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Exact solutions for the KdV6 and mKdV6 Equations via tanh-coth and sech Methods.

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Abstract

The tanh-coth method is used to seek solutions to obtain solutions to the new integrable sixth-order Korteweg-de Vries equation (KdV6). Following the analogy between the Korteweg-de Vries equation (KdV) and the modified Korteweg-de Vries equation (MKdV) we construct a new system equivalent to KdV6 from which exact solutions to original equation and derived, during the sech method.

Keywords: Tanh-coth method; sech method; Korteweg-de Vries equation (KdV); sixth-order Korteweg-de Vries equation (KdV6)

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1. Introduction

The search of exact solutions of nonlinear evolution equations which appear in many scientific fields, especially in physics attracted a huge number of research projects. Several direct and computational methods have been developed in the last few years with the aim of making further progress in this field, deriving more solutions as well as facilitating the calculations involved. The use of computer symbolic systems such as Mathematica and Maple have so far, helped with reducing the tediousness and complications of the calculations involved.

This paper, however, is devoted to the study of two forms of the new integrable sixth-order Korteweg-de Vries equation (KdV6)

$$(\partial_x^3 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0, \tag{1.1}$$

which was recently derived by Karasu et al. (2008) as an integrable particular case of the general sixth-order nonlinear wave equation

$$u_{xxxxxx} + \alpha u_x u_{xxxx} + \beta u_{xx}u_{xxx} + \gamma u_x^2 u_{xx} + \delta u_{tt} + \rho u_{xxxt} + \omega u_x u_{xt} + \sigma u_t u_{xx} = 0. \tag{1.2}$$

Here, α, β, γ, δ, ρ, ω, σ are arbitrary parameters, and $u = u(x,t)$ is a differentiable function. Using the change of variables

$$\begin{cases} v = u_x \\ w = u_t + u_{xxx} + 6u_x^2, \end{cases} \tag{1.3}$$

equation (1.1) converts to Korteweg–de Vries equation with a source Kupershmidt (2008) satisfying the third-order differential system

$$\begin{cases} v_t + v_{xxx} + 12vv_x - w_x = 0 \\ w_{xxx} + 8vw_x + 4wv_x = 0. \end{cases} \tag{1.4}$$

By means of the transformation

$$\begin{cases} v(x,t) = \tfrac{1}{2}u(x,-t) \\ w(x,t) = \tfrac{1}{2}w(x,t), \end{cases} \tag{1.5}$$

system (1.4) reduces to Kupershmidt (2008), Gómez and Salas (2008)

$$\begin{cases} u_t - 6uu_x - u_{xxx} + w_x = 0, \\ w_{xxx} + 4uw_x + 2u_x w = 0. \end{cases} \tag{1.6}$$

By the end of last year, Yao, Y. and Zeng, Y. (2008) successfully demonstrated the integrability of (1.6). Exact solutions to this last system have also been derived in Salas, A. H. and Gómez, C. A. (2008). In the same way, exact solutions to (1.1) have been obtained by the authors in Wazwaz (2008), Zhang et al. (2009). Our goal in this work is to derive soliton solutions to (1.6) using a simple method compared to those used in the previous works. In a similar way, following the analogy between KdV and modified Korteweg-de Vries equation (MKdV), we construct a new system from which exact solutions to original KdV6 can be derived.

2. Exact Solutions to Integrable KdV6 system (1.6) Using the tanh-coth Method

Using in (1.6) the wave transformation $\xi = \mu(x + \lambda t + \xi_0)$, being μ, λ constants to be determined latter and ξ_0 an arbitrary constant, integrating with respect to the new variable ξ, letting the integration constant equal to zero and simplifying, we obtain the system

$$\begin{cases} lu(\xi) - 3u^2(\xi) - \mu^2 u''(\xi) + w(\xi) = 0, \\ w'''(\xi) + 4u(\xi)w'(\xi) + 2u'(\xi)w(\xi) = 0. \end{cases} \tag{1.6a}$$

We seek solutions to system (1.6a) in the form

$$\begin{cases} u(\xi) = \sum_{i=-M}^{M} a_i \tanh^i(\xi) \\ w(\xi) = \sum_{i=-M}^{M} b_i \tanh^i(\xi), \end{cases}$$

called tanh-coth method. Balancing u'' with u^2 in the first equation in (1.6a) and w''' with uw' in the second equation of (1.6a) we obtain

$$M + 2 = 2M,$$

and

$$M + 3 = M + (M + 1),$$

from where $M = 2$. Therefore, solutions to (1.6a) take the form

$$\begin{aligned} u(\xi) &= a_{-2}\coth^2(\xi) + a_{-1}\coth(\xi) + a_0 + a_1\tanh(\xi) + a_2\tanh^2(\xi), \\ w(\xi) &= b_{-2}\coth^2(\xi) + b_{-1}\coth(\xi) + b_0 + b_1\tanh(\xi) + b_2\tanh^2(\xi), \\ &\xi = \mu(x + \lambda t + \xi_0),\ \xi_0 = \text{arbitrary constant}, \end{aligned} \tag{1.7}$$

where a_{-2}, a_{-1}, a_0, a_1, a_2, b_{-2}, b_{-1}, b_0, b_1, b_2, λ and μ are some constants to be determined. Substituting (1.7) into system (1.6a) and replacing hyperbolic functions tanh and coth with their exp form, we obtain a system of two polynomial equations in the variable $\zeta = \exp(\xi)$. Equating the coefficients of the different powers of ζ to zero yields an algebraic system in the unknowns a_i, b_i, λ and μ. Solving it with the aid of Mathematica and reversing to original variables, we get a pair of solutions to system (1.6):

First solution:

$$a_{-2}=0\,,a_{-1}=0\,,a_0=a_0\,,a_1=0\,,a_2=-2\mu^2\,,b_{-1}=0\,,b_{-2}=0\,,$$
$$b_0=-12a_0^2+2a_0\lambda+40a_0\mu^2-4\lambda\mu^2-32\mu^4\,,$$
$$b_1=0\,,\ b_2=-12a_0\mu^2+2\lambda\mu^2+16\mu^4:\ u_1(x,t)=a_0-2\mu^2\tanh^2(\mu(x+\lambda t+\xi_0)) \tag{1.8}$$

$$w_1(x,t)=-2\left(6a_0-\lambda-8\mu^2\right)\left(a_0-2\mu^2+\mu^2\tanh^2(\mu(x+\lambda t+\xi_0))\right) \tag{1.9}$$

Second solution:

$$a_{-2}=-2\mu^2\,,a_{-1}=0\,,a_0=a_0\,,a_1=0\,,a_2=0\,,$$
$$b_{-2}=-12a_0\mu^2+2\lambda\mu^2+16\mu^4,\ b_{-1}=0,\ b_0=-12a_0^2+2a_0\lambda+40a_0\mu^2-4\lambda\mu^2-32\mu^4\,,$$
$$b_1=0\,,\ b_2=0:\ u_2(x,t)=a_0-2\mu^2\coth^2(\mu(x+\lambda t+\xi_0)), \tag{1.10}$$

$$w_2(x,t)=-2\left(6a_0-\lambda-8\mu^2\right)\left(a_0-2\mu^2+\mu^2\coth^2(\mu(x+\lambda t+\xi_0))\right) \tag{1.11}$$

3. Solutions to a New System KdV6 Using the sech Method

A direct calculation shows that (1.1) reduces to

$$u_{xxxxxx}+20u_xu_{xxxx}+40u_{xx}u_{xxx}+120u_x^2u_{xx}+u_{xxxt}+4u_{xx}u_t+8u_xu_{xt}=0, \tag{1.12}$$

or, in equivalent form,

$$(\partial_x^2+4u_{xx}\partial_x^{-1}+8u_x)(u_{xt}+u_{xxxx}+12u_xu_{xx})=0. \tag{1.13}$$

Using the analogy between the KdV equation and the mKdV equation, and motivated by the structure of (1.13), the sixth-order modified Korteweg-de Vries equation (MKdV6) was introduced by Zhang et al. (2009) in the form

$$(\partial_x^3+8v_x^2\partial_x+8v_{xx}\partial_x^{-1}v_x\partial_x)(v_t+v_{xxx}+4v_x^3)=0. \tag{1.14}$$

They also showed that

$$\begin{cases}(\partial_x^3+8u_x\partial_x+4u_{xx})(u_t+u_{xxx}+6u_x^2)=2v_x+\sqrt{2}/(2i)\partial_x\\ \quad(\partial_x^3+8v_x^2\partial_x+8v_{xx}\partial_x^{-1}v_x\partial_x)(v_t+v_{xxx}+4v_x^3)=0,\end{cases} \tag{1.15}$$

where $v_x^2+\sqrt{2}/2iv_{xx}$ is the Miura transformation between the KdV6 equation (1.1) and the MKdV6 equation (1.14). Therefore, solving (1.14) and taking into account (1.15), solutions to (1.1) can be obtained. Setting $w_x = v_x^2$, the new mKdV6 equation is equivalent to the new system

$$\begin{cases} v_{xxxxxx}+20v_x^2 v_{xxxx}+80v_x v_{xx} v_{xxx}+20v_{xx}^3+120v_x^4 v_{xx}+v_{xxxt}+ \quad 8v_x^2 v_{xt}+4v_{xx}w_t=0, \\ w_{xx}-2v_x v_{xx}=0. \end{cases} \tag{1.16}$$

In order to solve this last system, let

$$\begin{aligned} &\xi=\mu(x+\lambda t+\xi_0),\ \xi_0=\text{arbitray constant;} \\ &v=v(x,t)=v(\xi) \text{ and } w=w(x,t)=w(\xi). \end{aligned} \tag{1.17}$$

Substituting (1.17) into (1.16) we obtain:

$$\begin{cases} \lambda\mu^4 v''''(\xi)+\mu^6 v^{(6)}(\xi)+4\lambda\mu^3 v''(\xi)w'(\xi)+20\mu^6 v''(\xi)^3+20\mu^6 v''''(\xi)v'(\xi)^2 \\ \quad +8\lambda\mu^4 v'(\xi)^2 v''(\xi)+120\mu^6 v'(\xi)^4 v''(\xi)+80\mu^6 v^{(3)}(\xi)v'(\xi)v''(\xi)=0. \\ w''(\xi)-2\mu v'(\xi)v''(\xi)=0. \end{cases} \tag{1.18}$$

Integrating the second equation in (1.18) gives

$$w'(\xi)-\mu v'(\xi)^2=c_1,$$

so that

$$w'(\xi)=c_1+\mu v'(\xi)^2,\ c_1=\text{arbitrary constant.} \tag{1.19}$$

Substituting (1.19) into the first equation of (1.18) and letting

$$v'(\xi)=u(\xi),$$

we finally get

$$\begin{aligned} &4c_1\lambda u'(\xi)+12\lambda\mu u(\xi)^2 u'(\xi)+120\mu^3 u(\xi)^4 u'(\xi)+20\mu^3 u'(\xi)^3 \\ &+80\mu^3 u(\xi)u'(\xi)u''(\xi)+\lambda\mu u'''(\xi)+20\mu^3 u(\xi)^2 u'''(\xi)+\mu^3 u''''(\xi)=0. \end{aligned} \tag{1.20}$$

Now, we integrate equation (1.20) once to obtain

$$\begin{aligned} &4c_1\lambda u(\xi)+\mu^3 u''''(\xi)+\mu u''(\xi)\left(20\mu^2 u(\xi)^2+\lambda\right) \\ &+20\mu^3 u(\xi)u''(\xi)^2+4\lambda\mu u(\xi)^3+24\mu^3 u(\xi)^5=c_2, \end{aligned} \tag{1.21}$$

where c_2 is the constant of integration.

Equation (1.21) is a polynomial ordinary differential equation in the unknown function $u(\xi)$ and their derivatives. To solve it, we may employ distinct methods. In this case, the sech method Hereman (2008) may be applied successfully to obtain an exact solution. We seek a solution to (1.21) in the form

$$u(\xi)=\sum_{i=0}^{M} a_i \operatorname{sech}^i(\xi).$$

Balancing u'''' with u^5 gives

$$M+4=5M,$$

so that $M=1$. Therefore, we seek solutions to (1.21) as

$$u(\xi)=a_0+a_1\operatorname{sech}(\xi).$$

Proceeding as before, we obtain two solutions to system (1.16), namely:

First solution: $a_0=0, a_1=\sqrt{2}, c_1=-\frac{9\mu}{40}, c_2=0, \lambda=-10\mu^2$:

$$\begin{cases} u(\xi)=\sqrt{2}\operatorname{sech}(\xi), \\ v(x,t)=2\sqrt{2}\tan^{-1}(\tanh(\frac{1}{2}\mu(x-10\mu^2 t+\xi_0)))+k_1, \\ w(x,t)=-\frac{9}{40}\mu^2(x-10\mu^2 t+\xi_0)+2\mu\tan(\mu(x-10\mu^2 t+\xi_0))+k_2 \end{cases}$$

Second solution: $a_0=0, a_1=\frac{1}{\sqrt{2}}, c_1=-\frac{\mu\left(\lambda+\mu^2\right)}{4\lambda}, c_2=0$:

$$\begin{cases} u(\xi)=\dfrac{1}{\sqrt{2}}\operatorname{sech}(\xi), \\ v(x,t)=\sqrt{2}\tan^{-1}\left(\tanh\left(\frac{1}{2}\mu\left(x+\lambda t+\xi_0\right)\right)\right)+k_1, \\ w(x,t)=\frac{1}{2}\mu\tanh\left(\mu\left(x+\lambda t+\xi_0\right)\right)-\frac{\mu^2\left(\lambda+\mu^2\right)}{4\lambda}\left(x+\lambda t+\xi_0\right)+k_2 \end{cases}$$

We believe that these traveling wave solutions are new in the open literature.

4. Conclusions

Two forms of the KdV6 equation have been studied from the point of view of its exact solutions using computational methods. The tanh-coth and the sech methods were employed to achieve the goals set for this work. Kink solutions were formally derived for each form. The results obtained here show that the method used can also be applied to other NLPDE's.

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