infty(0,T; H^{r+1}(\Omega))$, $(r \ge 2)$
$c_t, c_{tt}, c_{ttt} \in L^\infty(0,T; H^1(\Omega))$; $p_t, p_{tt} \in L^\infty(0,T; L^\infty(\Omega))$

(A3) For any $\phi \in L^2(\Omega)$, the boundary values problem:
$-\Delta\phi + \phi = \varphi, \quad x \in \Omega$
$\dfrac{\partial \phi}{\partial n} = 0, \qquad x \in \Gamma,$

there exists unique solution $\varphi \in H^2(\Omega)$ and a positive constant M such that $\|\varphi\|_2 \le M\|\phi\|$ [Manaa (2000)].

3. Finite Element Spaces

Consider a regular family $\{T_h\}$ of triangulation defined over Ω, where h is the longest diameter of a triangular element with the triangular T_h, we have a set of close triangles $\{e_i\}(1 \le i \le N_e)$ and a set of nodes $\{P_i\}(1 \le i \le N_P + M_P)$ where $P_i(1 \le i \le N_P)$ are interior nodes in Ω and $P_j(N_{P+1} \le j \le N_P + M_p)$ are boundary nodes on Γ. We put h_s to be the maximum side length of triangles and k to be minimum perpendicular length of triangles for all $e \in T_h$.

Definition 3.1. A family T_h of triangulations is of weakly acute type, if there exists a constant $\theta_0 > 0$ independent of h such that, the internal angle θ of any triangle $e_i \in T_h$ satisfies $\theta_0 \le \theta \le \pi/2$.

Definition 3.1.1. Let $\phi_i(p), (1 \le i \le M)$, be the continuous function in Ω s.t. $\phi_i(p)$, is linear on each $e \in T_h$ and $\phi_i(p_j) = \delta_{ij}$ for any nodal point p_j. We denote M_h be the linear span of $\phi_i, (1 \le i \le M)$, i.e., a finite dimensional subspace of $H^1(\Omega)$ and defined by: $M_h = \{z_h \mid \in C(\Omega); z_h \text{ is a linear function on } e, \forall e \in T_h\}$. Also, define a subspace of $H_0^1(\Omega)$ be: $M_{0h} = \{z_h \mid z_h \in M_h; z_h(P_k) = 0, k = M+1, ..., K\}$.

We associate the index set $\Lambda = \{j \neq i : P_j$ is adjacent to $P_i\}$. Let $P_i\, P_j\, P_k$, be three vertices of triangular element e and $\lambda_i, \lambda_j, \lambda_k$ be barycentric coordinates. We have the following definitions [see Hu and Tian (1992)].

Definition 3.2. With each vertex P_i belonging to triangle e, the barycentric subdivision Ω_i^e is given by: $\Omega_i^e = \{P \mid P \in e\ ;\ \lambda_i(P) \geq \lambda_j(P)\ ,\ \lambda_i(P) \geq \lambda_k(P), \forall P_j \in e\}$, and the barycentric domain Ω_i associated with vertex P_i in Ω is given by $\Omega_i = \cup \Omega_i^e\ , e \in T_h$.

Definition 3.3. With the characteristic function $\mu_i(x)$ of barycentric domain Ω_i, the mass lumping operator $\wedge$: $w \in C(\Omega) \to \hat{w} \in L_\infty(\Omega)$ is defined by $\hat{w}(p) = \sum_i^{N_P + M_P} w(p_i)\mu_i(p)$.

Using interpolation theory in Sobolev space [Ciarlet (1978)] and inverse inequality, with step-length h_c, we have the relation between $\hat{w}$ and w from the following lemma:

Lemma 3.1. There exists a constant C such that:

$$\|w - \hat{w}\|_{o,p} \leq Ch_c |w|_{1,p}, \qquad \forall w \in M_h\ , p \geq 1 \tag{7}$$

$$|w_h|_1 \leq Mh_c^{-1} \|w_h\|, \quad \forall\ w_h \in M_{0h}. \tag{8}$$

Lemma 3.2. There exist constants $C_1, C_2 > 0$ such that:

$$C_1 \|w\|_{0,p} \leq \|\hat{w}\|_{0,p} \leq C_2 \|w\|_{0,p}, \quad \forall w \in M_h. \tag{9}$$

Definition 3.4. Let $\{M_h\}$ be a family of finite dimensional subspaces of $C(\Omega)$, which is piecewise polynomial space of degree less or equal to r with step length h_P and the following property: for $P \in [1, \infty]$, $r \geq 2$, there exists a constant M such that for $0 \leq q \leq 2$ and $\phi \in w_P^{r+1}(\Omega)$:

$$\inf_{x \in \{M_h\}} \|\phi - x\|_{q,p} \leq Mh^{r+1-q} |\phi|_{r+1,p}.$$

Similarly, we define $\{N_h\}$ be a family of finite-dimensional subspace of $C(\Omega) \times C(\Omega)$, which is piecewise polynomial space of degree less or equal to *r-1* with the similar property as M_h and $0 \leq q \leq r-1$. We also assume the families $\{M_h\}$ and $\{N_h\}$ satisfy inverse inequalities:

$$\|\varphi\|_{L^\infty} \leq Mh_P^{-1} \|\varphi\|, \qquad \|\nabla \varphi\|_{L^\infty} \leq Mh_P^{-1} \|\nabla \varphi\|, \qquad \forall \varphi \in M_h \tag{10}$$

[Manaa (2000)].

4. Error Estimates

Let $\tau > 0$ is time step and $N_\tau = T/\tau$. We use a Galerkin finite element method for the pressure and velocity and partial upwind finite element scheme for brine, radionuclide, and heat.

Let $C^0 \in M_h$ be a $L^2(\Omega)$ - projection of c^0 in M_h:

$$(c^0 - C^0, z_h) = 0 \quad \forall z_h \in M_h .$$

We can get $P^0 \in V_h$ such that

$$\int_\Omega P^0 dx = 0 \,,\, \left(\frac{k(x)}{\mu(c^0)}\nabla P^0, \nabla v\right) = (-q^0, v) + (R_s^{'}, v) \;\;,\; \forall v \in V_h \; and \; U^0 \in W_h \; from$$

$$U^0 = -\frac{k(x)}{\mu(C^0)}\nabla P^0 = -a(C^0)\nabla P^0$$

If the approximate solution $\{P^m, U^m, C^m, C_i^m (i = 1,2,...,N), T^m\} \in V_h \times W_h \times M_h \times M_h^N \times R_h$ is known, we want to find $\{P^{m+1}, U^{m+1}, C^{m+1}, C_i^{m+1} (i = 1,2,...,N), T^{m+1}\} \in V_h \times W_h \times M_h \times M_h^N \times R_h$ at $t = t^{m+1}$, with five steps . Let (.,.) denote the inner product in $L^2(\Omega)$

Step 1.
Find C^{m+1} for $m = 0,1,\ldots,N_\tau - 1$, such that

$$(\hat{\phi} D_\tau \hat{C}^m, \hat{z}_h) + (E_c \nabla C^{m+1/2}, \nabla z) + R(U^m, C^{m+1/2}, z_h) = (\hat{g}(C^{m+1/2}), \hat{z}_h) \;\forall z_h \in M_h \;, \tag{11}$$

where

$$D_\tau C^m = (C^{m+1} - C^m)/\tau \;\;,\;\; C^{m+1/2} = (C^{m+1} + C^m)/2 \text{ and with } z_i = z_h(P_i) \;,$$

$$C_i^{m+1/2} = C^{m+1/2}(P_i) \,,$$

and

$$\beta_{ij}^m = \int_{\Gamma_{ij}} U^m . n_{ij} \, d\Gamma$$, here n_{ij} is the unit outer normal to Γ_{ij} .

The partial upwind coefficients should be required that [Hu and Tian (1992)].

$$(a) \quad \alpha_{ij}^m + \alpha_{ji}^m = 1$$

$$\begin{aligned} (b) \quad & \max\{1/2, 1 - \rho_{ij}^{-1}\} \le \alpha_{ij} \le 1, if\, \beta_{ij} \ge 0, \\ & \max\{1/2, 1 - \rho_{ij}^{-1}\} \le \alpha_{ji} \le 1, if\, \beta_{ij} < 0. \end{aligned} \tag{12}$$

Step 2.
Find C_i^{m+1}, for $m = 0,1,\ldots,N_\tau - 1$, such that:

$$(\hat{\varphi}\hat{K}_i D_\tau \hat{C}_i^m, \hat{z}_h) + (E_c \nabla C_i^{m+1}, \nabla z) + R(U^m, C_i^{m+1}, z_h) \\ = (\hat{f}_i(C^{m+1/2}, C_1^{m+1}, ..., C_N^{m+1}), \hat{z}_h) \ \forall z_h \in M_h^N, \tag{13}$$

where

$$R(U^m, C_i^{m+1}, z_h) = \sum_{i=1}^{M} z_i \sum_{j \in \Lambda i} \beta_{ij}^m (\alpha_{ij}^m C_i^{m+1/2} + \alpha_{ji}^m C_j^{m+1/2}).$$

Step 3.
Find T^{m+1} such that:

$$(d_2 D_\tau \hat{T}^m, \hat{r}_h) + (\tilde{E}_H \nabla T^{m+1/2}, \nabla r_h) + R(c_p U^m, T^{m+1/2}, r_h) \\ = (\hat{Q}(U^m, T^m, C^m, P^m), \hat{r}_h) \quad \forall r_h \in R_h, \tag{14}$$

where

$$R(c_p U^m, T^{m+1/2}, r_h) = \sum_{i=1}^{M} r_i \sum_{j \in \Lambda i} \beta_{ij}^m (\alpha_{ij}^m T_i^{m+1/2} + \alpha_{ji}^m T_j^{m+1/2}) \quad .$$

Step 4.
Find P^{m+1} such that:

$$\int_\Omega P^{m+1} dx = 0 \, , (\frac{k(x)}{\mu(C^{m+1})} \nabla P^{m+1}, \nabla v) = (-q^{m+1}, v) + (R_s^{'}, v) \ \ , \ \forall v \in V_h \quad . \tag{15}$$

Step 5.
Find U^{m+1} as:

$$U^{m+1} = -\frac{k(x)}{\mu(C^{m+1})} \nabla P^{m+1} = -a(C^{m+1}) \nabla P^{m+1}. \tag{16}$$

Lemma 4.1. Let $\overline{p} \in V_h$ be the elliptic projection of $p \in H^1(\Omega)$ into V_h defined by $(a(c)\nabla\overline{p}, \nabla v) = (a(c)\nabla p, \nabla v)$, $\forall v \in V_h$. Then, there exists a constant k_1 such that $\|p - \overline{p}\| + h_p \|\nabla p - \nabla \overline{p}\| \le k_1 \|p\|_{r+1} h_p^{r+1}$.

Proof:

See Quarteroni and Valli (1997).

Some Important Remarks:

1. $U^{m+1}.n = 0$ in Γ. (17)

2. If $\|U^l - u^l\| \le Ch_P \quad (0 \le l \le m)$, then $\|U^{m+1} - u^{m+1}\| \le Ch_P$.

and if $\left\| U^m - u^m \right\| \le Ch_P$, then $\left\| \nabla U^m \right\| \le C_1$. (18)

3. We will make the inductive assumption that if $\left\| U^l \right\|_{L_\infty} \le k^* \quad (0 \le l \le m)$, then

$$\left\| U^{m+1} \right\|_{L_\infty} \le k^* \tag{19}$$

4. From Manaa (2000), if T_h is regular triangulation of weakly acute type we have

$$\left| w_h \right|_1 \le \sqrt{6} / k \left\| \hat{w}_h \right\| \quad , \ \forall w_h \in M_h \tag{20}$$

Theorem 4.1. C satisfies discrete mass conservation law

$$\int_\Omega \hat{\varphi} D_\tau \hat{C}^m d\Omega = \int_\Omega \hat{g}^{m+1/2}(C) d\Omega \quad , \ m = 1, 2, \cdots, N_\tau . \tag{21}$$

Proof:

In (10), let $z_h = 1$, then $(Ec \nabla C^{m+1/2}, \nabla 1) = 0$ and

$$R(U^m, C^{m+1/2}, 1) = \sum_{i=1}^{M} 1 \sum_{j \in \Lambda_i} \beta_{ij}^m C_{ij}^{m+1/2} = \sum_{e \in T_h}^{M} \sum_{p_i, p_j \in e, i < j_i} (\beta_{ij}^m C_{ij}^{m+1/2} - \beta_{ij}^m C_{ij}^{m+1/2}) = 0 .$$

Then (21) holds.

Theorem 4.2. C_i satisfies discrete mass conservation law

$$\int_\Omega \hat{\phi} \hat{K}_i D_\tau \hat{C}_i^m d\Omega = \int_\Omega f_i^{m+1/2} d\Omega \quad , \ i = 1, 2, ..., N \tag{22}$$

Proof:

Same as Theorem 4.1.

Lemma 4.2. Let

$$\overline{c} : [0, t] \to M_h \ \ and \ \ \overline{c}_i : [0, t] \to M_h^N, i = 1, ..., N$$

such that

$$(Ec \nabla c - \overline{c}, \nabla z) - \lambda (c - \overline{c}, z) = 0, \ \forall z \in M_h ,$$
$$(Ec \nabla (c_i - \overline{c}_i), \nabla z) - \lambda ((c_i - \overline{c}_i), z) = 0, \ \forall z \in M_h^N \ , \ t \in J$$

and let

$$c-\overline{c}=\varsigma \ , c_i-\overline{c}_i=\varsigma_i \, ,$$

then,

$$\|\varsigma\|_1 \le Mh_c \ , \ \left\|\frac{\partial \varsigma}{\partial t}\right\| \le Mh_c^2 \ , \left\|\frac{\partial \varsigma}{\partial t}\right\|_1 \le Mh_c$$

and

$$\|\varsigma_i\|_1 \le Mh_c \ , \ \left\|\frac{\partial \varsigma_i}{\partial t}\right\| \le Mh_c^2 \ , \left\|\frac{\partial \varsigma_i}{\partial t}\right\|_1 \le Mh_c .$$

Proof:

See Ciarlet (1978).

Lemma 4.3. For all $z_h \in M_h$ and $\varsigma = c-\overline{c}$, $\xi = \overline{c} - C$,

$$\left|((u^m.\nabla c^{m+1/2}), z_h) - R(U^m, C^{m+1/2}, z_h)\right| \le M(h_c^2 + \left\|\xi^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2 + \left\|\nabla\varsigma^{m+1/2}\right\|^2 + \left\|u^m - U^m\right\|^2 + \left\|z_h\right\|^2) + \varepsilon\left\|\nabla z_h\right\|^2 , \tag{23}$$

where $\varepsilon > 0$ is arbitrary small constant.

Proof:

$$\begin{aligned}((u^m.\nabla c^{m+1/2}), z_h) - R(U^m, C^{m+1/2}, z_h) &= ((u^m - U^m).\nabla c^{m+1/2}, z_h) \\ &+ (U^m \nabla (c^{m+1/2} - C^{m+1/2}), z_h) + (U^m.\nabla C^{m+1/2}, z_h - \hat{z}_h) \\ &+ [(U^m.\nabla C^{m+1/2}, \hat{z}_h) - R(U^m, C^{m+1/2}, z_h)] = J1 + J2 + J3 + J4.\end{aligned}$$

With (A1) and (A2) we have:

$$\begin{aligned}J1 = ((u^m - U^m).\nabla c^{m+1/2}, z_h) &\le M\left\|u^m - U^m\right\|.\left\|z_h\right\| \\ &\le M(\left\|u^m - U^m\right\|^2 + \left\|z_h\right\|^2).\end{aligned}$$

Using (A1), (19), (20) and (9) we have:

$$J2=(U^m.\nabla(c^{m+1/2}-C^{m+1/2}),z_h)\le M[\left\|\nabla(c^{m+1/2}-\overline{c}^{m+1/2})\right\|+\left\|\nabla(\overline{c}^{m+1/2}-C^{m+1/2})\right\|].\left\|z_h\right\|$$
$$\le M[\left\|\nabla\varsigma^{m+1/2}\right\|+\left\|\nabla\xi^{m+1/2}\right\|].\left\|z_h\right\|$$
$$\le M[\frac{\sqrt{6}}{k}\left\|\hat{\varsigma}^{m+1/2}\right\|^2+\frac{\sqrt{6}}{k}\left\|\xi^{m+1/2}\right\|^2]+M\left\|z_h\right\|^2$$
$$\le M(\left\|\zeta^{m+1/2}\right\|^2+\left\|\xi^{m+1/2}\right\|^2+\left\|z_h\right\|^2).$$

From (A2), (19), (20), (7) and (8) we have

$$J_3=(U^m\nabla C^{m+1/2},z_h-\hat{z}_h)\le M\left\|U^m\right\|\left\|\nabla C^{m+1/2}\right\|.\left\|z_h-\hat{z}_h\right\|$$
$$\le M\left\|\nabla\xi^{m+1/2}\right\|\left\|z_h-\hat{z}_h\right\|+M\left\|\nabla\varsigma^{m+1/2}\right\|\left\|z_h-\hat{z}_h\right\|$$
$$+M\left\|\nabla c^{m+1/2}\right\|\left\|z_h-\hat{z}_h\right\|=k_1+k_2+k_3$$
$$k_1\le M[h_c^2+\left\|\xi^{m+1/2}\right\|^2]+\varepsilon\left\|\nabla z_h\right\|^2$$
$$k_2\le M[h_c^2+\left\|\nabla\varsigma^{m+1/2}\right\|^2+\left\|z_h\right\|^2]$$
$$k_3\le M[h_c^2+\left\|z_h\right\|^2]$$
$$J_3\le M[h_c^2+\left\|\xi^{m+1/2}\right\|^2+\left\|\nabla\varsigma^{m+1/2}\right\|^2+\left\|z_h\right\|^2]+\varepsilon\left\|\nabla z_h\right\|^2.$$

Using (11), (9), (18) and (19) implies

$$J4\le\left|\sum_{i=1}^{M}z_i\sum_{j\in\Lambda i}\int_{\Gamma_{ij}}(C^{m+1/2}-C_{ij}^{m+1/2})U^m.n_{ij}\,d\Gamma\right|+\left\|C^{m+1/2}\right\|\left\|\nabla.U^m\right\|\left\|\hat{z}_h\right\|.$$

Since $C_{ij}^{m+1/2}=\alpha_{ij}^m C_i^{m+1/2}+\alpha_{ji}^m C_j^{m+1/2}$,

$$J4\le\left|\sum_{e\in T_h}\sum_{P\,P_j\in e,i<j}(z_i-z_j)\int_{\Gamma_{ij}^e}(C^{m+1/2}-C_{ij}^{m+1/2})U^m.n_{ij}\,d\Gamma\right|+$$
$$M\left\|C^{m+1/2}-\overline{c}^{m+1/2}+\overline{c}^{m+1/2}-c^{m+1/2}+c^{m+1/2}\right\|\left\|\hat{z}_h\right\|$$

$$\le M\left|\sum_{e\in T_h}h_e\left|\nabla(z_h|e\;)\right|\sum_{PiP_j\varepsilon e,i<j}\int_{\Gamma_{ij}^e}(C^{m+1/2}-C_{ij}^{m+1/2})d\Gamma\right|$$
$$+M[\left\|\xi^{m+1/2}\right\|+\left\|\varsigma^{m+1/2}\right\|+\left\|c^{m+1/2}\right\|]\left\|z_h\right\|$$

and in element e we have $\left|C^{m+1/2}-C_{ij}^{m+1/2}\right|\le 2\left|\nabla(C^{m+1/2}|e)\right|h_e$, then

$$J4\le 3M\,2\sum_{e\in T_h}h_e^3\left\|\nabla(z_h|e)\right\|.\left\|\nabla(C^{m+1/2}|e)\right\|\;+M[\left\|\xi^{m+1/2}\right\|+\left\|\varsigma^{m+1/2}\right\|+\left\|c^{m+1/2}\right\|]\left\|z_h\right\|$$

$$\leq M(h_c^2 + \left\|\xi^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2 + \left\|\nabla\varsigma^{m+1/2}\right\|^2 + \left\|z_h\right\|^2) + \varepsilon\left\|\nabla z_h\right\|^2 /$$

Hence, J_1+J_2+J_3+J_4 Implies (23).

Lemma 4.4. For all $z_h \in M_h$ and $\varsigma_i = c_i - \bar{c}_i$, $\xi_i = \bar{c}_i - C_i$,

$$\begin{aligned}\left|((u^m.\nabla c_i^{m+1/2}), z_h) - R(U^m, C_i^{m+1/2}, z_h)\right| &\leq M(h_c^2 + \left\|\xi_i^{m+1/2}\right\|^2 + \left\|\varsigma_i^{m+1/2}\right\|^2 \\ &+ \left\|\nabla\varsigma_i^{m+1/2}\right\|^2 + \left\|u^m - U^m\right\|^2 + \left\|z_h\right\|^2) + \varepsilon\left\|\nabla z_h\right\|^2,\end{aligned} \tag{24}$$

where $\varepsilon > 0$ is arbitrary small constant.

Proof:

$$\begin{aligned}((u^m.\nabla c_i^{m+1/2}), z_h) - R(U^m, C_i^{m+1/2}, z_h) &= ((u^m - U^m).\nabla c_i^{m+1/2}, z_h) \\ &+ (U^m\nabla(c_i^{m+1/2} - C_i^{m+1/2}), z_h) + (U^m.\nabla C_i^{m+1/2}, z_h - \hat{z}_h) \\ &+ [(U^m.\nabla C_i^{m+1/2}, \hat{z}_h) - R(U^m, C_i^{m+1/2}, z_h)] = J1 + J2 + J3 + J4.\end{aligned}$$

Same as Lemma 4.3.

Lemma 4.5. There exists a positive constant k_2 such that:

$$\left\|\nabla P^{m+1} - \nabla\bar{P}^{m+1}\right\| \leq k_2\left\|C^{m+1} - c^{m+1}\right\| \tag{25}$$

Proof:

We have

$$(a(C^{m+1})\nabla P^{m+1}, \nabla\upsilon) = (q^{m+1/2}, v) + (R_s^{'}(C^{m+1}), v), \tag{26}$$

$$(a(c^{m+1})\nabla\bar{p}^{m+1}, \nabla\upsilon) = (q^{m+1/2}, v) + (R_s^{'}(c^{m+1}), v). \tag{27}$$

subtracting (27) from (26), we get

$$(a(C^{m+1})\nabla(P^{m+1} - \bar{P}^{m+1}), \nabla\upsilon) + ([a(C^{m+1}) - a(c^{m+1})]\nabla\bar{P}^{m+1}, \nabla\upsilon) = (R_s^{'}(C^{m+1}) - (R_s^{'}(c^{m+1}), v)$$

Let $\upsilon = P^{m+1} - \bar{P}^{m+1} \in V_h$, then

$$\left\|\nabla(P^{m+1} - \bar{P}^{m+1})\right\|^2 \leq \left|(a(C^{m+1})\nabla(P^{m+1} - \bar{P}^{m+1}), \nabla(P^{m+1} - \bar{P}^{m+1}))\right|$$

$$= \left| ([a(c^{m+1}) - a(C^{m+1})]\nabla\bar{P}^{m+1}, \nabla(P^{m+1} - \bar{P}^{m+1})) + (R_s^{'}(C^{m+1}) - (R_s(c^{m+1}), P^{m+1} - \bar{P}^{m+1}) \right|.$$

Using (A1), we have

$$\leq M \left\|\nabla \bar{P}^{m+1}\right\| \; \left\|C^{m+1}-c^{m+1}\right\| \; \left\|\nabla(P^{m+1}-\bar{P}^{m+1})\right\| + \left\|C^{m+1}-c^{m+1}\right\| \; \left\|P^{m+1}-\bar{P}^{m+1}\right\|$$

with lemma 4.1 and (A2) if h_P >0 is sufficiently small

$\left\|\nabla \bar{P}^{m+1}\right\| \leq \left\|\nabla P^{m+1}\right\| + k_1 \left\|P\right\|_{r+1} h_P^r < M$, so we have (25).

Lemma 4.6. There exists a positive constant k_3 such that

$$\left\|U^{m+1}-u^{m+1}\right\| = k_3(\left\|C^{m+1}-c^{m+1}\right\| + h_P^r) \tag{28}$$

Proof:

$$\left\|U^{m+1}-u^{m+1}\right\| = \left\|a(C^{m+1})\nabla P^{m+1} - a(c^{m+1})\nabla P^{m+1}\right\|$$

$$\leq \left\|a(C^{m+1})\nabla(P^{m+1}-p^{m+1})\right\| + \left\|a(C^{m+1})-a(c^{m+1})\right\| \; \left\|\nabla p^{m+1}\right\|.$$

From (A1) and (A2) , we have

$$\leq \text{const.}\left\|\nabla P^{m+1}-\nabla p^{m+1}\right\| + \left\|C^{m+1}-c^{m+1}\right\| \; \left\|\nabla p^{m+1}\right\|_{L_\infty}$$

we have $\left\|\nabla P^{m+1}-\nabla p^{m+1}\right\| \leq \left\|\nabla P^{m+1}-\nabla \bar{p}^{m+1}\right\| + \left\|\nabla \bar{P}^{m+1}-\nabla p^{m+1}\right\|$.

Using Lemma 4.1 and Lemma 4.5, we get

$\left\|\nabla P^{m+1}-\nabla p^{m+1}\right\| \leq k_2 \left\|C^{m+1}-c^{m+1}\right\| + k_1 \left\|P^{m+1}\right\|_{r+1} h_p^r$, from (A2), (28) holds.

Theorem 4.3. For all $m \leq l \leq N_\tau$ if $\tau \leq \tau_0$, then

$\left\|c^{l+1}-C^{l+1}\right\| \leq M(\tau + h_c + h_P^r)$, where M independent of τ and h_c .

Proof:

Multiply (2) by z_h and integrating by parts to obtain

$t=(m+1/2)\tau$. Let $w_{m+1/2}=w(.,(m+1/2)\tau)$ and $w^{m+1/2}=(w^{m+1}+w^m)/2$.

Then,

$$\begin{aligned}(\varphi D_\tau c^m, z_h) &+ (Ec\nabla c^{m+1/2}, \nabla z_h) + (u^m \nabla c^{m+1/2}, z_h) \\ &= (g(c^{m+1/2}), z_h) + (\varphi(D_\tau c^m - \frac{\partial c}{\partial t}|_{m+1/2}), z_h) \\ &\quad + ((Ec\nabla c^{m+1/2} - Ec\nabla c_{m+1/2}), \nabla z_h) + (\ (u^m \nabla c^{m+1/2} - u_{m+1/2} \nabla c_{m+1/2}), z_h).\end{aligned} \tag{29}$$

Let $e = c - C = (c - \bar{c}) + (C - \bar{c}) = \varsigma + \xi$. Subtract (10) from (29), to obtain:

$$\begin{aligned}(\hat{\varphi} D_\tau \hat{e}^m, \hat{z}_h) &+ (Ec\nabla e^{m+1/2}, \nabla z_h) = (R(U^m, C^{m+1/2}, z_h) \\ &- (u^m \nabla c^{m+1/2}, z_h)) + ((\hat{\varphi} D_\tau \hat{c}^m, \hat{z}_h) - (\varphi D_\tau c^m, z_h)) \\ &+ ((g(c^{m+1/2}), z_h) - (\hat{g}(C^{m+1/2}), \hat{z}_h)) + (\varphi(D_\tau c^m - \frac{\partial c}{\partial t}|_{m+1/2}), z_h) \\ &+ ((Ec\nabla c^{m+1/2} - Ec\nabla c_{m+1/2}), \nabla z_h) + ((u^m \nabla c^{m+1/2} - u_{m+1/2} \nabla c_{m+1/2}), z_h).\end{aligned}$$

Hence,

$$\begin{aligned}&(\hat{\varphi} D_\tau \hat{\xi}^m, \hat{z}_h) + (Ec\nabla \xi^{m+1/2}, \nabla z_h) = -(\hat{\varphi} D_\tau \hat{\varsigma}^m, \hat{z}_h) - (Ec\nabla \varsigma^{m+1/2}, \nabla z_h) \\ &-((u^m \nabla c^{m+1/2}, z_h) - R(U^m, C^{m+1/2}, z_h)) + ((g(c^{m+1/2}), z_h) - (\hat{g}(C^{m+1/2}), \hat{z}_h)) \\ &-((\varphi D_\tau c^m, z_h) - (\hat{\varphi} D_\tau \hat{c}^m, \hat{z}_h)) + ((\varphi(D_\tau c^m - \frac{\partial c}{\partial t}|_{m+1/2}), z_h) \\ &+((Ec\nabla c^{m+1/2} - Ec\nabla c_{m+1/2}), \nabla z_h) + ((u^m \nabla c^{m+1/2} - u_{m+1/2} \nabla c_{m+1/2}), z_h) \\ &\qquad = I1 + I2 + I3 + I4 + I5 + I6.\end{aligned} \tag{30}$$

In (30) , let $z_h = \xi^{m+1/2} \in M_h$, and using (A1) the Left Hand Side (LHS) is

$$\geq \frac{\hat{\phi}\left\|\hat{\xi}^{m+1}\right\|^2 - \left\|\hat{\xi}^m\right\|^2}{2\tau} + c_0\left\|\nabla \xi^{m+1/2}\right\|^2 .$$

From (A1), we have:

$$I1 = (\hat{\phi} D_\tau \hat{\varsigma}^m, \hat{z}_h) \leq M(\left\|D_\tau \hat{\varsigma}^m\right\|^2 + \left\|\hat{\xi}^{m+1/2}\right\|^2 .$$

Using (7) and (8), we have

$$\left\|D_\tau \hat{\varsigma}^m\right\| \leq \left\|D_\tau \varsigma^m\right\| + \left\|D_\tau \varsigma^m - D_\tau \hat{\varsigma}^m\right\| \leq \left\|D_\tau \varsigma^m\right\| + Mh_c\left|D_\tau \varsigma^m\right|_1 .$$

From (9) we have

$$\left\|\hat{\xi}^{m+1/2}\right\| \leq M\left\|\xi^{m+1/2}\right\| \leq M(\left\|\xi^{m+1}\right\| + \left\|\xi^m\right\|)$$

$$I1 \leq M(\left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2 + \left\|D_\tau \varsigma^m\right\|^2 + h_c^2\left\|D_\tau \varsigma^m\right\|_1^2) .$$

Using (A1), we have $I2 \le M\left\|\nabla \varsigma^{m+1/2}\right\|^2 + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2$.

From Lemma 4.3,

$$I3 \le M(h_c^2 + \left\|\xi^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2 + \left\|\nabla \varsigma^{m+1/2}\right\|^2 + \left\|u^m - U^m\right\|^2) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2.$$

Let $\theta = \phi\dfrac{c^{m+1} - c^m}{\tau} = \phi(\dfrac{\partial c}{\partial t}\Big|_{m+1/2} + 1/24\dfrac{\partial^3 c}{\partial t^3}\Big|_{\tau^2})$. Using (A1), (A2) we get [Raviart and Girault (1979))]

$$I4 \le Mh_c(1+\tau^2)|z_h|_1 + Mh_c(1+\tau^2)\|z_h\|$$

$$\le Mh_c\|z_h\| + Mh_c|z_h|_1 + Mh_c\tau^2\|z_h\| + Mh_c\tau^2|z_h|_1$$

$$\le M(h_c^2 + h_c^2\tau^4) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2$$

$$I5 \le M(h_c^2 + \left\|\xi^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2.$$

Let I6=$K1+K2+K3$. So,

$$K1 \le M(\tau^4 + \left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2),$$
$$K2 \le M\tau^2 + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2,$$
$$K3 \le M(\tau^2 + \left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2).$$

Then, we get

$$I6 \le M(\tau^2 + \left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|.$$

Relation (30) can be written now as

$$\frac{\hat{\varphi}\left\|\hat{\xi}^{m+1}\right\|^2 - \left\|\hat{\xi}^m\right\|^2}{2\tau} + c_0\left\|\nabla \xi^{m+1/2}\right\| \le M(\left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2 + \left\|D_\tau \varsigma^m\right\|^2 + h_c^2\left\|D_\tau \varsigma^m\right\|_1^2 + \left\|\nabla \varsigma^{m+1}\right\|^2 + \left\|\nabla \varsigma^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2 + \left\|u^m - U^m\right\|^2 + h_c^2 + \tau^2) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2.$$

Take the summation from *0* to *l,* where $m \le l \le N_\tau$, and $C^0 = \bar{c}^0$ so $\xi^0 = 0$

$$\hat{\phi}\left\|\hat{\xi}^{l+1}\right\|^2 \le M_1\sum_{m=0}^{l+1}\left\|\xi^m\right\|^2\tau + M_2\left\|D_\tau\varsigma\right\|^2_{L^2(L^2)} + M_2 h_c^2\left\|D_\tau\varsigma\right\|^2_{L^2(H^1)}$$
$$+M_2\sum_{m=0}^{l}\left\|\varsigma\right\|_1^2\tau + M_2(h_c^2+\tau^2) + M_3\sum_{m=0}^{l}\left\|u^m - U^m\right\|^2\tau$$

Where

$$\left\|w\right\|^2_{L^2{}_{\Delta(x)}} = \sum_{m=0}^{N_\tau}\left\|w^m\right\|_x^2\tau .$$

Using (7) and (A1), we get

$$\left\|\xi^{l+1}\right\| \le \left\|\hat{\xi}^{l+1}\right\| + Mh_c\left|\xi^{l+1}\right|_1 \le \left\|\hat{\xi}^{l+1}\right\| + Mh_c\frac{\sqrt{6}}{k}\left\|\hat{\xi}^{l+1}\right\| \le M\left\|\hat{\xi}^{l+1}\right\|$$

$$\hat{\phi}\left\|\hat{\xi}^{l+1}\right\| \ge M_0\left\|\hat{\xi}^{l+1}\right\|,$$

so

$$\left\|\xi^{l+1}\right\|^2 \le M_1\sum_{m=0}^{l+1}\left\|\xi^m\right\|^2\tau + M_3\sum_{m=0}^{l}\left\|u^m - U^m\right\|^2\tau + M_2(\left\|D_\tau\varsigma\right\|^2_{L^2_\Delta(L^2)} + h_c^2\left\|D_\tau\varsigma\right\|^2_{L^2_\Delta(H^1)}$$
$$+\left\|\varsigma\right\|^2_{L^2_{\Delta(H1)}} + h_c^2 + \tau^2). \tag{31}$$

From Lemma 4.6, we have

$$\sum_{m=0}^{l}\left\|u^m - U^m\right\|^2\tau \le M(\sum_{m=0}^{l}\left\|\xi^m\right\|^2\tau + \left\|\varsigma\right\|^2_{L^2_\Delta(L^2)} + h_p^{2r}) \tag{32}$$

From (31) and (32), we have

$$\left\|\xi^{l+1}\right\|^2 \le M_1\sum_{m=0}^{l+1}\left\|\xi^m\right\|^2\tau + M_2(h_c^2 + \tau^2 + h_p^{2r}) .$$

From Gronwall inequality, we get:

$$\left\|\xi^{l+1}\right\| \le M(h_c + \tau + h_P^r) .$$

It is $\left\|c^{l+1} - C^{l+1}\right\| \le M(h_c + \tau + h_P^r)$.

Theorem 4.4. For all $m \le l \le N_\tau$ if $\tau \le \tau_0$, then

$$\left\|c_i^{l+1} - C_i^{l+1}\right\| \le M(\tau + h_c + h_P^r),$$

where *M* independent of τ and h_c.

Proof:

Multiply (4) by z_h and integrate by parts to obtain

$t=(m+1/2)\tau$. Let $w_{m+1/2}=w(.,(m+1/2)\tau)$ and $w^{m+1/2}=(w^{m+1}+w^m)/2$.

Then

$$
\begin{aligned}
&(\varphi K_i D_\tau c_i{}^m, z_h)+(Ec\nabla c_i^{m+1/2},\nabla z_h)+(u^m\nabla c_i^{m+1/2}, z_h)\\
&\quad =(f_i(c^{m+1/2},c_1^{m+1/2},\ldots,c_N^{m+1/2}),z_h)+(\varphi K_i(D_\tau c_i{}^m-\frac{\partial c_i}{\partial t}|_{m+1/2}),z_h)\\
&\quad +((Ec\nabla c_i^{m+1/2}-Ec\nabla c_{i,m+1/2}),\nabla z_h)+((u^m\nabla c_i^{m+1/2}-u_{m+1/2}\nabla c_{i,m+1/2}),z_h).
\end{aligned}
\tag{33}
$$

Let $e=c_i-C_i=(c_i-\overline{c}_i)+(\overline{c}_i-C_i)=\varsigma_i+\xi_i$. Subtract (10) from (33) to obtain:

$$
\begin{aligned}
&(\hat{\varphi}K_i D_\tau \hat{e}_i{}^m,\hat{z}_h)+(Ec\nabla e_i{}^{m+1/2},\nabla z_h)=(R(U^m,C_i{}^{m+1/2},z_h)\\
&\quad -(u^m\nabla c_i^{m+1/2},z_h))-((\hat{\varphi}D_\tau\hat{c}_i{}^m,\hat{z}_h)-(\varphi D_\tau c_i{}^m,z_h))\\
&\quad +((f_i(c^{m+1/2},c_1^{m+1/2},\ldots,c_N^{m+1/2}),z_h)-(\hat{f}_i(C^{m+1/2},C_1^{m+1/2},\ldots,C_N^{m+1/2}),\hat{z}_h))+(\varphi K_i(D_\tau c_i{}^m-\frac{\partial c_i}{\partial t}|_{m+1/2}),z_h)\\
&\quad +((Ec\nabla c_i^{m+1/2}-Ec\nabla c_{i,m+1/2}),\nabla z_h)+((u^m\nabla c_i^{m+1/2}-u_{m+1/2}\nabla c_{i,m+1/2}),z_h).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(\hat{\varphi}\hat{K}_i D_\tau\hat{\xi}_i{}^m,\hat{z}_h)+(Ec\nabla\xi_i{}^{m+1/2},\nabla z_h)=-(\hat{\varphi}\hat{K}_i D_\tau\hat{\varsigma}_i{}^m,\hat{z}_h)-(Ec\nabla\varsigma_i{}^{m+1/2},\nabla z_h)\\
&-((u^m\nabla c_i^{m+1/2},z_h)-R(U^m,C_i{}^{m+1/2},z_h))-((\hat{f}_i(C^{m+1/2},C_1^{m+1/2},\ldots,C_N^{m+1/2}),\hat{z}_h)\\
&-(f_i(c^{m+1/2},c_1^{m+1/2},\ldots,c_N^{m+1/2}),z_h))-(\varphi K_i D_\tau c_i{}^m,z_h)-(\hat{\varphi}\hat{K}_i D_\tau\hat{c}_i{}^m,\hat{z}_h))\\
&+((\varphi(D_\tau c_i{}^m-\frac{\partial c_i}{\partial t}|_{m+1/2}),z_h)+((Ec\nabla c_i^{m+1/2}-Ec\nabla c_{i,m+1/2}),\nabla z_h)\\
&+((u^m\nabla c_i^{m+1/2}-u_{m+1/2}\nabla c_{i,m+1/2}),z_h)\ =I1+I2+I3+I4+I5+I6.
\end{aligned}
\tag{34}
$$

The rest of the proof is the same as in Theorem 4.3.

Theorem 4.5. With the assumption (A1)~ (A3), if $h_c=O(h_P),\tau=O(h_P)$, then

$$\|c-C\|_{L^\infty(l^2)}+\|u-U\|_{L^\infty(l^2)}\le M(\tau+h_c+h_P^r).$$

Proof:

We will prove, first the inductive assumption (19). We have

$$\left\|U^{m+1}\right\|_{L_\infty} \le \left\|a(C^m)\right\|_{L_\infty}\left\|\nabla P^{m+1}\right\|_{L_\infty} \le k_4\left\|\nabla P^{m+1}\right\|_{L_\infty} .$$

But,

$$\left\|\nabla P^{m+1}\right\|_{L_\infty} \le \left\|\nabla P^{m+1} - \nabla\bar{P}^{m+1}\right\|_{L_\infty} + \left\|\nabla\bar{P}^{m+1}\right\|_{L_\infty} \le Mh_P^{-1}\left\|\nabla P^{m+1} - \nabla\bar{P}^{m+1}\right\| + k_5 \le 2k_5.$$

When $h_c = o(h_P)$, $\tau = o(h_P)$ $(r \ge 2)$, $\left\|\nabla P^{m+1}\right\|_{L_\infty} \le k^*$,

$$\left\|U^{m+1} - u^{m+1}\right\| \le k_3(\left\|C^{m+1} - c^{m+1}\right\| + h_P^r) \ , \ \forall m = 0,1,\dots,N_\tau \tag{35}$$

and from Theorem 4.3,

$$\left\|c^{m+1} - C^{m+1}\right\| \le M(h_c + \tau + h_P^r) \ , \ \forall m = 0,1,\dots,N_\tau . \tag{36}$$

Equations (35) and (36) implies

$$\left\|c^{m+1} - C^{m+1}\right\| + \left\|u^{m+1} - U^{m+1}\right\| \le M(\tau + h_P^r + h_c) \ , \ \forall m = 0,1,\dots,N_\tau .$$

This will complete the proof.

5. Numerical Applications

Example 5.1.

A two dimensional diffusion-convection problem of interest is concerned with a description of sediment transport in channels. The governing differential equation is [Smith et al. (1973)]:

$$d_x \frac{\partial^2 c}{\partial x^2} + d_y \frac{\partial^2 c}{\partial y^2} - u_x \frac{\partial c}{\partial x} - u_y \frac{\partial c}{\partial y} = \frac{\partial c}{\partial t} \quad ,$$

where c is the sediment concentration, d_x and d_y are the sediment diffusion coefficients ,u_x and u_y are the fluid velocity component in the x and y directions. The boundary conditions of the problem are

$$\frac{\partial c}{\partial y} = \frac{u_y}{d_y} c = M_1 c, \qquad (M_1 \text{ is constant}), \quad \text{at} \quad \mathrm{y} = 0,$$

$$\frac{\partial c}{\partial y} = \frac{u_y}{d_y} = M_2 \quad (M_2 \text{ is constant}), \quad \text{at} \quad \mathrm{y} = \text{-56.0}.$$

We use a Kind of upwind finite element method to solve this problem with the mesh as triangular element with 82 nodes and take the values u_x=1. E-6, u_y=0.49 , d_x=0, d_y=0.0135, we take the time step τ=300 and the number of time steps N_τ =30 for θ =1 ,the time step τ=100 and the number of steps N_τ =90 for θ = 1/2 , and also the time step τ=300 and the number of steps N_τ =30 for θ = 2/3 . In figure (5.1) we show that we can solve this problem by decreasing the time step τ and increasing the number of time steps $N\tau$ for a kind of partial upwind finite element method.

Example 5.2.

In this example, we solve a purely convective problem in one dimension [Smith et al. (1973)]:

$$-u_y \frac{\partial c}{\partial y} = \frac{\partial c}{\partial t},$$

where c is the concentration in the region $-2 \le y \le 0$, subject to the boundary conditions

$$\begin{array}{lll} c = 1, & y = 0, & 0 \le t \le 0.2, \\ c = 0, & y = 0, & t > 0.2, \\ \dfrac{\partial c}{\partial y} = 0, & y = -2, & \textit{for all } t. \end{array}$$

We discretized this region into 200 triangular elements with 202 nodes and with the same distance between any two nodes is 0.02 in both directions x and y, and take u_y =1.0, the time step τ=0.04 and the number of time steps N_τ =25 for θ =0.5. Figure (5.2) shows the computed solution after one second and it draws between the concentration and coordinate y, we can see that while y convergence to zero the value of concentration convergence oscillation to solution.

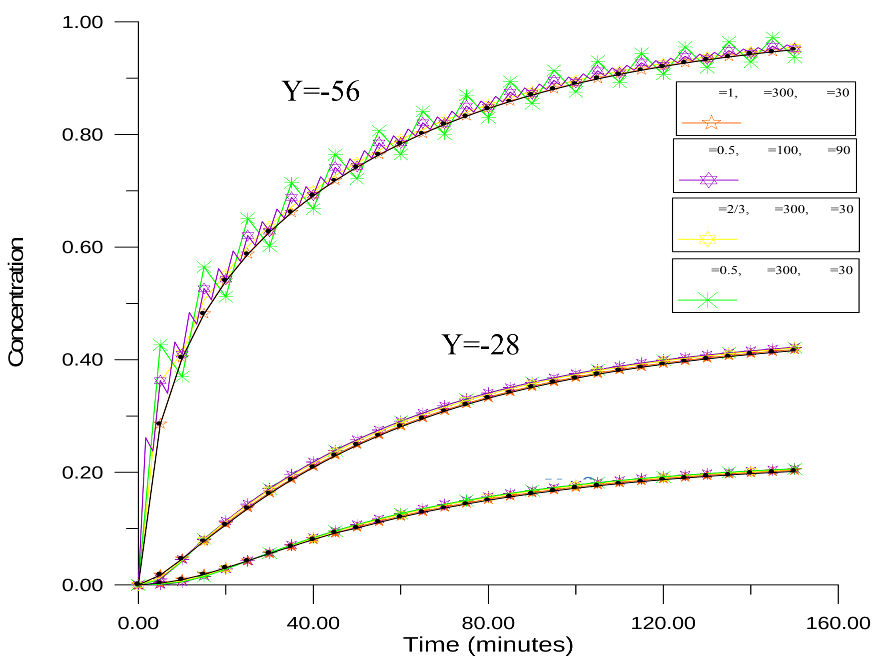

Figure 5.1. A kind of partial upwind F.E.M. for triangular element at Different values of θ in Example 5.1.

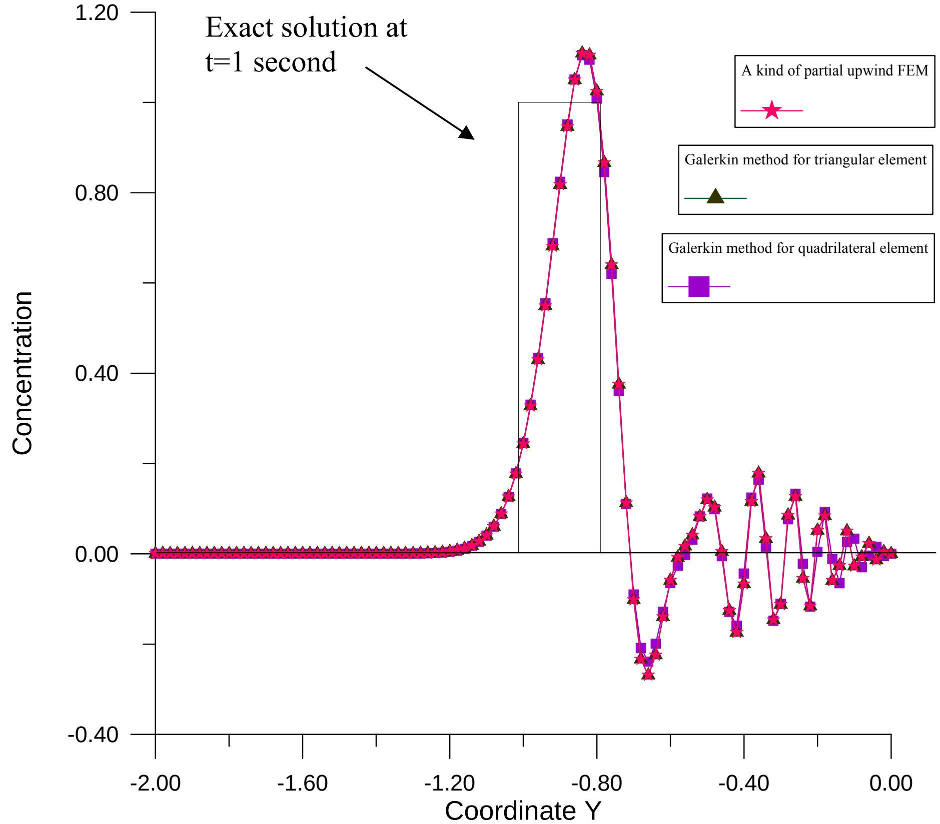

Figure 5.2. Solutions in Example 5.2 after one second for θ =0.5, τ =0.04 , N_τ =25

6. Conclusions

We used the system with large coupled of strongly non-linear partial differential equations arising from the contamination of nuclear waste in a porous media. For the incompressible case the method satisfied the discrete mass conservation law for approximate C, C_i and derived the error estimates in $L^\infty(0,\tilde{T}, L^2(\Omega))$. The two examples included clearly demonstrate certain aspects of the theory and illustrate the capabilities of a kind of partial upwind F.E.M.

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