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# Solutions of Nonlinear Second Order Multi-point Boundary Value Problems by Homotopy Perturbation Method

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## Abstract

In this paper, we present an algorithm for the numerical solution of the second order multi- point boundary value problem with suitable multi boundary conditions. The algorithm is based on the homotopy perturbation approach and the solutions are calculated in the form of a rapid convergent series. It is observed that the method gives more realistic series solutions that converge very rapidly in physical problems. Illustrative numerical examples are provided to demonstrate the efficiency and simplicity of the proposed method in solving this type of multi-point boundary value problems.

**Keywords:** Differential Equation; Multi-point Boundary Value Problem; Approximate Solution; Homotopy Perturbation Method

**MSC 2010 No.:** 30E25; 34B10; 34A34; 74H10

## 1. Introduction

Non-linear problems which have wide classes of application in science and engineering have usually been solved by perturbation methods. These methods have some limitations, e.g., the approximate solution involves a series of small parameters which poses difficulty since the majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters do lead to ideal solution while in most other cases, unsuitable choices lead to serious effects in the solutions. The homotopy perturbation method (HPM) employed here, is a new approach for finding the approximate solution that does not require small parameters, thus overcoming the limitations of the traditional perturbation techniques. The method was first proposed by He (1999) and successfully applied by other researchers like He (2000, 2005) boundary value problems by He (2006), in integro differential equation by El-Shahed (2005) , in non-Newtonian flow by Siddiqui et al. (2006), in linear PDEs of fractional order by Monami and Odibat (2007), Darvishi and Khani (2008), Belendez and Alvarez (2008), Mousa and Ragab (2008), Das and Gupta (2009) etc discussed the method to solve various linear and nonlinear problems.

The existence of positive solutions for multi-point boundary value problems (BVP) is one of the key areas of research these days owing to its wide application in engineering like in the modelling of physical problems involving vibrations occurring in a wire of uniform cross section and composed of material with different densities, in the theory of elastic stability and also its applications in fluid flow through porous media.

The nonlocal multipoint problems are mainly restricted to second order equations. In 2004, Agarwal and Kinguradze (2004) solved linear ordinary differential equations of higher order with singularities at several points. The existence of a solution for quasi-linear and resonance cases is discussed in the article of Cheung and Ren (2005) and Lin and Liu (2009). In 2006, Tatari and Dehghan (2006) used another mathematical tool the Adomian decomposition method (ADM) to obtain an approximate solution of multi-point BVP. [To the best of authors' knowledge the solution of multi-point BVP, using only the HPM is yet to be found].

In this, article the efficient mathematical tool HPM is used to solve the nonlinear second order multipoint BVP

$$\begin{cases} u''(x) + g(u,u') = f(x), \quad 0 \le x \le 1, \\ u(0) = \alpha, \qquad u(1) = \sum_{i=1}^{m} \alpha_i\, u(\eta_i) + \gamma, \end{cases} \tag{1}$$

where $\eta_i \in (0,1),\ i = 0,1,2,3,...m,\ \alpha$ and $\gamma$ are constants.

The problem (1) was recently solved by Geng and Cui (2009) in a rather complicated manner by combining the HPM and variational iteration method (VIM). The elegance of this paper lies in the simplicity of the solution, using only the HPM. Three examples are solved which show that only a few iterations are needed to obtain accurate approximate solutions. The error analysis represented graphically demonstrates the high reliability of the proposed method.

## 2. Method of Solution

According to the HPM, we construct the following homotopy of equation (1) as

$$H(v,p)=u''(x)-f(x)+p\,g(u,u')=0, \tag{2}$$

where $p\in[0,1]$ is an embedding parameter.

Now by applying the classical perturbation technique, we assume that the solution of the equation (2) can be expressed as a power series in $p$ as

$$u(x)=u_0(x)+pu_1(x)+p^2u_2(x)+p^3u_3(x)+.... \tag{3}$$

where, $u_n(x),\ n=0,1,2,3,..$ is the function to be determined by the following iterative scheme. When $p\to 1$, equation (3) becomes the approximate solution of equation (1). Substituting equation (3) in equation (2) and equating the like powers of $p$, we obtain the following set of nonlinear differential equations:

$$p^0:\begin{cases} u_0''(x)+f(x)=0, \\ u_0(0)=\alpha, \quad u_0(1)=\sum_{i=1}^{m}\alpha_i u_0(\beta_i)+\gamma, \end{cases} \tag{4}$$

$$p^1:\begin{cases} u_1''(x)+g(u,u')\big|_{p=0}=0, \\ u_1(0)=0, \quad u_1(1)=\sum_{i=1}^{m}\alpha_i u_1(\beta_i), \end{cases} \tag{5}$$

$$p^2:\begin{cases} u_2''(x)+\left.\dfrac{\partial g(u,u')}{dp}\right|_{p=0}=0, \\ u_2(0)=0, \quad u_2(1)=\sum_{i=1}^{m}\alpha_i u_2(\beta_i), \end{cases} \tag{6}$$

$$p^3:\begin{cases} u_3''(x)+\left.\dfrac{\partial^2 g(u,u')}{dp^2}\right|_{p=0}=0, \\ u_3(0)=0, \quad u_3(1)=\sum_{i=1}^{m}\alpha_i u_3(\beta_i), \end{cases} \tag{7}$$

$$\vdots$$

$$p^n : \begin{cases} u_n''(x) + \left.\dfrac{\partial^{n-1} g(u,u')}{dp^{n-1}}\right|_{p=0} = 0, \\ u_n(0) = 0, \quad u_n(1) = \sum_{i=1}^{m} \alpha_i u_n(\beta_i). \end{cases} \tag{8}$$

The above nonlinear equations can be easily solved and the components $u_n(x)$ can be completely determined, thus enabling the series solution to be entirely determined.

Finally, the approximate solution for $u(x)$ is obtained by truncating the series

$$u(x) = \lim_{N\to\infty} \phi_N(x), \tag{9}$$

where

$$\phi_N(x) = \sum_{n=0}^{N-1} u_n(x) \,.$$

## 3. Numerical Examples

In this section, to demonstrate the effectiveness of the HPM algorithm, four examples of the nonlinear systems are discussed.

**Example 1:** Consider the Nonlinear multi-point boundary value problem

$$\begin{cases} u''(x) + \dfrac{x^2(1-x)}{2} u'(x) + u^2(x) = f(x), \\ u(0) = 0, \quad u(1) = \sum_{i=-0}^{4} \left(\dfrac{1}{1+i}\right) u\left(\dfrac{i}{5}\right) + 0.708667, \end{cases}$$

with exact solution $u(x) = x^2$, when $f(x) = x^3 + 2.$

Here, applying the same method as the previous section, the $u_i(x), i = 0,1,2,...$ are calculated as

$$u_0(x) = x^2 + 0.099794\, x + 0.05x^5,$$

$$u_1(x) = 0.0821x + 0.00333\, x^4 - 0.04252x^5 - 0.002054\, x^8 + 0.00035\, x^9 - 0.000019\, x^{12},$$

$$u_2(x) = 0.01365\,x - 0.002055\,x^4 - 0.00616x^5 - 0.000143\ 9x^7 + 0.0016\,x^8 - 0.0003\ x^9$$
$$+ 0.000068\,x^{11}\ - 0.00001x^{12} + 0.000006\,x^{13} + 0.0000015\,x^{15} - 0.00000046x^{16}$$
$$+ 0.0000000055\ 4x^{19},$$

$$u_3(x) = 0.003025x - 0.0009x^4 - 0.0010237x^5 + 0.000075x^7 + 0.00035x^8 - 0.000043x^9$$
$$+ 0.000005\ x^{10} - 0.00005\ x^{11} + 0.000025\ x^{12} - 0.0000047\ x^{13} - 0.0000018\ x^{14} - 0.00000018\ x^{15}$$
$$- 0.00000007\ x^{16} + 0.00000009\ x^{17} - 0.00000007\ x^{18} + 0.00000004\ x^{19} - 0.000000009\ x^{20}$$
$$- 0.0000000006\ x^{22} + 0.0000000002\ x^{23} - 0.0000000000\ 014\ \ x^{26},$$

In the same manner, the rest of components can be obtained. Finally we get the approximate solution from equation (9), taking six terms of the series solutions. The series solution converges very rapidly. The rapid convergence means only few terms are required to get the approximate solution.

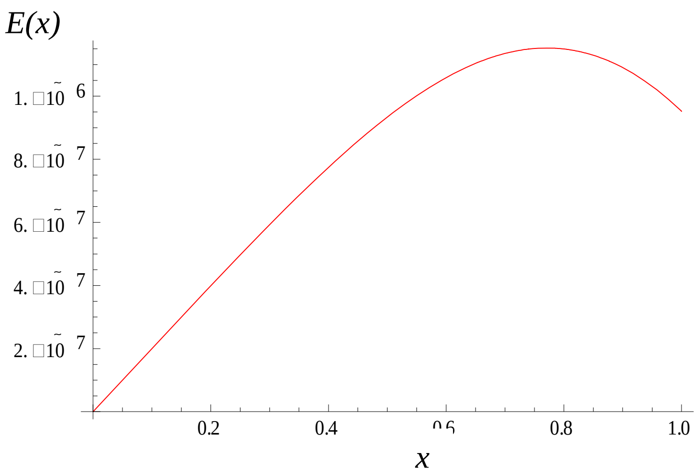


**Figure 1.** Plot of absolute error $E(x)$ vs. $x$ for Example 1

**Example 2.** Now we consider the following nonlinear multi-point boundary value problem

$$\begin{cases} u''(x) + x\,u(x)u'(x) - 2u^2(x) = f(x), \\ u(0) = 0, \quad u(1) = \sum_{i=0}^{4} \dfrac{1}{1+i} u\left(\dfrac{i}{5}\right) + 0.252, \end{cases}$$

with exact solution $u(x) = x(x-1),$ when $f(x) = x^3 - x^2 + 2.$

Proceeding as in the previous section, the values of $u_i(x), i = 0,1,2,...$ are obtained as

$$u_0(x) = -0.9399x + x^2 - 0.08333\,x^4 - 0.05\,x^5,$$

$$u_1(x) = -0.048923x + 0.07362x^4 - 0.047x^5 - 0.00187x^7 + 0.00466x^8 - 0.0021x^9 - 0.000154x^{10} + 0.00019x^{11} - 0.000057x^{12},$$

$$u_2(x) = -0.004288x + 0.00244613x^5 + 0.0046508x^7 - 0.0082383x^8 + 0.00326346x^9 + 0.000292119x^{10} - 0.0000863751x^{11} - 0.000258359x^{12} + 0.000102859x^{13} + 0.000044695x^{14} - 0.0000340023x^{15} + 0.0000074x^{16} + 0.000001123x^{17} - 0.00000065x^{18} + 0.000000125x^{19},$$

$$u_3(x) = -0.00118096x + 0.000871127x^4 - 0.000214393x^5 + 0.0000772396x^7 + 0.0000077x^8 - 0.000101922x^9 + 0.000069797x^{10} - 0.00038649x^{11} + 0.0004535x^{12} - 0.00011090 6x^{13} - 0.0000753132x^{14} + 0.00003843x^{15} + 0.00000533x^{16} - 0.00000385x^{17} - 0.0000375x^{18} + 0.00000263x^{19} - 0.000000327x^{20} - 0.00000029x^{21} + 0.000000129x^{22} - 0.0000000164x^{23} - 0.00000000447x^{24} + 0.0000000017x^{25} - 0.00000000024x^{26},$$

Similarly, the rest of the components are calculated and the approximate analytical solution of $u(x)$ can be obtained from equation (9).

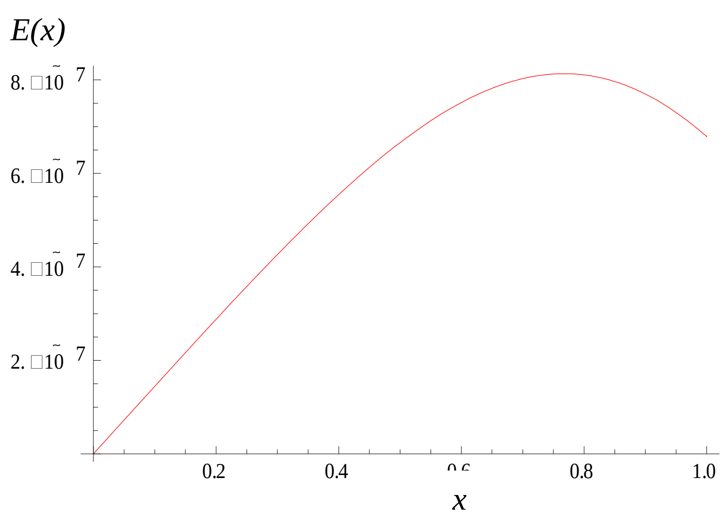


**Figure 2.** Plot of absolute error $E(x)$ vs. $x$ for Example 2

**Example 3.** Here, we consider the nonlinear multi-point boundary value problem as

$$\begin{cases} u''(x) + u(x)u'(x) = f(x), \\ u(0) = 0, \quad u(1) = \displaystyle\sum_{i=0}^{4} \frac{1}{1+i} u\left(\frac{i}{5}\right) + 0.3277, \end{cases}$$

with exact solution $u(x) = \sin x$, when $f(x) = (\cos x - 1)\sin x$.

In this case the values of $u_i(x), i = 0,1,2,...$ are calculated as

$$u_0(x) = \sin x + \frac{3e^{2x}}{8} - \frac{1}{4}\cos x \sin x + 0.01165x,$$

$$u_1(x) = -0.0086x - 0.000023x^3 - 0.00073x\cos 2x + 0.051\sin x + \cos(0.1165 + 0.251\sin x)$$
$$-0.02083\sin 3x + 0.00097\sin 4x,$$

Finally, the approximate analytical solution of the problem is obtained from equation (9).

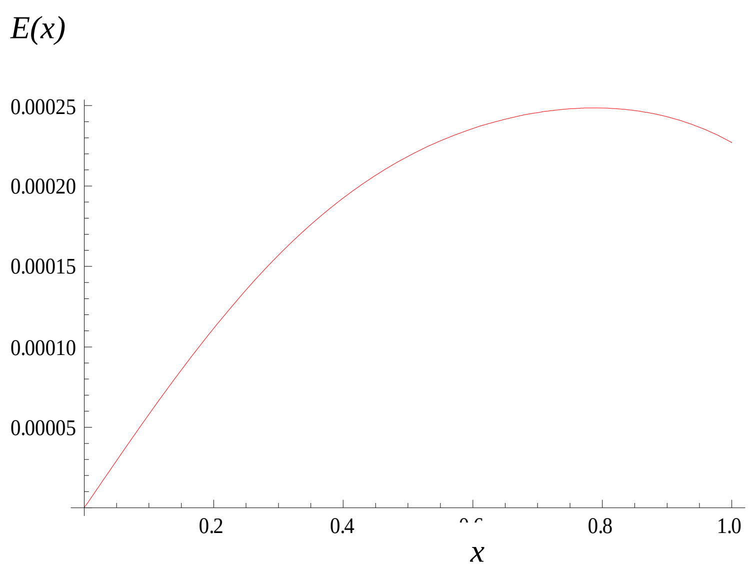


**Figure 3.** Plot of absolute error $E(x)$ vs. $x$ for Example 3

## 4. Numerical Results and Discussion

In this section, we discuss the implementation of our proposed algorithm and investigate the accuracy achieved by applying the homotopy perturbation method. The simplicity and accuracy of the proposed method is illustrated through numerical examples. The absolute error is computed $E(x) = |u_{exact}(x) - u_{approx}(x)|$ for all three examples, where $u_{exact}(x)$ is the exact solution and $u_{approx}(x)$ is an approximate solution of the problem. This was done by truncating the series (9). The errors $E(x)$ for all the considered examples are depicted in Figures 1-3. In each case the error is increases with an increasing $x$, except when $x$ is close to unity, when a decreasing tendency of the error is observed. The accuracy maybe improved by introducing more terms to the approximate solution. [Six iterations were performed in each case].

## 5. Conclusion

In this paper the powerful mathematical tool the HPM is successfully applied to solve the nonlinear multipoint BVPs. The three examples included aptly demonstrate the level of accuracy achieved in the approximation of the solutions. This method yields more realistic series solutions that converge rapidly to the exact solutions. The study shows that the method leads to quantitatively more reliable results with less computational work.

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