



Remarks on the Stability of Some Size-Structured Population Models VI: The Case When the Death Rate Depends on Juvéniles Only and the Growth Rate Depends on Size Only and the Case When Both Rates Depend on Size Only

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Abstract

We continue our study of size-structured population dynamics models when the population is divided into adults and juveniles, started in El-Doma (to appear 1) and continued in El-Doma (to appear 2). We concentrate our efforts in two special cases, the first is when the death rate depends on juveniles only and the growth rate depends on size only, and, the second is when both the death rate and the growth rate depend on size only. In both special cases we assume that the maximum size for an individual in the population is infinite. We identify three demographic parameters and show that they determine sufficient conditions for the (in)stability of a nontrivial steady state. We also give examples that illustrate the stability results. The results in this paper generalize previous results, for example, see Calsina, et al. (2003), El-Doma (2006), and El-Doma (2008).

Keywords: Adults; Juveniles; Population; Size-structure; Stability; Steady State

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1. Introduction

IN this paper, we continue our study of a size-structured population dynamics model that divides the population at any time t into adults with size larger than the maturation size $T \geq 0$, we denote by $A(t)$, and juveniles with size smaller than the maturation size, we denote by $J(t)$, started in El-Doma (to appear 1) and continued in El-Doma (to appear 2). The vital rates i.e., the birth rate, the death rate, and the growth rate, depend on size, adults, and juveniles, and accordingly, the model takes into account the limited resources as well as the intra-specific competition between adults and juveniles.

In this paper, we concentrate our efforts in the study of two special cases, the first is when the death rate depends on juveniles only and the growth rate depends on size only, and, the second is when both the death rate and the growth rate depend on size only. In both special cases we assume that the maximum size for an individual in the population is infinite.

While there are some reasons to assume that the death rate depends on adults only, for example, see El-Doma (to appear 2), there is little evidence that juveniles only can be the cause of death in a population for many species, but can not be excluded, however, it is certainly the case if the death rate is a constant independent of both juveniles and adults.

The assumption that the death rate depends on juveniles only will also allow us to generalize stability results given, for example, in Gurney, et al. (1980) and Weinstock, et al. (1987) for the classical age-structured population dynamics model of Gurtin, et al. (1974), which corresponds to problem (1) in El-Doma (to appear 1) when $V \equiv 1$, and $T = 0$.

The motivation for the second special case, i.e., for assuming that both the death rate and the growth rate depend on size only, is that such study will relate to other related models where juveniles are not considered, for example, see El-Doma (2008). Indeed the results show interesting generalization.

In our study of the local asymptotic stability of the, nontrivial steady states which are given by Theorem 2.1 (2) in El-Doma (to appear 1), we identify three demographic parameters via which we determine sufficient conditions for the (in)stability of a nontrivial steady state. We also give examples that illustrate our stability results.

This is the last paper of our series, which started by El-Doma (to appear 1) in which we studied problem (1) with general vital rates. Further stability results are given in El-Doma (to appear 2) for the case when, $V(a, J, A) = V(a)$, $\mu(a, J, A) = \mu(A)$.

The organization of this paper as follows: in section 2 we obtain stability results, and give examples that illustrate some of our theorems; in section 3 we conclude our results.

2. Stability of the Nontrivial Steady States

The case: $l = +\infty, V(a, J, A) = V(a), \mu(a, J, A) = \mu(J), \int_0^\infty \frac{d\tau}{V(\tau)} = +\infty$

We note that, in this case, if $\mu(J_\infty) = 0$, then from equation (4) in El-Doma (to appear 1), we obtain $P_\infty = +\infty$, therefore, we assume that, $\mu(J_\infty) > 0$, throughout the paper.

We also note that from Corollary 3.6 in El-Doma (to appear 1), we obtain the following condition for the local asymptotic stability of a nontrivial steady state:

$$\begin{aligned} & \int_T^l \frac{e^{-\mu(A_\infty) \int_0^a \frac{d\tau}{V(\tau)}}}{V(a)} \left| \left[\beta(a, J_\infty, A_\infty) + \delta \right] \right| da + \\ & |\gamma| \int_0^T \frac{e^{-\mu(A_\infty) \int_0^a \frac{d\tau}{V(\tau)}}}{V(a)} da + \int_0^T \int_0^a F(a, \sigma) \left| g_J(\sigma, J_\infty, A_\infty) \right| d\sigma da \\ & + \int_0^T \int_T^l \int_0^a \frac{e^{-\mu(A_\infty) \int_0^b \frac{d\tau}{V(\tau)}}}{V(b)} F(a, \sigma) \left| g_J(\sigma, J_\infty, A_\infty) \left(\beta(a, J_\infty, A_\infty) + \delta \right) \right| d\sigma dadb \\ & + \int_T^l \int_0^T \int_0^a \frac{e^{-\mu(A_\infty) \int_0^b \frac{d\tau}{V(\tau)}}}{V(b)} F(a, \sigma) \left| g_J(\sigma, J_\infty, A_\infty) \left(\beta(b, J_\infty, A_\infty) + \delta \right) \right| d\sigma dadb \\ & < 1, \end{aligned} \tag{1}$$

where δ , and γ are given, respectively, by equations (22), (23) in El-Doma (to appear 1).

From Theorem 3.2 in El-Doma (to appear 1), we obtain the following condition for the instability of a nontrivial steady state:

$$\begin{aligned} & \delta A_\infty + \gamma J_\infty + \mu'(J_\infty) \left\{ \delta \left[A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right] - \right. \\ & \left. J_\infty \int_T^l \int_0^a \beta(a, J_\infty, A_\infty) \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right\} > 0. \end{aligned} \tag{2}$$

Also, in this case, by straightforward calculations in the characteristic equation (11) in El-Doma (to appear 1), we obtain the following characteristic equation:

$$\begin{aligned} 1 = & \int_0^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-[\xi + \mu(J_\infty)] \int_0^a \frac{d\tau}{V(\tau)}} da + \frac{\xi \gamma [1 - e^{-[\xi + \mu(J_\infty)]m}]}{[\xi + \mu(J_\infty)][\xi + \chi^* e^{\mu(J_\infty)m} (1 - e^{-\mu(J_\infty)m})]} \\ & + \frac{\delta}{[\xi + \mu(J_\infty)]} \left\{ 1 - \frac{[\xi + \chi^* e^{\mu(J_\infty)m}] [1 - e^{-[\xi + \mu(J_\infty)]m}]}{[\xi + \chi^* e^{\mu(J_\infty)m} (1 - e^{-\mu(J_\infty)m})]} \right\}, \quad \xi \neq 0, \end{aligned} \tag{3}$$

where m is given by equation (5) in El-Doma (to appear 2), and χ^* is defined as follows

$$\chi^* = \frac{p_\infty(0)V(0)\mu'(J_\infty)e^{-\mu(J_\infty)m}}{\mu(J_\infty)}. \quad (4)$$

The stability results that we are going to obtain are in terms of the following Three demographic parameters:

$$\begin{aligned} \delta &= \int_T^\infty \beta_A(a, J, A)p_\infty(a)da, \\ \gamma &= \int_T^\infty \beta_J(a, J, A)p_\infty(a)da, \\ \chi^* &= \frac{p_\infty(0)V(0)\mu'(J_\infty)e^{-\mu(A_\infty)m}}{\mu(A_\infty)}, \\ &= P_\infty\mu'(J_\infty)e^{-\mu(J_\infty)m}. \end{aligned}$$

We note that δ can be interpreted as the total change in the birth rate, at the steady state, due to a change in adults only. Also, note that γ can be interpreted similarly.

If $T = 0$, then $\chi^* = P_\infty\mu'(P_\infty)$ and therefore, it can be interpreted as the total change in the death rate, at the steady state, due to a change in the population, for example, see Weinstock, et al. (1987). If $T \neq 0$, then χ^* can be interpreted as the total change in the death rate, at the steady state, due to a change in juveniles only. Note that the factor $e^{-\mu(J_\infty)m}$ in the formula defining χ^* when $T \neq 0$, is the probability of survival up size T .

We expect that $\delta < 0$, $\gamma < 0$, and $\chi^* \geq 0$ are conditions that imply the local asymptotic stability of a nontrivial steady state, for example, see El-Doma (2008), for the special case when, $T = 0$. On the other hand, from (2.2), it is easy to see that if $\delta = 0$, $\gamma \geq 0$, and $\chi^* \leq 0$, with both not equal to zero, then a nontrivial steady state is unstable. We also note that condition (2) is different than condition (2) in El-Doma (to appear 2) since $\left[A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right] \leq 0$. Which indicates a difference in response to changes in the demographic parameters δ, γ, χ^* .

Now, we let $\xi = x + iy$ in the characteristic equation (2.3), and we obtain the following pair of equations:

$$\begin{aligned} 1 &= \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-[x+\mu(J_\infty)] \int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da + \frac{\gamma}{\Delta^*} \left\{ [x(x + \mu(J_\infty)) + y^2] \times \right. \\ &\left. \left[\left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right) \left(1 - e^{-[x+\mu(J_\infty)]m} \cos ym \right) + ye^{-[x+\mu(J_\infty)]m} \sin ym \right] + \right. \\ &\left. \left. \mu(J_\infty)y \left[y \left(1 - e^{-[x+\mu(J_\infty)]m} \cos ym \right) - e^{-[x+\mu(J_\infty)]m} \sin ym \left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right) \right] \right\} + \end{aligned}$$

$$\frac{\delta}{\Delta^*} \left\{ [x + \mu(J_\infty)] \left[\left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right) \left((x + \chi^* e^{\mu(J_\infty)m}) e^{-[x+\mu(J_\infty)]m} \cos ym - \chi^* \right) + y e^{-[x+\mu(J_\infty)]m} \left(y \cos ym - \chi^* \sin ym \right) \right] - y \left[(x + \chi^* e^{\mu(J_\infty)m}) \left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right) \times e^{-[x+\mu(J_\infty)]m} \sin ym + y \left(y e^{-[x+\mu(J_\infty)]m} \sin ym - \chi^*(1 - e^{-[x+\mu(J_\infty)]m} \cos ym) \right) \right] \right\}, \quad (5)$$

$$\int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-[x+\mu(J_\infty)] \int_0^a \frac{d\tau}{V(\tau)}} \sin y \int_0^a \frac{d\tau}{V(\tau)} da = \frac{\gamma}{\Delta^*} \left\{ [x(x + \mu(J_\infty)) + y^2] \times \left[\left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right) e^{-[x+\mu(J_\infty)]m} \sin ym - y(1 - e^{-[x+\mu(J_\infty)]m} \cos ym) \right] + \mu(J_\infty)y \left[\left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right) (1 - e^{-[x+\mu(J_\infty)]m} \cos ym) + y e^{-[x+\mu(J_\infty)]m} \sin ym \right] \right\} - \frac{\delta}{\Delta^*} \left\{ [x + \mu(J_\infty)] \left[\left(x + \chi^* e^{\mu(J_\infty)m} \right) \left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right) e^{-[x+\mu(J_\infty)]m} \sin ym + y \left(y e^{-[x+\mu(J_\infty)]m} \sin ym - \chi^*(1 - e^{-[x+\mu(J_\infty)]m} \cos ym) \right) \right] + y \left[\left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right) \times \left((x + \chi^* e^{\mu(J_\infty)m}) e^{-[x+\mu(J_\infty)]m} \cos ym - \chi^* \right) + y e^{-[x+\mu(J_\infty)]m} \left(y \cos ym - \chi^* \sin ym \right) \right] \right\}, \quad (6)$$

where Δ^* is defined as

$$\Delta^* = [(x + \mu(J_\infty))^2 + y^2] \left[\left(x + \chi^*(e^{\mu(J_\infty)m} - 1) \right)^2 + y^2 \right].$$

In the following theorem, we describe the stability of a nontrivial trivial steady state when, $T = 0$, and $\delta < 0$ or $\delta > 0$. The special case when $\delta < 0$ is important for our future study because it establishes the stability of a nontrivial steady state when $T = 0$, and then we can vary our parameter T to other values to see if a change in stability occurs.

Theorem 2.1 Suppose that, $T = 0$. Then a nontrivial steady state is locally asymptotically stable if $\delta < 0$, and is unstable if $\delta > 0$.

Proof. Suppose that $\delta < 0$. We note that in this case the characteristic equation (2.3) becomes

$$1 = \int_0^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-[x+\mu(J_\infty)] \int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da + \frac{\delta[x + \mu(J_\infty)]}{[(x + \mu(J_\infty))^2 + y^2]}. \quad (7)$$

From equation (7), and equation (6) in El-Doma (to appear 1), we see that for $x \geq 0$ the right-hand side of equation (2.7) is strictly less than one since $\mu(J_\infty) > 0$ and $\delta < 0$. Therefore, the

characteristic equation (2.3) is not satisfied for any ξ with, $Re\xi \geq 0$. Accordingly, a nontrivial steady state is locally asymptotically stable.

Now, suppose that $\delta > 0$. Then from Theorem 3.2 in El-Doma (to appear 1), we obtain the result. This completes the proof of the theorem.

In the following result, we describe the stability of a nontrivial steady state in the special case when, $\delta = \gamma = 0$.

Theorem 2.2 Suppose that, $\delta = \gamma = 0$. Then a nontrivial steady state is locally asymptotically stable if $\chi^* > 0$, and, is unstable if $\chi^* < 0$.

Proof. We note that if $T = 0$, then the stability part is obtained in Theorem 3.6 in El-Doma (2008), and the instability part can be obtained easily from Theorem 3.2 in El-Doma (2008).

We also, note that in this case if $T > 0$, then the characteristic equation (2.3) can be rewritten in the following form:

$$\left[1 + \frac{\chi^*(e^{\mu(J_\infty)^m} - 1)}{\xi}\right] \left[1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a, J_\infty, A_\infty) e^{-\xi \int_0^a \frac{d\tau}{V(\tau)}} da\right] = 0, \quad \xi \neq 0. \quad (8)$$

From equation (2.8), we see that if $T > 0$, and $\chi^* < 0$, then $\xi = -\chi^*(e^{\mu(J_\infty)^m} - 1) > 0$ is a root of (2.8), accordingly, we obtain instability.

On the other hand, if $\chi^* > 0$, then $\xi = -\chi^*(e^{\mu(J_\infty)^m} - 1) < 0$, is a root of (2.8), and the only other possible root of (2.8) is when

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a, J_\infty, A_\infty) e^{-\xi \int_0^a \frac{d\tau}{V(\tau)}} da = 0, \quad \xi \neq 0. \quad (9)$$

Now, suppose that $\xi = x + iy, x \geq 0$, then by equation (6) in El-Doma (to appear 1), it is easy to see that the only possible root of equation (2.9) is, $\xi = 0$, and Theorem 3.3 in El-Doma (to appear 1) shows that, $\xi = 0$, is not a root of the characteristic equation (11) in El-Doma (to appear 1) since, $\chi^* > 0$. According, a nontrivial steady state is locally asymptotically stable if $\chi^* > 0$. This completes the proof of the theorem.

We note that the following conditions are for crossing the imaginary axis, for example, see Thieme, et al. (1993), and Iannelli (1995), stem from the fact that by Theorem 2.1 if $T = 0$, and, $\delta < 0$, then all the roots of the characteristic equation lie to the left of the imaginary axis, and by further conditions, for example, see Theorem 3.3 in El-Doma (to appear 1), they can only cross the imaginary axis to the right-half plane as T increases by crossing the imaginary axis when, $y \neq 0$:

$$1 = \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da + \frac{\gamma}{\Delta_0^*} \left\{ y^2 \left[\chi^*(e^{\mu(J_\infty)^m} - 1) \times \right. \right. \\ \left. \left. \left(1 - e^{-\mu(J_\infty)^m} \cos ym\right) + ye^{-\mu(J_\infty)^m} \sin ym \right] + \mu(J_\infty)y \left[y \left(1 - e^{-\mu(J_\infty)^m} \cos ym\right) - \right. \right.$$

$$\begin{aligned} & \left. \chi^*(1 - e^{-\mu(J_\infty)m}) \sin ym \right] \} + \frac{\delta}{\Delta_0^*} \left\{ \mu(J_\infty) \left[\chi^{*2}(e^{\mu(J_\infty)m} - 1) \right] (\cos ym - 1) + \right. \\ & y e^{-\mu(J_\infty)m} (y \cos ym - \chi^* \sin ym) \left. \right] - y \left[\chi^{*2}(e^{\mu(J_\infty)m} - 1) \right] \sin ym + \\ & y \left(y e^{-\mu(J_\infty)m} \sin ym - \chi^*(1 - e^{-\mu(J_\infty)m} \cos ym) \right) \left. \right] \}, \end{aligned} \tag{10}$$

$$\begin{aligned} & \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{V(\tau)}} \sin y \int_0^a \frac{d\tau}{V(\tau)} da = \frac{\gamma}{\Delta_0^*} \left\{ y^2 \left[\chi^*(e^{\mu(J_\infty)m} - 1) e^{-\mu(J_\infty)m} \times \right. \right. \\ & \sin ym - y(1 - e^{-\mu(J_\infty)m} \cos ym) \left. \right] + \mu(J_\infty) y \left[\chi^*(e^{\mu(J_\infty)m} - 1) (1 - e^{-\mu(J_\infty)m} \cos ym) + \right. \\ & y e^{-\mu(J_\infty)m} \sin ym \left. \right] \} - \frac{\delta}{\Delta_0^*} \left\{ \mu(J_\infty) \left[\chi^{*2}(e^{\mu(J_\infty)m} - 1) \sin ym + y \left(y e^{-\mu(J_\infty)m} \sin ym \right. \right. \right. \\ & \left. \left. - \chi^*(1 - e^{-\mu(J_\infty)m} \cos ym) \right) \right] + y \left[\chi^{*2}(e^{\mu(J_\infty)m} - 1) (\cos ym - 1) + \right. \\ & \left. y e^{-\mu(J_\infty)m} (y \cos ym - \chi^* \sin ym) \right] \left. \right\}, \end{aligned} \tag{11}$$

where Δ_0^* is defined as

$$\Delta_0^* = [\mu(J_\infty)^2 + y^2] \left[\chi^{*2}(e^{\mu(J_\infty)m} - 1)^2 + y^2 \right]. \tag{12}$$

We note that if we set $\chi^* = \chi = 0$, where χ is given by equation (4) in El-Doma (to appear 2), then the pair of equations for the crossing conditions for the case, $\mu(a, J, A) = \mu(A)$, and the case, $\mu(a, J, A) = \mu(J)$, become identical, and this is also true for the characteristic equations, therefore, we obtain the following three theorems from their corresponding case when, $\mu(a, J, A) = \mu(A)$ given in El-Doma (to appear 2):

Theorem 2.3 Suppose that, $\delta = \gamma < 0 = \chi^*$. Then a nontrivial steady state is locally asymptotically stable.

Theorem 2.4 Suppose that, $[e^{\mu(J_\infty)m} + 1]\gamma < \delta < 0, \delta \neq \gamma$, and, $\chi^* = 0$. Then a nontrivial steady state is locally asymptotically stable if $|\delta - \gamma|$, is sufficiently large, and, $\delta - \gamma$, have the appropriate sign.

Theorem 2.5 Suppose that, $\delta < 0, \delta \neq \gamma, \chi^* = 0, \delta A_\infty + \gamma J_\infty < 0$, and, $\sin ym \neq 0$. Then the result of Theorem 2.4 holds.

In the next result, we give a corollary to Theorem 2.3 for the case when $\mu(J)$ is a constant independent of J.

Corollary 2.6 Suppose that, $\delta = \gamma < 0$, and, $\mu(J)$ is a constant. Then a nontrivial steady state is locally asymptotically stable.

Proof. This result follows immediately from Theorem 2.3 since, in this case, $\chi^* = 0$. This completes the proof of the theorem.

In order facilitate our writing, we define $\frac{1}{D^*}$, N^* , and L^* as follows

$$\frac{1}{D^*} = e^{-\mu(J_\infty)m}(\delta - \gamma)[y^2 + \mu(J_\infty)\chi^*] + \gamma\mu(J_\infty)\chi^* + \delta\chi^{*2}(e^{\mu(J_\infty)m} - 1), \quad (13)$$

$$N^* = \left[e^{-\mu(J_\infty)m}(\delta - \gamma)(\mu(J_\infty) - \chi^*) - \gamma\chi^* \right] y^2 + \delta\chi^{*2}\mu(J_\infty)(e^{\mu(J_\infty)m} - 1), \quad (14)$$

$$L^* = e^{-\mu(J_\infty)m}(\delta - \gamma)(\mu(J_\infty) - \chi^*) - \gamma\chi^*. \quad (15)$$

We note that in what follows, we assume that, $T > 0$, since when $T = 0$, and $\delta < 0$, then a nontrivial steady state is local asymptotically stable by Theorem 2.1. Accordingly, $m > 0$.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $L^* = \frac{1}{D^*} = 0$, and $\delta < 0$.

Theorem 2.7 Suppose that, $\delta < 0$, $L^* = \frac{1}{D^*} = 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. We start by supposing that $\mu(J_\infty) = \chi^*$, then from $L^* = 0$, we obtain $\chi^*\gamma = 0$, and therefore, $\gamma = 0$; and from $\frac{1}{D^*} = 0$, we obtain $e^{-\mu(J_\infty)m}\delta[\mu(J_\infty)^2 + y^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)] = 0$, which is impossible. Accordingly, $\mu(J_\infty) \neq \chi^*$, and we can divide in the equation for $L^* = 0$ to obtain

$$e^{-\mu(J_\infty)m}\delta = e^{-\mu(J_\infty)m}\gamma + \frac{\gamma\chi^*}{\mu(J_\infty) - \chi^*}. \quad (16)$$

Now, we can use equation (2.16) in the equation for $\frac{1}{D^*} = 0$, to obtain the following two identical equations:

$$\frac{\chi^*\gamma}{[\mu(J_\infty) - \chi^*]} \left\{ y^2 + \left(\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)m} - 1) \right)^2 - \mu(J_\infty)\chi^*(e^{\mu(J_\infty)m} - 1) \right\} = 0, \quad (17)$$

$$\frac{\chi^*\gamma}{[\mu(J_\infty) - \chi^*]} \left\{ y^2 + \mu(J_\infty)^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)^2 + \mu(J_\infty)\chi^*(e^{\mu(J_\infty)m} - 1) \right\} = 0. \quad (18)$$

By using $L^* = 0$, $\mu(J_\infty) \neq \chi^*$, we obtain that $\gamma = 0$ is not possible in equations (2.17)-(2.18). Also, from equation (2.18), it is easy to see that $y^2 + \mu(J_\infty)^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)^2 + \mu(J_\infty)\chi^*(e^{\mu(J_\infty)m} - 1) > 0$ when $\chi^* > 0$; and also, from equation (2.17), it is easy to see

$y^2 + \left(\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)m} - 1)\right)^2 - \mu(J_\infty)\chi^*(e^{\mu(J_\infty)m} - 1) > 0$ when $\chi^* < 0$. Accordingly, only $\chi^* = 0$ is possible in equations (2.17)-(2.18), and from $L^* = 0$, we obtain $\delta = \gamma$, and hence, we obtain the result from Theorem 2.3. This completes the proof of the Theorem.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $L^* = N^* = 0$, and $\delta < 0$.

Theorem 2.8 Suppose that, $\delta < 0, L^* = N^* = 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. We note that in this case, we obtain that $\mu(J_\infty)\chi^{*2}(e^{\mu(J_\infty)m} - 1) = 0$, which implies that $\chi^* = 0$, since otherwise $m = 0$ and the result follows from Theorem 2.1. Hence using $\chi^* = 0$ in $L^* = 0$, we obtain $\delta = \gamma$, and accordingly, the local asymptotic stability follows from Theorem 2.3. This completes the proof of the Theorem.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $L^* = 0$, and $\delta = \gamma < 0$.

Theorem 2.9 Suppose that, $L^* = 0$, and, $\delta = \gamma < 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. From $L^* = 0$, we obtain that $\chi^* = 0$, and hence the result follows from Theorem 2.3. This completes the proof of the Theorem.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $\frac{1}{D^*} = N^* = 0$, and $\delta = \gamma < 0$.

Theorem 2.10 Suppose that, $\frac{1}{D^*} = N^* = 0$, and, $\delta = \gamma < 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. From $N^* = 0$, we obtain $\chi^*[\chi^*\mu(J_\infty)(e^{\mu(J_\infty)m} - 1) - y^2] = 0$. Therefore, either $\chi^* = 0$, and hence the result follows from Theorem 2.3, or $y^2 = \chi^*\mu(J_\infty)(e^{\mu(J_\infty)m} - 1)$, and in this case, by using $\frac{1}{D^*} = 0$, we obtain $\chi^*[\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)m} - 1)] = 0$. We note that, in this case, if we assume that $\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)m} - 1) = 0$, then we obtain that $y^2 = -\chi^{*2}(e^{\mu(J_\infty)m} - 1)^2 < 0$, which is impossible. This completes the proof of the Theorem.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $\frac{1}{D^*} = N^* = 0$, and $\delta < 0, \delta A_\infty + \gamma J_\infty < 0$.

Theorem 2.11 Suppose that, $\frac{1}{D^*} = N^* = 0, \delta < 0$, and, $\delta A_\infty + \gamma J_\infty < 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. We note that when $\frac{1}{D^*} = L^* = 0$, then by Theorem 2.7, we obtain the result. Therefore, we only need to consider the case $L^* \neq 0$. Accordingly, we obtain the following equations for

y^2 from $N^* = 0$:

$$y^2 = -\frac{\mu(J_\infty)\chi^{*2}(e^{\mu(J_\infty)m} - 1)\delta}{L^*}. \tag{19}$$

We note that, in this case, if $\delta = \gamma$, then we obtain the result from Theorem 2.10. Accordingly, we assume that $\delta \neq \gamma$, and therefore, from $\frac{1}{D^*} = 0$, we obtain

$$y^2 = -\mu(J_\infty)\chi^* - \frac{[\gamma\mu(J_\infty)\chi^* + \delta\chi^{*2}(e^{\mu(J_\infty)m} - 1)]}{e^{-\mu(J_\infty)m}(\delta - \gamma)}. \tag{20}$$

Now, using equations (2.19)-(2.20), we obtain

$$\begin{aligned} & \left[e^{-\mu(J_\infty)m}(\delta - \gamma) \right]^2 \mu(J_\infty)\chi^*(\mu(J_\infty) - \chi^*) + e^{-\mu(J_\infty)m}(\delta - \gamma) \left[\mu(J_\infty)\chi^*(\mu(J_\infty) - 2\chi^*)\gamma \right. \\ & \left. - \chi^{*3}(e^{\mu(J_\infty)m} - 1)\delta \right] - \gamma\chi^* \left[\gamma\mu(J_\infty)\chi^* + \chi^{*2}(e^{\mu(J_\infty)m} - 1)\delta \right] = 0. \end{aligned} \tag{21}$$

From equation (2.19), we deduce that $L^* > 0$, and therefore, if $\mu(J_\infty) = \chi^*$, then $L^* = -\chi^*\gamma > 0$ implies $\gamma < 0$, and hence we obtain a contradiction by using equation (2.21). Therefore, $\mu(J_\infty) \neq \chi^*$, and we can divide in equation (2.21) to obtain the following, after some tedious calculations:

$$\left[\mu(J_\infty)L^* - \chi^{*2}(e^{\mu(J_\infty)m} - 1)\delta \right] [L^* + \mu(J_\infty)\gamma] = 0, \text{ or } \chi^* = 0.$$

Then it is easy to see that $\chi^* = 0$ is the only solution of equation (2.21), since if $L^* + \mu(J_\infty)\gamma = 0$, then $\gamma < 0$, and using $L^* + \mu(J_\infty)\gamma = (\mu(J_\infty) - \chi^*)[e^{-\mu(J_\infty)m}(\delta - \gamma) + \gamma] \neq 0$ for $\gamma < 0$. But in this case by the assumptions $\delta < 0, \delta A_\infty + \gamma J_\infty < 0$, crossing with $y = 0$ is not possible by Theorem 3.3 in El-Doma (to appear 1), and the result is completed by using Theorem 2.3. This completes the proof of the theorem.

In the following result, we describe the stability of a nontrivial steady state when, $\delta = \gamma < 0, N^* = 0$, and $\chi^* \leq 0$.

Theorem 2.12 Suppose that $\delta = \gamma < 0, N^* = 0$, and, $\chi^* \leq 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. From $N^* = 0$, we obtain $\delta\chi^* \left[\mu(J_\infty)\chi^*(e^{\mu(J_\infty)m} - 1) - y^2 \right] = 0$, which implies that $\chi^* = 0$, since $\chi^* \leq 0$. Now, $\chi^* = 0$, and $\delta = \gamma$ implies $\frac{1}{D^*} = 0$, and accordingly, the result follows from Theorem 2.10. This completes the proof of the theorem.

We note that, as before, we need to impose a condition that will ensure the crossing of the imaginary axis with $y \neq 0$. The following is such a condition:

$$\begin{aligned} & \delta A_\infty + \gamma J_\infty + \frac{\chi^* e^{\mu(J_\infty)m}}{P_\infty} \left\{ \delta \left[A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right] - \right. \\ & \left. J_\infty \int_T^l \int_0^a \beta(a, J_\infty, A_\infty) \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right\} < 0. \end{aligned} \tag{22}$$

In the following result, we describe the stability of a nontrivial steady state when, $\frac{1}{D^*} = 0$.

Theorem 2.13 Suppose that, $\delta < 0$, and, $\frac{1}{D^*} = 0$. Then a nontrivial steady state is locally asymptotically stable in each of the following cases:

1)

$$N^* [\cos ym - e^{\mu(J_\infty)m}] + \delta\mu(J_\infty) [y^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)^2] \leq 0, \tag{23}$$

when $\delta \neq \gamma$, and condition (2.22) holds, where y^2 is given by

$$y^2 = -\mu(J_\infty)\chi^* - \frac{[\gamma\mu(J_\infty)\chi^* + \delta\chi^{*2}(e^{\mu(J_\infty)m} - 1)]}{e^{-\mu(J_\infty)m}(\delta - \gamma)}. \tag{24}$$

2) $\delta = \gamma$, and, $\chi^* \geq 0$.

3) $\delta = \gamma, \chi^* < 0, y = \frac{2n\pi}{m}, n = \pm 1, \pm 2, \pm 3, \dots$, and, condition (2.22) holds.

Proof. To prove 1, suppose that $\delta \neq \gamma$, then from equation (2.10), we obtain

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{v(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da = \tag{25}$$

$$\frac{N^*}{\Delta_0^*} [\cos ym - e^{\mu(J_\infty)m}] + \frac{\delta\mu(J_\infty)}{\Delta_0^*} [y^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)^2].$$

Now, using equations (25), (6) in El-Doma (to appear 1), we obtain (23). Also, from the equation for $\frac{1}{D^*} = 0$, we obtain equation (24). This proves 1.

To prove 2, suppose that $\delta = \gamma$, then from $\frac{1}{D^*} = 0$, we obtain

$$\chi^* [\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)m} - 1)] = 0, \tag{26}$$

which implies $\chi^* = 0$, since $\chi^* \geq 0$, and therefore, we obtain the result from Theorem 2.3. This proves 2.

To prove 3, we assume that $\delta = \gamma$, then from equation (2.26), we obtain that $\mu(J_\infty) = -\chi^*(e^{\mu(J_\infty)m} - 1)$, since $\chi^* < 0$, and in this case, using condition (2.23), we obtain the following condition for the local asymptotic stability of a nontrivial steady state:

$$\delta\chi^* [y^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)^2] [1 - \cos ym] \leq 0.$$

Accordingly, the result follows. This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial steady state when, $N^* = 0$.

Theorem 2.14 Suppose that, $\delta < 0, N^* = 0, \frac{1}{D^*} \neq 0$, and, $\sin ym \neq 0$. Then a nontrivial steady state is locally asymptotically stable if $L^* = 0$. If $L^* \neq 0$, then a nontrivial steady state is locally

asymptotically stable if inequality (2.22) holds, and, the following inequality holds:

$$1 + \left\{ 1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da - \frac{\mu(J_\infty)\delta}{\Delta_0^*} [y^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)^2] \right\} \times \left\{ \frac{e^{\mu(J_\infty)} - \delta D^* [y^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)^2] - \cos ym}{\sin ym} \right\} \leq 0, \tag{27}$$

where y^2 is given by

$$y^2 = -\frac{\mu(J_\infty)\chi^{*2}(e^{\mu(J_\infty)m} - 1)\delta}{L^*}. \tag{28}$$

Proof. We start by noting that if $L^* = 0$, then Theorem 2.8 gives the result.

If $L^* \neq 0$, then from the equation for $N^* = 0$, we obtain (2.28).

From equation (2.11), we obtain the following inequality for the local asymptotic stability:

$$1 + \frac{y \cos ym}{D^* \Delta_0^*} - \frac{y}{\Delta_0^*} [(\delta - \gamma)\mu(J_\infty)\chi^* + \gamma(\mu(J_\infty)\chi^* e^{\mu(J_\infty)m} - y^2) + \chi^{*2}(e^{\mu(J_\infty)m} - 1)\delta] \leq 0. \tag{29}$$

We also note that from equation (2.10), we obtain

$$1 = \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da + \frac{\delta\mu(J_\infty)}{\Delta_0^*} [y^2 + \chi^{*2}(e^{\mu(J_\infty)m} - 1)^2] - \frac{y \sin ym}{\Delta_0^* D^*}. \tag{30}$$

Now, we can solve for y in equation (2.30), since $\sin ym \neq 0$, and $\frac{1}{D^*} \neq 0$, then use it in inequality (2.29) to obtain (2.27). This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial steady state when, $N^* \neq 0$, $\frac{1}{D^*} \neq 0$, $\sin ym \neq 0$, and condition (2.22) holds.

Theorem 2.15 Suppose that, $\delta < 0$, $N^* \neq 0$, $\frac{1}{D^*} \neq 0$, $\sin ym \neq 0$, and, condition (2.22) holds. Then a nontrivial steady state is locally asymptotically stable if

$$\frac{N^*}{\Delta_0^* \sin ym} \left\{ \left[\sin ym + \frac{\Delta_0^*}{2N^*} \right]^2 + \left[\cos ym + \frac{W^*}{2N^*} \right]^2 \right\} - \frac{1}{4N^* \Delta_0^* \sin ym} \left\{ \Delta_0^{*2} + \hat{W}^{*2} \right\} \leq 0, \tag{31}$$

where, W^* , \hat{W}^* , are defined as follows

$$\begin{aligned}
 W^* &= -\left\{ \Delta_0^* \left[1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da \right] \right. \\
 &\quad \left. - \mu(J_\infty) \delta [y^2 + \chi^{*2} (e^{\mu(J_\infty)m} - 1)^2] + N e^{\mu(J_\infty)m} + \right. \\
 &\quad \left. D^* N^* \left[(\delta - \gamma) \mu(J_\infty) \chi^* + (\mu(J_\infty) \chi^* e^{\mu(J_\infty)m} - y^2) \gamma + \chi^{*2} (e^{\mu(J_\infty)m} - 1) \delta \right] \right\},
 \end{aligned} \tag{32}$$

$$\begin{aligned}
 \hat{W}^* &= \Delta_0^* \left[1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da \right] \\
 &\quad - \mu(J_\infty) \delta [y^2 + \chi^{*2} (e^{\mu(J_\infty)m} - 1)^2] + N e^{\mu(J_\infty)m} - \\
 &\quad D^* N^* \left[(\delta - \gamma) \mu(J_\infty) \chi^* + (\mu(J_\infty) \chi^* e^{\mu(J_\infty)m} - y^2) \gamma + \chi^{*2} (e^{\mu(J_\infty)m} - 1) \delta \right].
 \end{aligned} \tag{33}$$

Proof. From equation (2.10), we obtain

$$\begin{aligned}
 1 &= \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da + \frac{N^*}{\Delta_0^*} \left[\cos y m - e^{\mu(J_\infty)m} \right] \\
 &\quad + \frac{\delta \mu(J_\infty)}{\Delta_0^*} \left[y^2 + \chi^{*2} (e^{\mu(J_\infty)m} - 1)^2 \right] - \frac{y \sin y m}{\Delta_0^* D^*}.
 \end{aligned} \tag{34}$$

Since $\sin ym \neq 0$, and $\frac{1}{D^*} \neq 0$, we can solve for y in equation (2.34), and then use it in equation (2.11), in a manner similar to that in Theorem 2.14, to obtain (2.31). This completes the proof of the theorem.

In the following result, we describe the stability of a nontrivial steady state when, $\sin ym = 0$. The proof of this result is similar to Theorem 2.13 and therefore, is omitted.

Theorem 2.16 Suppose that, $\delta < 0$, $\sin ym = 0$, and, condition (2.22) holds. Then a nontrivial steady state is locally asymptotically stable if condition (2.23) is satisfied when, $y = \frac{n\pi}{m}$, $n = \pm 1, \pm 2, \pm 3, \dots$

In the following result, we give a corollary to Theorem (2.16), and the proof is obvious and therefore, is omitted.

Corollary 2.17 Suppose that, $\delta < 0 = \sin ym = N^*$, and, condition (2.22) holds. Then a nontrivial steady state is locally asymptotically stable.

In the following result, we describe the stability of a nontrivial steady state when, $\sin ym = 0$, and $L^* = 0$.

Theorem 2.18 Suppose that, $\delta < 0$, $\sin ym = L^* = 0$, and, inequality (2.22) holds. Then a nontrivial steady state is locally asymptotically stable if $\frac{ym}{\pi}$ is an even integer. On the other hand, if $\frac{ym}{\pi}$ is an odd integer, then a nontrivial steady state is locally asymptotically stable if $[y^2 - 2\chi^{*2}(e^{\mu(J_\infty)^m} - 1)] \geq 0$, where, y^2 , satisfies, $y^2 = (\frac{n\pi}{m})^2$, $n = 1, 3, 5, \dots$

Proof. The proof of the theorem follows from inequality (2.23), because if $L^* = 0$, then $N^* = \delta\mu(J_\infty)\chi^{*2}(e^{\mu(J_\infty)^m} - 1)$. Accordingly, since $\sin ym = 0$, the result follows from considering $\cos ym = 1$, and $\cos ym = -1$. This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial steady state when, $\delta < 0$, $\sin ym = 0$, and $\delta = \gamma$. The proof is reminiscent of Theorem 2.18.

Theorem 2.19 Suppose that, $\delta < 0$, $\sin ym = 0$, $\delta = \gamma$, and, inequality (2.22) holds. If $\frac{ym}{\pi}$ is an even integer, then a nontrivial steady state is locally asymptotically stable if $\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)^m} - 1) \geq 0$. On the other hand, if $\frac{ym}{\pi}$ is an odd integer, then a nontrivial steady state is locally asymptotically stable if $y^2[\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)^m} + 1)] - 2\mu(J_\infty)\chi^{*2}(e^{\mu(J_\infty)^m} - 1) \geq 0$, when, $y^2 = (\frac{n\pi}{m})^2$, $n = 1, 3, 5, \dots$

Proof. The proof follows from inequality (2.23), because if $\delta = \gamma$, then $N = -\delta\chi^*y^2 + \delta\mu(J_\infty)\chi^{*2}(e^{\mu(J_\infty)^m} - 1)$. Accordingly, the result follows from inequality (2.23) by considering $\cos ym = 1$, and $\cos ym = -1$. This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial steady state when, $\delta < 0$, $\sin ym = 0$, $\chi^* \geq 0$, and $\delta = \gamma$.

Corollary 2.20 Suppose that, $\delta < 0$, $\sin ym = 0$, $\chi^* \geq 0$, $\delta = \gamma$, and, inequality (2.22) holds. If $\frac{ym}{\pi}$ is an even integer, then a nontrivial steady state is locally asymptotically stable. On the other hand, if $\frac{ym}{\pi}$ is an odd integer, then a nontrivial steady state is locally asymptotically stable if $y^2[\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)^m} + 1)] - 2\mu(J_\infty)\chi^{*2}(e^{\mu(J_\infty)^m} - 1) \geq 0$, when, $y^2 = (\frac{n\pi}{m})^2$, $n = 1, 3, 5, \dots$

Proof. The result follows from Theorem 2.19, since in this case $\mu(J_\infty) + \chi^*(e^{\mu(J_\infty)^m} - 1) \geq 0$. This completes the proof of the corollary.

The case: $l = +\infty$, $V(a, J, A) = V(a)$, $\mu(a, J, A) = \mu(a)$, $\int_0^\infty \frac{\mu(\tau)}{V(\tau)} d\tau = +\infty$

We note that, in this case, using Corollary 3.5 in El-Doma (to appear 1), we obtain the following condition for a nontrivial steady state to be locally asymptotically stable:

$$\int_T^l \frac{\pi(a)}{V(a)} \left| \left[\beta(a, J_\infty, A_\infty) + \delta \right] \right| da + |\gamma| \int_0^T \frac{\pi(a)}{V(a)} da < 1, \quad (35)$$

where $\pi(a) = e^{-\int_0^a \frac{\mu(\tau)}{V(\tau)} d\tau}$.

Also from Theorem 3.2 in El-Doma (to appear 1), we obtain the following condition for the instability of a nontrivial steady state:

$$J_\infty \delta + A_\infty \gamma > 0. \tag{36}$$

Also, in this case, by straightforward calculations in the characteristic equation (11) in El-Doma (to appear 1), we obtain the following characteristic equation:

$$1 = \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a) e^{-\xi \int_0^a \frac{d\tau}{v(\tau)}} da + \gamma \int_0^T \frac{\pi(a)}{V(a)} e^{-\xi \int_0^a \frac{d\tau}{v(\tau)}} da + \delta \int_T^l \frac{\pi(a)}{V(a)} e^{-\xi \int_0^a \frac{d\tau}{v(\tau)}} da. \tag{37}$$

In the next result, we obtain a general stability result that generalizes the one given in El-Doma (2008), for the special case when juveniles are not considered. We also note that we can retain the result in El-Doma (2008) and obtain the characteristic equation for that case which resembles that of cannibalism, for example, see Iannelli (1995), Bekkal-Brikci, et al. (2007) and El-Doma (2007).

Theorem 2.21 Suppose that, $\delta < 0$, and, $\gamma \leq 0$. Then a nontrivial steady state is locally asymptotically stable in each of the following cases:

- 1) $y=0$.
- 2)

$$\begin{aligned}
 & (\gamma - \delta) \pi(T) \left\{ \frac{\sin ym}{y} + \frac{\mu(T)}{y^2} (1 - \cos ym) \right\} \\
 & + \frac{\gamma}{y^2} \int_0^T \frac{\pi(a)}{V(a)} \left[\mu(a)^2 - V(a) \mu'(a) \right] (1 - \cos y \int_0^a \frac{d\tau}{V(\tau)}) da \\
 & + \frac{\delta}{y^2} \int_T^l \frac{\pi(a)}{V(a)} \left[\mu(a)^2 - V(a) \mu'(a) \right] (1 - \cos y \int_0^a \frac{d\tau}{V(\tau)}) da \leq 0, \quad y \neq 0.
 \end{aligned} \tag{38}$$

Proof. To prove 1, we note that if we set $\xi = x + iy$ in the characteristic equation (2.37), we obtain

$$\begin{aligned}
 1 &= \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a) e^{-x \int_0^a \frac{d\tau}{v(\tau)}} \cos \left(y \int_0^a \frac{d\tau}{V(\tau)} \right) da + \gamma \int_0^T \frac{\pi(a)}{V(a)} e^{-x \int_0^a \frac{d\tau}{v(\tau)}} \cos \left(y \int_0^a \frac{d\tau}{V(\tau)} \right) da \\
 &+ \delta \int_T^l \frac{\pi(a)}{V(a)} e^{-x \int_0^a \frac{d\tau}{v(\tau)}} \cos \left(y \int_0^a \frac{d\tau}{V(\tau)} \right) da,
 \end{aligned} \tag{39}$$

$$0 = \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a) e^{-x \int_0^a \frac{d\tau}{v(\tau)}} \sin \left(y \int_0^a \frac{d\tau}{V(\tau)} \right) da + \gamma \int_0^T \frac{\pi(a)}{V(a)} e^{-x \int_0^a \frac{d\tau}{v(\tau)}} \sin \left(y \int_0^a \frac{d\tau}{V(\tau)} \right) da$$

$$+\delta \int_T^l \frac{\pi(a)}{V(a)} e^{-x \int_0^a \frac{d\tau}{V(\tau)}} \sin \left(y \int_0^a \frac{d\tau}{V(\tau)} \right) da. \quad (40)$$

Now, suppose that $x \geq 0, y = 0$, then from equations (6) in El-Doma (to appear 1) and (39), we obtain

$$\begin{aligned} 1 &= \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a) e^{-x \int_0^a \frac{d\tau}{V(\tau)}} da + \gamma \int_0^T \frac{\pi(a)}{V(a)} e^{-x \int_0^a \frac{d\tau}{V(\tau)}} da + \delta \int_T^l \frac{\pi(a)}{V(a)} e^{-x \int_0^a \frac{d\tau}{V(\tau)}} da \\ &\leq 1 + \gamma \int_0^T \frac{\pi(a)}{V(a)} e^{-x \int_0^a \frac{d\tau}{V(\tau)}} da + \delta \int_T^l \frac{\pi(a)}{V(a)} e^{-x \int_0^a \frac{d\tau}{V(\tau)}} da \\ &< 1. \end{aligned}$$

Accordingly, the characteristic equation (2.37) is not satisfied for any $x \geq 0$, and $y = 0$. This proves 1.

To prove 2, suppose that $x \geq 0$, and $y \neq 0$, and observe that (2.39) can be rewritten in the following form:

$$\begin{aligned} 1 &= \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a) e^{-x \int_0^a \frac{d\tau}{V(\tau)}} \cos \left(y \int_0^a \frac{d\tau}{V(\tau)} \right) da + \\ &(\gamma - \delta) e^{-\int_0^T \left[\frac{x + \mu(\tau)}{V(\tau)} \right] d\tau} \left\{ \frac{\sin ym}{y} + \left[\frac{x + \mu(T)}{y^2} \right] (1 - \cos ym) \right\} \\ &+ \frac{\gamma}{y^2} \int_0^T \frac{e^{-\int_0^a \left[\frac{x + \mu(\tau)}{V(\tau)} \right] d\tau}}{V(a)} \left[(x + \mu(a))^2 - V(a)\mu'(a) \right] \left(1 - \cos y \int_0^a \frac{d\tau}{V(\tau)} \right) da \\ &+ \frac{\delta}{y^2} \int_T^l \frac{e^{-\int_0^a \left[\frac{x + \mu(\tau)}{V(\tau)} \right] d\tau}}{V(a)} \left[(x + \mu(a))^2 - V(a)\mu'(a) \right] \left(1 - \cos y \int_0^a \frac{d\tau}{V(\tau)} \right) da. \quad (41) \end{aligned}$$

Accordingly, the result follows from inequality (2.38), equation (6) in El-Doma (to appear 1), and $y \neq 0$. This completes the proof of the theorem.

Here, we note that from equation (2.41) if $T = 0$, then the second and the third terms in the right-hand side of equation (2.41) are zeros, and therefore, it is easy to see that a nontrivial steady state is locally asymptotically stable in this case if $V(a)\mu'(a) \leq \mu(a)^2$, as has been shown in El-Doma (2008).

In the following result, we deduce the stability of a nontrivial steady state when, $V(a)\mu'(a) \leq \mu(a)^2$, and, $\delta = \gamma < 0$. We note that the proof of this result follows directly from Theorem 2.21, therefore, we omit the proof.

Corollary 2.22 Suppose that, $\delta = \gamma < 0$. Then a nontrivial steady state is locally asymptotically stable if $V(a)\mu'(a) \leq \mu(a)^2$.

In the next result, we deduce another stability result from Theorem 2.21, the proof is straightforward, and therefore, is omitted.

Corollary 2.23 Suppose that, $\delta < 0, \gamma \leq 0, V(a)\mu'(a) \leq \mu(a)^2$, and, $y = \frac{n\pi}{m}, n = \pm 1, \pm 2, \pm 3, \dots$. Then a nontrivial steady state is locally asymptotically stable if n is even. On the other hand if n is odd, then a nontrivial steady state is locally asymptotically stable if $(\gamma - \delta) \leq 0$.

Example 1: In this example, we consider an example considered in Cushing, et al. (1991) and El-Doma (to appear 1). Their interest is to determine the juvenile competitive effects on adult's fertility. They assumed that $\beta(a, J, A) = \beta(a, W), W = \alpha J + A, \alpha > 0; W_\infty = \alpha J_\infty + A_\infty, \mu(a, J, A) = \mu(a)$, and $V(a, J, A) = 1$, where the constant, α , measures the depressive effects of juveniles on adult's fertility.

Theorem 3.2 in El-Doma (to appear 1) gives the following condition for a nontrivial steady state to be unstable:

$$[\alpha J_\infty + A_\infty] \int_T^l \beta_W(a, W_\infty) p_\infty(a) da > 0.$$

Accordingly, from Theorem 3.2 in El-Doma (to appear 1), for a nontrivial steady state to be locally asymptotically stable we must have $\int_T^l \beta_W(a, W_\infty) p_\infty(a) da \leq 0$.

Now, we can use Corollary 2.22 to deduce the local asymptotic stability of a nontrivial steady state, for example, when $\alpha = 1, \int_T^l \beta_W(a, W_\infty) p_\infty(a) da < 0$, and $\mu'(a) \leq \mu(a)^2$.

We note that in El-Doma (to appear 1) we showed that if α is large i.e., when adult's fertility is adversely affected by competition from juveniles, then it is a destabilizing effect that can induce instability. This is in agreement with Cushing, et al. (1991), and the references therein.

Example 2: In this example, we consider the case when $\beta(a, J, A) = \beta_0, \mu(a, J, A) = \mu(J), V(a, J, A) = V(a)$, where β_0 is a constant.

In this case from equation (6) in El-Doma (to appear 1), we obtain

$$\mu(J_\infty) e^{\mu(J_\infty) \int_0^T \frac{d\tau}{V(\tau)}} = \beta_0. \tag{42}$$

Also, from equation (8) in El-Doma (to appear 1), we obtain

$$J_\infty = P_\infty \left[1 - e^{-\mu(J_\infty) \int_0^T \frac{d\tau}{V(\tau)}} \right]. \tag{43}$$

From equation (9) in El-Doma (to appear 1), we obtain

$$P_\infty = A_\infty e^{\mu(J_\infty) \int_0^T \frac{d\tau}{V(\tau)}}. \tag{44}$$

Now, if we can solve for J_∞ from equation (2.42), then P_∞ is determined from equation (2.43), and accordingly, A_∞ can be obtained from equation (2.44).

Therefore, using Theorem 2.2, we obtain that a nontrivial steady state is locally asymptotically stable if $\chi^* > 0$, and is unstable if $\chi^* < 0$.

Example 3: In this example, we consider the case when $\beta(a, J, A) = \beta_0(a)e^{-c_1 P}$, $\mu(a, J, A) = \mu(J)$, $V(a, J, A) = V(a)$, where c_1 is a positive constant.

In this case from equation (6) in El-Doma (to appear 1), we obtain

$$1 = e^{-c_1 P_\infty} \int_T^\infty \frac{\beta_0(a)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{v(\tau)}} da. \quad (45)$$

Also, from equation (8) in El-Doma (to appear 1), we obtain equation (2.43).

Now, using equation (2.45), we can see that if

$$\int_T^\infty \frac{\beta_0(a)}{V(a)} e^{-\mu(0) \int_0^a \frac{d\tau}{v(\tau)}} da > 1, \quad (46)$$

and

$$\int_T^\infty \frac{\beta_0(a)}{V(a)} e^{-\mu(J_\infty) \int_0^a \frac{d\tau}{v(\tau)}} da < +\infty, \quad (47)$$

then a nontrivial steady state exists.

Using equation (2.4), we obtain

$$\chi^* = A_\infty \mu'(J_\infty). \quad (48)$$

Accordingly, using Theorem 2.3, we obtain that a nontrivial steady state is locally asymptotically stable if $\chi^* = 0$.

Regarding Example 1 - Example 3, we note that we can use Theorem 2.3 in El-Doma (to appear 1), and Corollary 3.10 in El-Doma (to appear 1) to show that these steady states as well as their stability results remain unchanged if each of the vital rates is multiplied by any positive function $f(J, A) \in C^1(\mathbb{R}^{+2})$.

3. Conclusion

In this paper, we continued our study of size-structured population dynamics models which divide the population into adults and juveniles, started in El-Doma (to appear 1) and continued in El-Doma (to appear 2). The present study included two special cases, the first is when the death rate depends on juveniles only and the growth rate depends on size only; and the second is when both the death rate and the growth rate depend on size only. In both special cases we assumed that the maximum size for an individual in the population is infinite.

We note that in our first paper, El-Doma (to appear 1), in this series of three papers, we studied the general model with general vital rates, determined the steady states and obtained general conditions for the (in)stability of the (non)trivial steady states as well as several special cases. We also illustrated our results by examples.

In the second paper, El-Doma (to appear 2), in this series, we studied the same model as in El-Doma (to appear 1), with the additional assumptions that the death rate depends on adults only, and the growth rate depends on size only. We studied the stability of the steady states, and identified three demographic parameters, namely, δ and is given by equation (22) in El-Doma (to appear 1), which can be interpreted as the total change in the birth rate, at the steady state, due to a change in adults only; γ and is given by equation (23) in El-Doma (to appear 1), which can be interpreted as the total change in the birth rate, at the steady state, due to a change in juveniles only; and finally, χ , and is given by equation (4) in El-Doma (to appear 2), which can be interpreted as the total change in the death rate, at the steady state, due to a change in adults only.

We obtained several sufficient conditions for the (in)stability of the nontrivial steady states via these demographic parameters. We also illustrated our results by examples.

In this paper, we studied the stability of the steady states in the two special cases, and also determined three demographic parameters, namely, δ, γ, χ^* , where χ^* , is given by equation (2.4), which can be interpreted as the total change in the death rate, at the steady state, due to a change in juveniles only.

We obtained several sufficient conditions for the (in)stability of the nontrivial steady states via these demographic parameters. One of the interesting results that we obtained is that if $\chi = \chi^* = 0$, then the characteristic equations are identical for the two special cases studied, respectively, in El-Doma (to appear 2) and the present paper i.e., the first is when the death rate depends on adults only, and the second case is when the death rate depends on juveniles only; accordingly, the stability results are the same, for example, see Theorem 2.3 - Theorem 2.5. In general, there are many results about the first case that can not be obtained in the second case, for example, see El-Doma (to appear 2). However, there is another similarity between these two special cases: we noticed that there are two variables and one parameter, namely, $\frac{1}{D^*}, N^*, L^*$, given, respectively, by equations (2.13)-(2.15), and a corresponding set of two variables and one parameter, namely, $\frac{1}{D}, N, L$, given, respectively, by equations (23), (29), (30), in El-Doma (to appear 2), and relations like, for example, $\frac{1}{D^*} = N^* = 0$, gives a stability result, and similarly the corresponding relation $\frac{1}{D} = N = 0$, gives a stability result, for example, Theorem 2.7 in this paper corresponds to Theorem 2.12 in El-Doma (to appear 2).

We can also see that there is a basic difference in these two special cases with regards to their response to the three demographic parameters in each case, for example, from (2.2) in this paper, we obtained that a nontrivial steady state is unstable if

$$\delta A_\infty + \gamma J_\infty + \frac{\chi^* e^{\mu(A_\infty)m}}{P_\infty} \left\{ \gamma \left[J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right] - A_\infty \int_T^l \int_0^a \beta(a, J_\infty, A_\infty) \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right\} > 0.$$

Also from (2) in El-Doma (to appear 2), we obtained that a nontrivial steady state is unstable if

$$\delta A_\infty + \gamma J_\infty + \frac{\chi^* e^{\mu(A_\infty)m}}{P_\infty} \left\{ \gamma \left[J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right] - A_\infty \int_T^l \int_0^a \beta(a, J_\infty, A_\infty) \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right\} > 0.$$

The basic difference between the two conditions is that

$$\left[J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right] \leq 0,$$

while

$$\left[J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da \right] \geq 0.$$

Consequently, we could not prove in this paper, a corresponding lemma to Lemma 2.4 in El-Doma (to appear 2).

We note that this case linked our study of the stability of our size-structured population dynamics model to the study of the classical Gurtin-MacCamy's age-structured population dynamics model given in Gurtin, et al. (1974), specifically, the studies for the stability given in Gurney, et al. (1980) and Weinstock, et al. (1987), in fact, the characteristic equation for this special case, when juveniles are not considered i.e. when, $T = 0$, has the same qualitative properties as the characteristic equation of the Gurtin-MacCamy's age-structured population dynamics model, for example, see El-Doma (2008).

Our stability results generalized those given in El-Doma (2008), for the special case when juveniles are not considered. We also note that we can retain the result in El-Doma (2008) and obtain the characteristic equation for that case which resembles that of cannibalism, for example, see Iannelli (1995), Bekkal-Brikci, et al. (2007) and El-Doma (2007). Also, we illustrated our stability results by several examples.

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